On the narrowness of transition fronts in accretion discs

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Accepted 1985 June 20. Received 1985 June 14

Summary. We consider the transition region in a bi-stable accretion disc between the high- and low-viscosity regions. We demonstrate that it is incorrect to calculate the evolution of such discs by regarding the transition region as being a region of negligible thickness across which appropriate jump conditions may be applied. Such treatments must always result in failure to conserve mass or angular momentum. This failure is most serious when there is a slow-moving interface between high- and low-viscosity regions, a situation that occurs when the transition front does not propagate through the whole disc. Thus the transition front cannot in general be considered to be narrow compared to the disc radius.

The model for dwarf nova outbursts under the most active consideration at present is the disc instability model. The viability of the model has fundamental implications covering the variability of disc-like flows to be found in other areas of astrophysics, for example in other binary systems (e.g. X-ray binaries, symbiotic stars), in star-forming regions, proto-planetary systems, or in active galactic nuclei. To obtain a global cyclic instability in such a dissipative system as an accretion disc, it is necessary for the vertically integrated viscosity, \( \mu \), to be, for some range of parameters, a two-valued function of the surface density, \( \Sigma \) (Bath & Pringle 1982). If parameters are chosen suitably, the disc whilst on the whole evolving on the viscous time-scale, is able to jump, on the thermal time-scale, between the upper value of \( \mu \) (high viscosity and therefore hot) and the lower value of \( \mu \) (low viscosity and therefore cold). For dwarf nova discs attention centres on the possibility of \( \mu(\Sigma) \) being two-valued in the region in the disc where hydrogen ionization/recombination occurs with the disc jumping between hot, optically thick and cool, optically thin states (Meyer & Meyer-Hofmeister 1981; Faulkner, Lin & Papaloizou 1983).

Extensive numerical computations of the disc instability and the related outburst behaviour have been carried out by a number of authors (Lin, Papaloizou & Faulkner 1985; Mineshige & Osaki 1985; Smak 1984). The computations are slowed by the fact that in the transition front which separates the hot from the cold regions of the disc, the computation must be carried out on...
the thermal time-scale. This is in general a factor of \(~(H/R)^2\) less than the viscous time-scale, where \(H\) is the disc thickness and \(R\) the radius and \(H \ll R\) (e.g. Pringle 1981). Moreover the width, \(w\), of the front might be extremely narrow, in principal necessitating finer zoning than has been used hitherto. The reason for this is as follows. If the front width is given by a balance between viscous flow across the front on a time-scale \(w^2/\nu\), where \(\nu\) is the local kinematic viscosity, and the thermal time-scale of the jump \(t_{\text{th}}\), then an estimate of \(w\) can be made by equating these two time-scales. In a steady disc not undergoing ionization, \(t_{\text{th}}=\alpha^{-1}\Omega^{-1}\), where \(\alpha\) is the usual Shakura–Sunyaev parameter and \(\Omega\) the local angular velocity in the disc. Across the jump, ionization energy increases \(t_{\text{th}}\), and we may take \(t_{\text{th}}=f\alpha^{-1}\Omega^{-1}\), where typically \(f=3-10\). Using the above value, one obtains \(w \sim \sqrt{\nu t_{\text{th}}} \sim f^{1/2} H \ll R\) for an \(\alpha\) model disc.

The fact that this argument implies \(w \ll R\), has led Meyer (1984) to suggest that transition fronts might be modelled by applying appropriate jump conditions across them, in the same manner as shockwaves can be treated by jump conditions in supersonic flow. To obtain the jump conditions Meyer takes the basic disc evolution equation appropriate to \(\Omega \propto R^{-3/2}\), viz.

\[
\frac{\partial \Sigma}{\partial t} = \frac{3}{R} \frac{\partial}{\partial R} \left( R^{1/2} \frac{\partial}{\partial R} \left( R^{1/2} \mu \right) \right)
\]

and approximates it by assuming that for quantities such as \(\Sigma\), \(\mu\), within the front \(\partial/\partial R \sim w^{-1} \gg R^{-1}\). In this approximation, equation (1) becomes

\[
\frac{\partial \Sigma}{\partial t} = \frac{3}{R} \frac{\partial^2 \mu}{\partial R^2}.
\]

However the basic equation (1) is actually derived from two fundamental conservation equations (e.g. Pringle 1981): one for mass conservation,

\[
\frac{\partial \Sigma}{\partial t} + \frac{1}{R} \frac{\partial}{\partial R} (R \Sigma v_R)=0,
\]

where \(v_R\) is the radial velocity in the disc, and the other for angular momentum conservation

\[
\frac{\partial \Sigma}{\partial t} + \frac{1}{R^{3/2}} \frac{\partial}{\partial R} \left( R^{3/2} \Sigma v_R \right) = -\frac{1}{R^{3/2}} \frac{\partial}{\partial R} \left( \frac{\Sigma}{3} R^1/2 \mu R^{1/2} \right).
\]

Equation (1) is derived from (2) and (3) by eliminating \(v_R\), which is easily shown to be given by

\[
v_R = -\frac{3}{R^{1/2} \Sigma} \frac{\partial}{\partial R} (\mu R^{1/2}).
\]

Let us suppose that the front occupies the region between the two radii \(R_1(t)\) and \(R_2(t)\) with \(w=R_2-R_1>0\). We also suppose that quantities such as \(\Sigma\), \(\mu\) and their derivatives have well-defined left- and right-hand limits denoted by \((\Sigma)_-\) and \((\Sigma)_+\), etc. respectively. Multiplying equation (1) by \(R\) and integrating through the front gives

\[
\frac{\partial}{\partial t} \int_{R_1}^{R_2} \Sigma \, RdR = 3 \left[ R^{1/2} \frac{\partial}{\partial R} \left( R^{1/2} \mu \right) \right]_+ - 3 \left[ R^{1/2} \frac{\partial}{\partial R} \left( R^{1/2} \mu \right) \right]_-=
\]

\[
+ (\Sigma R)_+ \frac{dR_2}{dt} - (\Sigma R)_- \frac{dR_1}{dt}.
\]
Similarly equation (3) gives
\[
\frac{\partial}{\partial t} \int_{R_1}^{R_2} \Sigma R^{3/2} dR = 3 \left\{ \frac{\partial}{\partial R} \left( R^{1/2} \mu \right) \right\}_+ - 3 \left\{ \frac{\partial}{\partial R} \left( R^{1/2} \mu \right) \right\}_- + (\Sigma R^{3/2})_+ \frac{dR_2}{dt} - (\Sigma R^{3/2})_- \frac{dR_1}{dt} - \frac{1}{2} (\mu R^{1/2})_+ + \frac{1}{2} (\mu R^{1/2})_-. \tag{6}
\]

If now the front has negligible thickness so that \((R)_+ = R_2 = R_1 = (R)_-\) and if there is no source or sink of mass at the interface, equation (5) requires that
\[
3 \left\{ \frac{\partial}{\partial R} \left( R^{1/2} \mu \right) \right\}_+ - 3 \left\{ \frac{\partial}{\partial R} \left( R^{1/2} \mu \right) \right\}_- + R_1^{1/2} \left\{ (\Sigma)_+ \frac{dR_1}{dt} - (\Sigma)_- \frac{dR_1}{dt} \right\} = 0. \tag{7}
\]

It then follows from equation (6) that
\[
\frac{\partial}{\partial t} \int_{R_1}^{R_2} (\Sigma R^{3/2} dR) = \frac{1}{2} R_1^{1/2} ((\mu)_- - (\mu)_+). \tag{8}
\]

As \((\mu)_+ \neq (\mu)_-\), it follows from equation (8) that the front interface is a source or sink of angular momentum if the front is assumed to have negligible thickness. For a narrow transition front that propagates with speed \(U = dR_1/dt = dR_2/dt\) conserving its mass content, the left-hand side of equation (8) is approximately equal to
\[
\frac{1}{2} R_1^{-1/2} U \int_{R_1}^{R_2} R \Sigma dR = \frac{1}{2} R_1^{1/2} U \Sigma^\prime,
\]
where \(\Sigma^\prime\) is an appropriate mean value of \(\Sigma\) in the front. In this case we derive from equations (5) and (6) that to order \(w^{-1}\)
\[
U = \frac{3}{wD} \{(\mu)_- - (\mu)_+\}, \quad \text{where} \quad D = \Sigma^\prime - (\Sigma_+ + \Sigma_-)/2 \tag{9}
\]
which relates the front speed to its thickness. In fact equation (9) can be shown from equations (5) and (6) to apply to any front that is assumed to be narrow. If the viscosity relaxes between the two states on a thermal time-scale, then we must have \(U \approx w/t_{th}\). It then follows from equation (9) that \(w^2 \approx t_{th} ((\mu)_+ - (\mu)_-)/\Sigma^\prime\) giving an estimate for \(w\) in agreement with our earlier argument.

We emphasize that the above discussion shows the specific need to treat in detail the front structure in order that physical quantities be correctly conserved and the front propagation be correctly described. This is particularly serious for a slowly moving interface which according to equation (9) must be wide. Such an interface occurs when a spontaneously generated front does not manage to propagate to a disc boundary, and can persist for long time periods.

An implication of the above results is that Meyer's (1984) application of the narrow transition front to the calculation of dwarf nova outburst (Meyer & Meyer-Hofmeister 1984) is incorrect. As a corollary one might note that any artificial smoothing involved in numerical calculations is not likely to have a disastrous effect on the results as long as the techniques do conserve mass and angular momentum. However, the necessity for calculating disc instabilities using timesteps appropriate to the short thermal time-scale is not easily circumvented.
References


