On gravitational field energy of a charged black hole

N. Dadhich  Department of Mathematics, University of Poona, Pune–411 007, India
V. Chellathurai  Department of Mathematics, University of Poona, Pune–411 007, India, and Spicer Memorial College, Pune–411 007, India

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Summary. By using the Lynden-Bell & Katz invariant definition for gravitational field energy for static spherical systems, we compute the field energy density for a charged sphere. It turns out that the field energy gets decreased by the presence of charge and in particular for a maximally charged \((m=e)\) black hole it is equal to \(1/2 \, Mc^2\). Unlike the Schwarzschild black hole, wherein the entire energy lies outside the hole, in this case it is shared by both interior and exterior of the hole, and for \(m = e\) it is shared equally. We also establish a complementary relationship between the Penrose and the Lynden-Bell & Katz definitions for energy inside and outside of a charged black hole.

1 Introduction

In a recent paper, Lynden-Bell & Katz (1985) have given an invariant definition of gravitational field energy density for a static spherically symmetric spacetime. They define the total energy of a gravitating source as

\[ Mc^2 = E_m + E_f \]

where \(E_m\) is termed as the matter energy and \(E_f\) is the gravitational field energy. In analogy with special relativity matter energy is defined by

\[ E_m = \int_{\Sigma} T^i_{\Sigma} \xi^i \sqrt{-g} \, d\Sigma \quad (1) \]

where \(\xi^i\) is a time-like Killing vector and \(\Sigma\) is any space-like surface on which the energy is to be evaluated. Then

\[ E_f = Mc^2 - E_m. \]

For a Schwarzschild black hole they have shown that its entire energy lies outside the horizon, i.e., \(E_f = Mc^2\). They also argue that \(E_m\) is the source of gravitation rather than the total energy \(Mc^2\).
In this note we wish to compute the field energy of a Reissner-Nordstrom black hole or a charged sphere. For this purpose we write the Reissner-Nordstrom metric in the isotropic form as
\[ ds^2 = e^{2\nu} dt^2 - e^{2\mu} \left( d\rho^2 + \rho^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right) \]  
where
\[ e^\nu = \left(1 - \frac{m^2 - e^2}{4\rho^2} \right) e^{-\mu}, \quad e^\mu = 1 + \frac{m^2 - e^2}{4\rho^2}, \quad m = \frac{GM}{c^2}, \quad e = \sqrt{GQ/c^2}. \]  
For the general metric (2) the field energy density is given by
\[ \frac{g^2}{8\pi G} = -\frac{c^4}{2\pi G} e^{-3\mu}(e^{\mu/2})'(e^{\nu+\mu/2})' \]  
where a dash denotes a derivative with respect to \( \rho \). For the metric coefficients (3), equation (4) gives
\[ g^2 = e^{-3\mu} \frac{G^2 M^2}{\rho^4} \left[ (1-\alpha^2) + \alpha^2 e^{-2\mu} \left(1 - \frac{m^2(1-\alpha^2)}{4\rho^2}\right) \right] \]  
where \( \alpha^2 = e^2/m^2 \). For \( e = 0 \), it reduces to Schwarzschild expression
\[ g = \frac{GM}{\rho^2(1+m/2\rho)^2}. \]  
Now integrating equation (5) over the space-like surface,
\[ E_t = \int \frac{g^2}{8\pi G} \sqrt{-g} d^3x = \frac{GM^2}{2\rho} \left(1 - \alpha^2(1-e^{-\mu})\right). \]  
Here the integration is taken from \( \rho \) to \( \rho = \infty \). For \( \alpha = 0 \), we get the usual expression \( E_t = GM^2/2\rho \), which for \( \rho = m/2 \) (in the isotropic coordinates \( r = 2m \) corresponds to \( \rho = m/2 \)) it gives \( E_t = Mc^2 \), implying the whole energy of a Schwarzschild black hole lies outside the horizon.

2 Results

In our case, the isotropic coordinate \( \rho \) is related to the curvature coordinate \( r \) as
\[ 2\rho = (r - m) + \sqrt{r^2 - 2mr + e^2}, \]  
which would mean that horizon \( r_+ = m + \sqrt{m^2 - e^2} \) occurs at \( 2\rho = \sqrt{m^2 - e^2} \). Thus the total field energy lying outside a charged black hole is given by writing \( 2\rho = \sqrt{m^2 - e^2} \) in equation (6),
\[ E_t = Mc^2 \left(1 - \frac{\alpha^2}{2(1+\sqrt{1-\alpha^2})}\right) = \frac{Mc^2}{2} \left(1+\sqrt{1-\alpha^2}\right). \]  
For the limiting case of \( \alpha^2 = 1 \), \( E_t = \frac{1}{2}Mc^2 \) and we can write,
\[ \frac{1}{2} Mc^2 \leq \frac{Mc^2}{2} \left(1+\sqrt{1-\alpha^2}\right) \leq Mc^2. \]  
That is, for a maximally charged black hole its total energy is equally shared by its interior and exterior. The presence of charge pushes the electrostatic contribution to the total energy into the interior of the hole. This can be seen as follows: from equation (7), the energy inside the horizon,
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\[ E_m = \frac{M c^2 - E \gamma}{(1 - \sqrt{1 - \alpha^2})} = \frac{e^2}{2r_+}. \]  

(9)

For a charged black hole we have the relation (Misner, Thorne & Wheeler 1973) connecting mass and irreducible mass as

\[ m = m_{irr} + \frac{e^2}{4m_{irr}}, \]  

(10)

which implies \( m_{irr} \leq m \leq 2m_{irr} \) (i.e. \( m = m_{irr} \) for \( e = 0 \) and \( m = 2m_{irr} \) for \( e = m \)). This shows that maximum electrostatic contribution to the hole’s mass is one half of the total mass. Manipulating (and keeping in mind \( m_{irr} \leq m \leq 2m_{irr} \)) equation (10), we get

\[ m = \frac{2m_{irr}}{1 + \sqrt{1 - \alpha^2}}, \]

and hence electrostatic contribution to the hole’s mass will be given by \( m - m_{irr} = m(2(1 - \sqrt{1 - \alpha^2}) = e^2/2r_+ \), which is the same as \( E_m \) given in equation (9). (Here we take \( G = c = 1 \).)

This shows that it is the electrostatic contribution to the total energy that lies in the interior of the hole. As charge goes to zero so does the energy in the interior and then \( E = Mc^2 \) as in the Schwarzschild case.

The presence of charge on the hole reduces the gravitational field energy as it weakens the strength of the gravitational field (Dadhich 1985). It is also reflected in decrease in surface gravity of the hole, which is zero for the case \( e = m \).

The Penrose mass (Tod 1983) for a charged black hole is given by

\[ m - \frac{e^2}{2r_+} = m \left( 1 - \frac{\alpha^2}{2(1 + \sqrt{1 - \alpha^2})} \right), \]

which is exactly equal to \( E_m \) in equation (7). However, the very important difference between the two being that the former is in the interior and the latter is in the exterior of the hole. In the latter case, the mass inside the horizon is

\[ m \left( 1 - \sqrt{1 - \alpha^2} \right) = \frac{e^2}{2r_+}. \]

In the Penrose definition \( e^2/2r_+ \) is outside and \( m - e^2/2r_+ \) is inside the horizon. In the Lynden-Bell & Katz definition, the situation is reversed. And the two thus bear a complementary relationship to each other, which is true in general. In the Schwarzschild case, the interior has \( m \) and the exterior is vacuous (Penrose) while the interior is vacuous and exterior has \( m \) (Lynden-Bell & Katz).

3 Conclusions

From the Lynden-Bell & Katz definition, we observe that the energy lying inside the hole could be extracted out, which for the maximally charged hole will be half of its total energy. For a rotating black hole, the maximum extractable energy is 29 per cent of the total energy (Misner et al. 1973). The question then is to verify the above observations for a Kerr black hole by generalizing the definition for an axially symmetric static spacetimes.
References


