The stability of a differentially rotating cylinder of an incompressible perfect fluid

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Accepted 1988 February 18. Received 1988 February 5; in original form 1987 December 3

Summary. The stability criterion for differentially rotating fluid cylinders given by Goldreich, Goodman & Narayan is re-examined. A rotation law of the form \( \Omega_0 \propto r^{-q} \) is assumed for the unperturbed state, where \( \Omega_0 \), \( r \) and \( q \) are the angular velocity, the distance from the central star and a constant, respectively. We find that, even for \( q > 1/3 \), a cylinder of an incompressible fluid is unstable provided that the half thickness of the cylinder, \( a \) is finite. The growth rate of a small perturbation is proportional to \( a^2 \) as long as \( a < 3 - q^2 \).

1 Introduction

Recently, it has been made clear by many authors that fluid discs and tori differentially rotating around a point mass are unstable to global non-axisymmetric perturbations (Blaes 1985; Blaes & Glatzel 1985; Drury 1985; Glatzel 1987a, b; Goldreich & Narayan 1985; Goldreich, Goodman & Narayan 1986; Goodman, Narayan & Goldreich 1987; Hanawa 1987; Hawley 1987; Kato 1987; Kojima 1986; Papaloizou & Pringle 1984, 1985, 1987; Zurek & Benz 1986). The first step was the discovery that a torus having constant specific angular momentum distribution is unstable (Papaloizou & Pringle 1984; Kojima 1986). One of the most interesting problems was then what rotation law a rotating system finally approaches after a redistribution of the angular momentum caused by the instabilities. The solution of this problem would give an important constraint on models of accretion discs around black holes, the solar nebula and nebulae around stars (e.g. \( \beta \)-Pictoris).

To this end, Papaloizou & Pringle (1985) examined the stability of a torus of a compressible fluid with a rotation law given by \( \Omega_0 \propto r^{-q} \), where \( \Omega_0 \), \( r \) and \( q \) are the angular velocity, the radius and a constant, respectively. They found that a slender torus with \( q < 1/3 \) is stable. This result was also supported by three-dimensional computer simulations: Zurek & Benz (1986) showed that a torus with \( q = 2 \) finally approaches one with \( q = 1/3 \) owing to the redistribution of angular momentum. Furthermore, Goldreich et al. (1986) claimed that a slender torus was stable independent of the polytropic index provided that \( q < 1/3 \) (even for a two-dimensional incompressible fluid).

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In this paper, we show that the incompressible cylinder is unstable not only when \( q > \sqrt{3} \) but also when \( q \leq \sqrt{3} \), even in the limit of a thin cylindrical shell (this corresponds to the limit of a slender torus). In Section 2, we derive the basic equations. In Section 3, the growth rate of perturbations obtained by using the analytical solution for the limit of a thin cylindrical shell is given and is compared with a numerically calculated one.

2 Basic equations

We consider a non-viscous incompressible fluid which has cylindrical geometry and constant density, \( \rho_0 \). We use cylindrical coordinates \((r, \phi, z)\). Self-gravity, the \( z \)-components of velocity and pressure gradient are neglected and only an external potential

\[
\Psi = -\frac{GM}{r},
\]

is considered.

For the unperturbed state, we consider an angular velocity profile as

\[
\Omega_0 = \Omega_p \left( \frac{r}{r_p} \right)^{-q},
\]

where \( q, r_p \) and \( \Omega_p \) are constants. The unperturbed pressure profile is then given by

\[
P_0(r) = \rho_0 \left[ \frac{\Omega_p^2 r_p^2}{2 - 2q} \left( \frac{r}{r_p} \right)^{2 - 2q} + \frac{GM}{r} + E \right],
\]

where \( E \) is a constant. In the following, \( r_p \) is taken to be the radius of the pressure maximum point. Units are chosen in order that \( r_p = GM = 1 \) (then, \( \Omega_p = 1 \)). If there is no external pressure, the radii of the inner and outer boundaries, \( r_+ \) and \( r_- \) are given by

\[
P_0(r_{\pm}) = 0.
\]

2.1 Perturbation equations

We assume that the perturbed quantities have the form

\[
\dot{Q}(r, \phi, t) = (Q_1(r) \exp [i(\omega t + m\phi)]),
\]

where \( m \) is an integer, \( \omega \) is a complex number and the bracket denotes the real part of a complex number. Then the linear perturbation equations are

\[
\frac{i\omega \nu_1 + 2\Omega_0 v_{r1}}{r} = -\frac{d}{dr} \frac{P_1}{\rho_0},
\]

\[
\frac{i\omega v_{\phi 1} + \frac{1}{r} \frac{d}{dr} v_{r1}}{r} = -\frac{imP_1}{r\rho_0},
\]

\[
\frac{1}{r} \frac{d}{dr} (rv_{r1}) + \frac{imv_{\phi 1}}{r} = 0,
\]

where \((v_{r1}, v_{\phi 1})\) and \( P_1 \) are the perturbations of velocity and pressure. Further, \( \Omega \) and the specific angular momentum \( l \) are given by

\[
\omega = \omega + m\Omega_0
\]

\[
l = r^2 \Omega_0.
\]
From equations (2.6) and (2.7), we have

\[ v_\tau = i \left( \frac{\delta}{D} \frac{d \chi_1}{dr} + \frac{2m \Omega_0}{r} \chi_1 \right), \quad (2.11) \]

\[ v_\phi = -\frac{1}{D} \left( \frac{1}{r} \frac{d \chi_1}{dr} + \frac{m \delta}{r} \chi_1 \right), \quad (2.12) \]

where

\[ D = \omega^2 - \frac{2\Omega_0}{r} \frac{dl}{dr}, \quad (2.13) \]

and

\[ \chi_1 = \frac{P_i}{\rho_0}. \quad (2.14) \]

Substituting equations (2.11) and (2.12) into (2.8), we obtain a second-order ordinary differential equation for \( \chi_1 \):

\[ \frac{d^2 \chi_1}{dr^2} + \frac{D}{r} \frac{d}{dr} \left( \frac{d \chi_1}{dr} \right) - \left( \frac{m^2}{r^2} - \frac{2mD}{\delta r} \frac{d}{dr} \frac{\Omega_0}{D} \right) \chi_1 = 0. \quad (2.15) \]

### 2.2 Boundary Condition

In this paper, we use the free boundary condition

\[ P(r_B + x, t) = 0, \quad (2.16) \]

where \( P \) is the pressure, \( r_B \) is the position vector of an unperturbed fluid element at the boundary and \( x \) is the Lagrangian displacement. The linear part of this equation can be written as

\[ \eta \omega \frac{d \chi_1}{dr} + \left( D \omega + \frac{2m \Omega_0 \eta}{r} \right) \chi_1 = 0, \quad (2.17) \]

where

\[ \eta = \frac{1}{\rho_0} \frac{d P_0}{dr} = -\frac{1}{r^2} + r \Omega_0^2. \quad (2.18) \]

In the following section, we solve equations (2.15) with (2.17) analytically in the limit of a thin cylindrical shell (hereafter, we call this solution the thin limit solution) and compare it with a numerical solution.

### 3 Thin Limit Solutions and Numerical Solutions

#### 3.1 Thin Limit Solutions

In the case of a thin cylindrical shell, an approximate value of the half thickness of the cylindrical shell is obtained from equations (2.3) and (2.4) as

\[ a = \sqrt{2P_p/(2q-3)} \rho_0, \quad (3.1) \]

where \( P_p = P_0(1) \) is the maximum pressure in the unperturbed cylinder. Then, we introduce a new
variable
\[ \xi = -\frac{r-1}{a}, \] (3.2)
and approximate positions of the inner and outer boundaries are
\[ \xi_\pm = ±1 + \frac{q+1}{3} a. \] (3.3)

Now we assume that the angular frequency has the form
\[ \omega = -m + ga \cdot s(a), \] (3.4)
where \( s(a) \) is a function of \( a \) and is finite as \( a \to 0 \). In this case, \( \bar{\omega} \) has the form
\[ \bar{\omega} = \bar{\omega}_0 a + \bar{\omega}_2 a^2 + \ldots, \] (3.5)
where \( \bar{\omega}_0 = -q m (\xi - s) \) and \( \bar{\omega}_2 = q (q+1) m \xi^2 / 2 \). Although \( s \) is a function of \( a \), the above expansion of \( \bar{\omega} \) does not lead to a wrong result as far as \( s = O(1) \). Then, the approximate form of the free boundary condition (2.17) becomes
\[ (2q-3) \left( \bar{\omega}_1 + (\bar{\omega}_2 - (q+1) \xi \bar{\omega}_1) a \right) \frac{d\xi}{d\xi} + 2(2-q) \left( \bar{\omega}_1 + (\bar{\omega}_2 - 2q \xi \bar{\omega}_1) a \right) = 0, \] (3.6)
at \( \xi = \xi_\pm \).

In order to obtain an approximate solution of equation (2.15), we define a new variable \( \zeta \) as
\[ \zeta = 2a(\xi - s). \] (3.7)
Note that \( \zeta = O(a) \). Expanding in terms of \( a \) and retaining only the lowest and the next order terms, we have an approximate form of equation (2.15) as
\[ \frac{d^2 \chi_1}{d\zeta^2} + \frac{1+2q}{2} \frac{d\chi_1}{d\zeta} - \frac{\chi_1}{\zeta} = 0. \] (3.8)
This equation has the so-called corotation singularity at \( \zeta = 0 \) and is solved by an expanding method. Retaining only the zero- and first-order terms, we have
\[ \chi_1 = \frac{A}{a} \left( \zeta + \frac{1-2q}{4} \zeta^2 \right) + B(1 + \zeta \log \zeta). \] (3.9)
where \( A \) and \( B \) are constants.

Substituting equation (3.9) into equation (3.6) and retaining only zeroth- and first-order terms in \( a \), we have
\[ f_A + g_\beta B = 0, \]
\[ f_- A + g_\beta B = 0, \] (3.10)
where
\[ f_\pm = -(2-q)[2q \sigma_\pm - (2q-3)] \sigma_\pm \]
\[ + a \left[ q(2q-1)(2-q) \sigma_\pm - (4q^2 - 11q + 3) \sigma_\pm + \frac{q+1}{3} (5q^2 - 28q + 30) \sigma_\pm + \frac{q+1}{6} (q+4)(2q-3) \right], \] (3.11)
and
\[ g_\pm = \pm(2-q)q\sigma_\pm \mp (2q-3) \]
\[ + a \left\{ q(2q^2-2q-3)\sigma_\pm + \frac{q+1}{6} (q^2+18q-30)+(2-q)[2q\sigma_\pm-(2q-3)]\sigma_\pm \log (\pm 2a\sigma_\pm) \right\}, \]
(3.12)

where \( \sigma_\pm = 1 \mp s \). The condition for the existence of non-trivial solutions for \( A \) and \( B \) is
\[ f_+ g_- - f_- g_+ = 0. \]
(3.13)

From the zeroth order term of equation (3.13), we obtained the zeroth-order part of \( s \) as
\[ s_0 = \frac{\sqrt{3(3-q^2)}}{q}. \]
(3.14)

Substituting this equation into (3.4), we have
\[ \omega = -m + ma/3(3-q^2). \]
(3.15)

This equation is consistent with the results of Papaloizou & Pringle (1985), and Goldreich et al. (1986): in the limit \( a \to 0 \), the cylinder is stable when \( q \leq \sqrt{3} \).

Now, we solve equation (3.13) including the first-order terms in \( a \) in the case of \( q \leq \sqrt{3} \). Since \( s_0 \) is real and \( |s_0| < 1 \), we assume
\[ -1 < s_R < 1, \]
(3.16)

and
\[ |s_I| \ll |1 - s_R|, \]
(3.17)

where \( s_R \) and \( s_I \) are the real and imaginary parts of \( s \), respectively. Then, from Lin’s rule that a growing mode solution can be integrated along the real axis (Lin 1945), we have
\[ \log |2a(\pm 1-s)| = \log |2a(\pm 1-s)| + \left( \frac{\pi}{2} \pm \frac{\pi}{2} \right) i \]
(3.18)

Retaining the lowest-order terms of the real and imaginary parts, in equation (3.13) and noting that
\[ \lim_{a \to 0} a \ln a = 0, \]
(3.19)

we have an approximate solution of (3.13) as
\[ s = \sqrt{\frac{3(3-q^2)}{q^2} - \frac{3(4q^2-9)^2(2-q)\pi}{2q^4}} a i. \]
(3.20)

Substituting this solution into equation (3.4), we have
\[ \omega = -m + ma \sqrt{\frac{3(3-q^2)}{q^2} - \frac{3(4q^2-9)^2(2-q)\pi}{2q^4}} a i. \]
(3.21)

When \( a \ll 3-q^2 \), this solution is further simplified as
\[ \omega = -m + ma \sqrt{\frac{3(3-q^2)}{q^2} - \frac{3(4q^2-9)^2(2-q)\pi}{4q^2(3-q^2)}} m a i. \]
(3.22)
From this equation, we see that the thin cylinder is unstable not only when $q>\sqrt{3}$ but also when $q<\sqrt{3}$. The growth rate of linear perturbations is proportional to $a$ for $q>\sqrt{3}$ (see equation 3.15) and to $a^2$ for $q<\sqrt{3}$.

3.2 Numerical Solution

We have calculated the eigenvalue $\omega$ for various values of $a$ by numerical integration of equation (2.15) with the boundary condition (2.17). Fig. 1 shows the imaginary part of the eigenvalue, i.e. the growth rate, obtained by numerical integration and for comparison also approximate solution according to equation (3.22). These values agree fairly well as far as $a<3-q^2$. Fig. 2 shows the eigenvalue $\omega$ as a function of the rotation parameter, $q$. The growth rate of the instability vanishes only when $q\rightarrow 3/2$.

4 Conclusion and discussion

In this paper, we have examined the stability of a thin cylindrical shell of an incompressible fluid with an unperturbed angular frequency distribution as $\Omega_0 \propto r^{-q}$. We found that the cylinder is unstable not only for $q>\sqrt{3}$, but also for $q<\sqrt{3}$. The growth rate of the instability vanishes only when $q\rightarrow 3/2$. Thus, we suggest that a disc of a two-dimensional incompressible fluid finally approaches the Keplerian disc, owing to angular momentum transport by the non-axisymmetric modes.

Now we imagine a cylinder of a compressible fluid under large external pressure. Such a situation may happen when a nebula around a star gets the coronal pressure at its inner surface and the ram pressure of accreting matter at its outer surface. In this case, a compressible fluid behaves as if it were an incompressible fluid; that is, $|\rho_1/\rho_0| \leq \epsilon$, where $\rho_0$ and $\rho_1$ are unperturbed and perturbed densities, respectively, and $\epsilon$ is a characteristic order parameter of the perturbed
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Figure 2. The growth rate $|\omega|_i$ of the $m=2$ mode as a function of $q$ in the cases of $a=0.1$ and $a=0.01$. The solid line represents the value obtained numerically; the dashed and the dash-dotted lines represent the values obtained from equations (3.22) and (3.15), respectively.

quantities (Sekiya 1983; Miyama, Narita & Hayashi 1987). Thus, a cylinder of a compressible fluid under large external pressure should be also unstable even in the case where $q<\sqrt{3}$. Miyama & Sekiya have confirmed this conjecture by numerical integration in the case of $N=3$, where $N$ is the polytropic index.

On the other hand, a slender cylinder of a compressible fluid without external pressure (free boundary condition) is stable for $q<\sqrt{3}$ in the case where the value of polytropic index $N$ is at least 0.5 (Miyama & Sekiya, in preparation). As was shown by Goldreich et al. (1986), the three-dimensional polytropic index $n$ is related to the two-dimensional one $N$ by $n=N-0.5$. Thus, a three-dimensional slender torus without external pressure is stable for $q<\sqrt{3}$ even if $n=0$ (incompressible). For a radially wide disc, however, the so-called acoustic mode is unstable even for $q<\sqrt{3}$ (Glatzel 1987b; Goldreich & Narayan 1985; Hanawa 1987; Kato 1987; Miyama & Sekiya, in preparation; Narayan, Goldreich & Goodman 1987).

In order to find the final rotation law of differentially rotating systems, non-linear simulations with sufficient precision and under a wide range of initial and boundary conditions and polytropic indices must be performed.

Acknowledgments

Dr Hanawa kindly sent us some copies of relevant papers. We thank the referee for valuable suggestions and linguistic comments. Numerical computations were carried out using FACOM M780 and VP200 at the Data Processing Center of Kyoto University. This work was supported by a Grant-in-Aid for Scientific Research of the Ministry of Education, Science and Culture of Japan (Nos 60300013, 62611518 and 62740144).

References

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