Tidal resonances in binary star systems – II. Slowly rotating stars

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Summary. The potential energy of tidal interactions in a binary system, in which the component stars are allowed to rotate, is formulated as a perturbation Hamiltonian which couples the dynamics of the rotating stars' oscillations and orbital motion in a self-consistent way. This extends previous work (Alexander) which considered only non-rotating stars. A method is given for finding a canonical transformation which puts the quadratic Hamiltonian, describing the linearized oscillations of the rotating star, into separable form. In this way, the action-angle formalism, used previously for discussing tidal resonances in the non-rotating case, can be extended to rotating stars. Results are presented for the case when $O(\Omega^2)$ terms in the Hamiltonian for the star's oscillations are neglected, where $\Omega$ is the frequency of the star's rotation. The behaviour of a two-mode system, which represents a good approximation in the vicinity of tidal resonance, is examined in detail. The procedure for treating an arbitrary number of modes is also described. The extension of the analysis to include $O(\Omega^2)$ terms in the Hamiltonian, which must take into account the equilibrium distortion of the star, is discussed. The methods of the present paper apply equally well to this more general case.

1 Introduction

In a previous paper (Alexander 1987; hereafter referred to as Paper I), a self-consistent approach to the problem of tidal resonances in close binary systems was formulated for the case in which the stars were assumed to be non-rotating. By 'self-consistent' is meant that the normal modes of oscillation of the star interact with the orbital motion in such a way that the dynamical variables describing both star and orbit are coupled (through tidal interaction) and allowed to vary together. Thus one is able to dispense with the assumption of a fixed orbit, unperturbed by tidal effects caused by the star's oscillations, which can result in unrealistic behaviour in the vicinity of resonances between a normal mode in one of the stars and the orbital (Keplerian) frequency or some harmonic of it. This is due to the fact that these tidal effects cause subtle changes in the Keplerian frequency and its harmonics, thus modifying the

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degree of ‘tuning’ of the resonance as well as time-scales associated with passage through resonance (Henrard 1982), compared with the fixed orbit case. The assumption of a fixed orbit when investigating resonances has previously been questioned by others (e.g. Savonije & Papaloizou 1983).

Since the investigation of resonance phenomena is most effectively carried out using action-angle formalism (for example, Lichtenberg & Lieberman 1983), it is fortunate that the non-rotating star case studied in Paper I readily yields an integrable zero-order Hamiltonian (consisting of the normal mode and orbital dynamics, without tidal coupling). However, the inclusion of stellar rotation introduces cross-terms (arising from Coriolis and centrifugal effects) into the part of the zero-order Hamiltonian describing the normal modes (Paper I, equation 82). It is the presence of these cross-terms that makes the solution of the Hamilton–Jacobi equations intractable in these canonical variables.

The object of the present paper is to show that the zero-order Hamiltonian for the rotating star case is integrable, just as in the non-rotating case. This is achieved by a suitable canonical transformation, the construction of which is given for the general case of the inclusion of arbitrary numbers of modes. The calculations are carried out for the case in which \( O(\Omega) \) terms, arising from Coriolis effects, are included, where \( \Omega \) is the angular velocity of rotation of the star. The splitting of degenerate eigenmodes by the rotation, as predicted by the standard perturbation theory (Cowling & Newing 1949; Ledoux & Walraven 1958; Unno et al. 1979), is thereby recovered. Furthermore, resonances may be studied self-consistently in the same way as was done in Paper I for non-rotating stars. The method applies also when \( O(\Omega^2) \) terms are included, which arise from centrifugal terms and the distortion of the star from spherical form (Smeyers & Denis 1971; Smeyers 1980; Smeyers, Craeynest & Martens 1981; Martens & Smeyers 1982).

In Section 2, the Hamiltonian for the rotating star’s normal mode oscillations is given, and the method described for achieving a transformation to canonical variables in which the Hamilton–Jacobi equation for these oscillations can be solved. In Section 3, the potential energy of tidal interaction is derived, and the Hamiltonian for the total system cast in action-angle variables of the zero-order problem. Section 4 applies the analysis to resonances in a simple two-mode system, and the motion in the vicinity of resonance is derived. Finally, Section 5 discusses the extension of the analysis to include \( O(\Omega^2) \) terms, and the effects of dissipation on the evolution of the system through resonance.

2 Hamiltonian for slowly rotating star

In the notation of Paper I, the first-order equations of motion for the Lagrangian displacement \( \xi \) from equilibrium in the corotating frame of reference of the star with angular velocity \( \Omega \), can be written (Paper I, equation 80):

\[
\rho_0 \left[ \frac{\partial^2 \xi}{\partial t^2} + 2\Omega \times \frac{\partial \xi}{\partial t} + \Omega \times (\Omega \times \xi) \right] + \mathcal{L} \xi = 0
\]  

(1)

where we have assumed the equilibrium configuration to be in rigid rotation and have neglected meridional circulation. Here, \( \mathcal{L} \) is the linear, self-adjoint operator satisfying the corresponding equations of motion for a non-rotating star, with real eigenvalues \( \omega_n^2 \) and corresponding orthonormal eigenvectors \( \{ \xi_n \} \):

\[
\mathcal{L} \xi_n = \rho_0 \omega_n^2 \xi_n
\]  

(2)
The equations of motion (1) may be derived from a Lagrangian $L$, with kinetic energy function $T$ given by:

$$T = \frac{1}{2} \rho_0 \left( \frac{\partial \xi}{\partial t} + \Omega \times \xi \right)^2 d^3 r$$

and potential energy function $V$:

$$V = \frac{1}{2} \left[ \xi \cdot \mathcal{L} \xi \right] d^3 r$$

The displacement $\xi$ may be represented as a superposition of normal modes $\xi_n$ of the non-rotating star:

$$\xi(r, t) = \sum_n q_n(t) \xi_n(r)$$

where the sum extends over both spheroidal and toroidal modes (see Paper I, equations 16 and 84):

Spheroidal: $\xi_n^{\prime} = [\xi_n^{R}(r) \hat{e}_r + \xi_n^{S}(r) r \hat{\nabla}] Y_{lm}(\theta, \phi)$

Toroidal: $\xi_n^{\prime} = \xi_n^{\tau}(r) \times \hat{\nabla} Y_{lm}(\theta, \phi)$

in spherical polar coordinates $(r, \theta, \phi)$, where $\{ Y_{lm} \}$ are the spherical harmonic functions with normalization:

$$\int Y_{lm} Y_{l^\prime m^\prime}^* d\Sigma = \delta_{ll^\prime} \delta_{mm^\prime}.$$

The spheroidal and toroidal modes form a complete set of eigenvectors (see Dyson & Schutz 1979). The toroidal oscillations are not excited in a non-rotating star, that is, they lie in the kernel of the operator $\mathcal{L} = \mathcal{L} \xi$ (toroidal) = 0.

In terms of the representation (6), the Lagrangian $L = T - V$ becomes:

$$L = \frac{1}{2} \sum_n (q_n^2 + \omega_n^2 q_n^2) + \frac{1}{4} \sum_{n,k} B_{nk} q_n q_k + O(\Omega^2)$$

in which:

$$B_{nk} = 2 \int \rho_0 \xi_n \cdot \Omega \times \xi_k d^3 r = -B_{kn}$$

represents the $O(\Omega^2)$ antisymmetric matrix element for the Coriolis operator, and $O(\Omega^2)$ terms include the centrifugal potential and equilibrium distortion of the star from spherical form due to rotation. The Hamiltonian $H_a$ equivalent to the Lagrangian $L$ is:

$$H_a = \frac{1}{2} \sum_n (p_n^2 + \omega_n^2 q_n^2) - \frac{1}{4} \sum_{n,k} [B_{nk} p_n q_k + \frac{1}{2} B_{nk} q_n q_k]$$
where

\[ p_n = \frac{\partial L}{\partial \dot{q}_n} = \dot{q}_n + \frac{1}{2} \sum_k B_{nk} q_k \]

denotes the generalized momentum conjugate to \( q_n \).

One may represent the \((nlm)\) component of the (complex) displacement as a sum of spheroidal and toroidal parts given by equations (7) and (8):

\[ \xi_{nlm}' = \xi_{nlm}^{(sph.)} + \xi_{nlm}^{(tor.)} \]  \hspace{1cm} (13)

for which the matrix elements can be shown to be:

\[ B_{nlm,n'l'm'} = 2\Omega(\beta^{(1)} + im \beta^{(2)}) \delta_{m_l,m'} \]  \hspace{1cm} (14)

where,

\[ \beta^{(1)} = \left\{ f_{l+1} \int \rho_0 r^2 [ I_{nlm}^{(R)} \xi_{lm}^{(R)} + (l+2) \xi_{nlm}^{(R)} \xi_{lm}^{(T)} + l(l+2) \xi_{nlm}^{(S)} \xi_{lm}^{(T)} - \xi_{nlm}^{(S)} \xi_{lm}^{(T)} \xi_{nlm}^{(R)} \xi_{lm}^{(S)} \xi_{nlm}^{(T)} ] dr \right\} \delta_{l,l-1} + \left\{ f_{l} \int \rho_0 r^2 [ l^2 - 1 ] ( \xi_{nlm}^{(S)} \xi_{lm}^{(T)} - \xi_{nlm}^{(S)} \xi_{lm}^{(T)}) \right\} \delta_{l,l+1} \]  \hspace{1cm} (15)

\[ \beta^{(2)} = \left\{ \rho_0 r^2 [ \xi_{nlm}^{(R)} \xi_{lm}^{(R)} + \xi_{nlm}^{(R)} \xi_{lm}^{(S)} + \xi_{nlm}^{(R)} \xi_{lm}^{(T)} + \xi_{nlm}^{(R)} \xi_{lm}^{(T)} \xi_{nlm}^{(S)} \xi_{lm}^{(T)} ] dr \right\} \delta_{l,l} \]  \hspace{1cm}

and

\[ f_l = \left( \frac{l^2 - m^2}{4l^2 - 1} \right)^{1/2} \]  \hspace{1cm} (16)

In order to keep the generalized coordinates \( \{ q_n \} \) in (10) real, one may choose real basis functions as in Paper I:

\[ \xi_{nlm}^{(c)} = \frac{1}{\sqrt{2}} (\xi_{nlm} + \xi_{nlm}' \)  \hspace{1cm} (17)

\[ \xi_{nlm}^{(o)} = \frac{1}{i\sqrt{2}} (\xi_{nlm}' - \xi_{nlm}) \]

(where, for convenience of notation, passive indices have been omitted). The spheroidal and toroidal basis functions may be written (see equations 7 and 8):

Spheroidal: \[ \xi_{nlm}^{(o)} = (\xi_{nlm}^{(R)} \mathbf{e}_r + \xi_{nlm}^{(S)} \mathbf{r} \nabla) Y_{lm}^{(o)} \]

Toroidal: \[ \xi_{lm}^{(o)} = \xi_{lm}^{(T)} \mathbf{r} \times \nabla Y_{lm}^{(o)} \]  \hspace{1cm} (18)
for $0 \leq m \leq l$, $\sigma = e$, o with $(m = 0$, $\sigma = o)$ excluded. Here, $Y_{lm}^{(o)}$ denotes, as in Paper I, the same transformation on $Y_{lm}$ as in (17) above. It should be noted that, for the toroidal components $\{\xi_{lm}^{(o)}\}$ in (18), the requirement that a general (toroidal) displacement:

$$\xi = \sum_{l = 1}^{\infty} \sum_{-m \leq n \leq m} q_{lm} \xi_{lm}$$

expressed in terms of complex $q_{lm}$ and $\xi_{lm}$ be real, means that $\xi$ can be written, without loss of generality, in terms of real $q_{lm}^{(o)}$, $\xi_{lm}^{(o)}$ as:

$$\xi = \sum_{l = 1}^{\infty} \sum_{0 \leq m < l, \sigma} q_{lm}^{(o)} \xi_{lm}^{(o)}$$

where the prime on the second summation denotes exclusion of $(m = 0$, $\sigma = o)$.

In terms of the real basis (18), one finds the matrix elements of $B$ to be:

$$B_{nm, n' m'}^{(c, e)} = 2 \Omega e_{m} \beta^{(1)}_{nn'} \delta_{mm'}$$

$$B_{nm, n' m'}^{(o)} = 2 \Omega \beta^{(1)}_{nn'} \delta_{mm'}$$

$$B_{nm, n' m'}^{(t)} = -B_{nm, n' m'}^{(o)} = -2 m \Omega \beta^{(2)}_{nn'} \delta_{mm'}$$

in which $e_{m} = 2$ if $m = 0$ and 1 if $m > 0$. Also, note that the orthonormalization condition (3) imposes the scaling requirement:

$$\int \rho_{0} r^{3} [(\xi_{n}^{R})^{2} + l(l + 1)[(\xi_{n}^{S})^{2} + (\xi_{lm}^{T})^{2}]] \, dr = 1$$

(20)

From (15) one deduces that, for purely spheroidal or toroidal displacements, only even–odd terms are non-zero:

Spheroidal: $B_{nm, n' m'}^{(c, o)} = -2 m \Omega \beta_{nn'} \delta_{nn'}$ (21)

where

$$\beta_{nn'} = \int \rho_{0} r^{2} (\xi_{n}^{R} \xi_{n'}^{S} + \xi_{n'}^{R} \xi_{n}^{S} + \xi_{n}^{S} \xi_{n'}^{S}) \, dr$$

(22)

Toroidal: $B_{lm, l' m'}^{(t)} = -\frac{2 m \Omega}{l(l + 1)} \delta_{ll'} \delta_{mm'}$. (23)

In (23), the normalization condition (20) and definition (15) of $\beta^{(2)}$ have been used.

From the above ‘selection rules’ for the operator $B$, it follows that the Coriolis effect gives rise to cross-terms in the Hamiltonian $H_{e}$ in (12), by coupling modes $(nlm\sigma)$ and $(n' l' m' \sigma')$ for $l' = l$, $l \pm 1$. In order to solve the Hamilton–Jacobi equation, and thus be able to employ action-angle formalism as in Paper I, it is necessary to find a transformation from canonical variables $\{q_{n}, p_{n}\}$ to new canonical variables $\{Q_{n}, P_{n}\}$ in which the Hamiltonian is separable – in particular, in which it takes the form:

$$H_{e}(Q, P) = \sum_{n} \nu_{n}(P_{n}^{2} + Q_{n}^{2})$$

(24)

where $\{\nu_{n}\}$ represent the new frequencies – arising, for example, from lifting the $(2l+1)$-fold degeneracy, with respect to azimuthal index $m$, of the normal modes of the non-rotating star.
The technique for achieving this in the (generic) case in which the frequencies $\omega_n$ of the non-rotating star contain only this $(2N+1)$-fold degeneracy (that is, the $\omega_n$ are distinct for different $nl$) is particularly simple (see Laub & Meyer 1974) since, as the matrix elements (19) show, the Hamiltonian $H_\ast$ is already separable with respect to the azimuthal index $m$.

The equations of motion for $N$ coupled normal modes, which follow from the Hamiltonian (12), may be written as the linear system:

$$\frac{dx}{dt} = EAx$$  \hspace{1cm} (25)

where

$$x = (q_1, p_1, \ldots, q_N, p_N)^T \in \mathbb{R}^{2N}$$

$$E = \text{diag}(J_2, J_2, \ldots, J_2) \in \mathbb{R}^{2N \times 2N}$$  \hspace{1cm} (26)

$$J_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and $A$ is a constant, symmetric matrix defining the quadratic form of $H_\ast$:

$$H_\ast = \frac{1}{2} x^T Ax.$$  \hspace{1cm} (27)

Denoting $EA$ by $K \in \mathbb{R}^{2N \times 2N}$, it is easy to show that the eigenvalues and eigenvectors of $K$ occur in pairs: $\dot{\lambda}_k$, $x_k$ and $\dot{\lambda}_k$, $\bar{x}_k$, $k = 1, \ldots, N$, where $\dot{\lambda}_k = -\lambda_k$. If, furthermore, one assumes the (distinct) eigenvalues are pure imaginary: $\dot{\lambda}_k = i\nu_k$, $\dot{\lambda}_k = -i\nu_k$, corresponding to real frequencies of oscillation $\nu_k$, then the corresponding eigenvectors may be written:

$$x_k = u_k + i\nu_k,$$

$$\bar{x}_k = u_k - i\nu_k$$  \hspace{1cm} (28)

where $u_k, v_k \in \mathbb{R}^{2N}$.

Define the antisymmetric 2-form $\omega : \mathbb{C}^{2N} \times \mathbb{C}^{2N} \to \mathbb{C}$ by:

$$\omega(x_1, x_2) = x_1^T Ex_2 = -\omega(x_2, x_1)$$  \hspace{1cm} (29)

Then, as may be shown, $K$ has the property:

$$\omega(Kx_1, x_2) + \omega(x_1, Kx_2) = 0 \quad \text{for all } x_1, x_2 \in \mathbb{C}^{2N}$$  \hspace{1cm} (30)

The eigenvectors of $K$ have the following important properties (Laub & Meyer 1974), which are easily proved using (29) and (30):

$$\omega(x_k, x_l) = \omega(\bar{x}_k, \bar{x}_l) = 0 \quad \text{for all } k, l = 1, \ldots, N$$

$$\omega(x_k, \bar{x}_l) = 0 \quad \text{unless } k = l.$$  \hspace{1cm} (31)

Using the representation (28), one may then derive the equivalent properties:

$$\omega(u_k, u_l) = \omega(v_k, v_l) = 0 \quad \text{for all } k, l = 1, \ldots, N$$

$$\omega(u_k, v_l) = 0 \quad \text{unless } k = l.$$  \hspace{1cm} (32)

By scaling the eigenvector pair $(u_k \pm iv_k)$ by the factor $|\omega(u_k, v_k)|^{-1/2}$ and, if $\omega(u_k, v_k) < 0$, interchanging the column vectors $u_k, v_k$ in the matrix $X$ defined by:

$$X = [u_1, v_1, \ldots, u_N, v_N] \in \mathbb{R}^{2N \times 2N}$$  \hspace{1cm} (33)
it follows from (32) that $X$ is symplectic, that is:

$$\omega(u_k, u_j) = \omega(v_k, v_j) = 0, \quad \omega(u_k, v_j) = \delta_{kj}$$  \hspace{1cm} (34)

or, equivalently.

$$X^T E X = E$$  \hspace{1cm} (35)

which implies also that

$$X E X^T = E.$$  \hspace{1cm} (36)

With $x = x(t)$ given by (26), if $y$ now is defined as

$$y = X^{-1} x$$  \hspace{1cm} (37)

with components:

$$y = (Q_1, P_1, \ldots, Q_N, P_N)^T$$  \hspace{1cm} (38)

then the equations of motion (25) may be put in the form:

$$\frac{dy}{dt} = \Lambda y,$$  \hspace{1cm} (39)

where

$$\Lambda = \text{diag}(\Lambda_1, \ldots, \Lambda_N)$$

$$\Lambda_k = \begin{pmatrix} 0 & v_k \\ -v_k & 0 \end{pmatrix}.$$  \hspace{1cm} (40)

Equivalently, using (38):

$$\frac{d}{dt} \begin{pmatrix} Q_k \\ P_k \end{pmatrix} = \begin{pmatrix} 0 & v_k \\ -v_k & 0 \end{pmatrix} \begin{pmatrix} Q_k \\ P_k \end{pmatrix}, \quad k = 1, \ldots, N$$  \hspace{1cm} (41)

The fact that the matrix $X$ (and hence also $X^{-1}$) is symplectic ensures that the transformation (37) (as well as its inverse) is canonical. Indeed, (35) and (36) are, respectively, the Lagrange and Poisson Bracket relations for a canonical transformation. It then follows that $(Q_k, P_k, k = 1, \ldots, N)$ are canonical coordinates, and equations (41) are Hamilton's equations of motion with Hamiltonian $H_u(Q, P)$ given by equation (24).

In the real basis (18), the orthogonality properties of toroidal and spheroidal functions imply that the zero-order part of the matrix $A$ (defined by equation 27), which is independent of $\Omega$, is diagonal. Non-diagonal terms are introduced through matrix elements of $B$ defined in equation (19). Furthermore, the matrix elements of $B$ between different $m$ vanish (a consequence of rotational symmetry), and, if pure spheroidal or toroidal oscillations are present, between different $l$ as well. Thus, in general, matrix $K = EA$ is block-diagonal with each block $K^{(m)}$ associated with a fixed value of $m$, and the procedure above for finding the symplectic transformation can be applied to each block separately.

The eigenvalues of matrix $K^{(m)}$ can most conveniently be found using (first-order) perturbation theory (see, for example, Messiah 1965; Wilkinson 1965), by writing:

$$K^{(m)} = K_0 + \epsilon K_1,$$  \hspace{1cm} (42)
where $K_0$ refers to the non-rotating problem and $K_1$ is $O(\Omega)$. By virtue of the fact that $K_0$ depends on indices $n, l$ and $\sigma = e, o$, whereas the eigenvalue of $K_0$ (which correspond to the normal mode frequencies of the non-rotating star) depend only on $n, l$, it follows that each eigenvalue of $K_0$ will be doubly-degenerate – except of course when $m = 0$. This degeneracy will in general be lifted by the perturbation $K_1$, leading to a distinct set of eigenvalues for $K^{(m)}$, as was assumed in deriving the symplectic transformation above.

Let $x_0, y_0$ denote right and left eigenvectors of $K_0$, belonging to the eigenspace $\mathcal{E}_0$ of the degenerate eigenvalue $\omega_0$. Write:

$$x = x_0 + \epsilon x' \quad \text{where } y_0^T x' = 0$$

$$\omega = \omega_0 + \epsilon \omega'. \quad (43)$$

Substituting (42) and (43) into the eigenvalue equation:

$$K^{(m)} x = \omega x$$

and equating coefficients of $\epsilon^0, \epsilon^1$, one obtains:

$$(K_0 - i\omega_0)x_0 = 0$$

$$K_0 x' + K_1 x_0 = i(\omega_0 x' + \omega' x_0) \quad (44)$$

For $k = 0, 1, \ldots, n$, let $\{x_{ka}, y_{ka}, k = 1, \ldots, g_k\}$ denote a basis of right eigenvectors for the $g_k$-dimensional subspace $\mathcal{E}_k$ of the degenerate eigenvalue $\omega_k$, and $\{y_{ka}, k = 1, \ldots, g_k\}$ the corresponding dual basis of left eigenvectors. Here, $n$ is such that:

$$\sum_{k=0}^{n} g_k = 2N,$$

where $N$ is the number of normal modes being considered. Within each subspace $\mathcal{E}_k$ the bases $\{x_{ka}, y_{ka}, k = 1, \ldots, g_k\}$ may conveniently be chosen to satisfy

$$y_{ka}^T x_{\beta} = \delta_{a\beta}$$

so that

$$y_{ja}^T x_{k\beta} = \delta_{jk} \delta_{a\beta}. \quad (45)$$

A left eigenvector $y_k$ of $K_0$ (or $K^{(m)}$) can readily be expressed in terms of a right eigenvector $x_k$.

Using $K = EA$ and the properties of $E$ and $A$ (see (26) and (27)), one may easily show that:

$$y_k = E \tilde{x}_k \quad (46)$$

where $\tilde{x}_k$ belongs to eigenvalue $(-i \omega_k)$ and is given by the second equation (28).

With these choices:

$$P_0 = \sum_{a=1}^{g_o} x_{0a} y_{0a}^T$$

defines a projection operator onto the eigenspace $\mathcal{E}_0$:

$$P_0 x_0 = x_0$$

Projecting the second equation (44) on to $\mathcal{E}_0$, one finds:

$$(P_0 K_1 P_0) x_0 = i \omega' x_0. \quad (47)$$
Let \((K_1)_{aa}\) denote the components of \((P_0, K, P_0)\) (or equivalently, of \(K_1\)) in the basis \(\{x_{0a}\}\). Then, equation (47) is an eigenvalue problem in \(\mathcal{E}_0\), the possible values \(\omega'\) for the first-order correction to the eigenfrequency being given by the eigenvalues \((i\omega')\) of the matrix \([\{K_1\}_{aa}']\) of order \(g_0\). (In the present application, \(g_0 = 2\) for \(m > 0\); \(g_0 = 1\) for \(m = 0\).) Note also that these first-order corrections to the eigenvalues of \(K_0\) occur in positive-negative pairs (see remark following equation 27). Hence, first-order perturbation theory predicts that the eigenvalues of \(K^{(m)}\) are of the form \(\pm i(\omega_0 + \omega')\), \(\pm i(\omega_0 - \omega')\), and that the degeneracy of \(K_0\) is removed if \(\omega' \neq 0\) when \(m > 0\).

The first-order perturbation \(x'\) of the eigenvector can be found as follows (Wilkinson 1965).

Write:

\[
x' = \sum_{k=1}^{n} \sum_{a=1}^{g_k} t_{ka}x_{ka}
\]

and substitute this into the second equation (44):

\[
\sum_{k=1}^{n} i(\alpha_k - \omega_0) \sum_{a=1}^{g_k} t_{ka}x_{ka} + K_1x_0 = i\omega'x_0
\]

Pre-multiplying by \(y_{ji}^T\) and using (45), one obtains:

\[
i(\omega_j - \omega_0) t_{ji} + y_{ji}^TK_1x_0 = 0
\]

which may be solved for \(t_{ji}\) and substituted in (48) to obtain \(x'\).

The application of the above analysis to the simple but important case of a 2-mode system (fixed \(n, l, m;\) and \(\sigma = e\) or \(o\)) will be given in Section 4 below. In this system \(K^{(m)}\) reduces to a \(4 \times 4\) matrix for which the eigenvalues can be found exactly. However, in more general cases, the perturbation approach given here will be useful.

3 Coupling with the tidal potential

The tidal potential \(\phi_T(r, \theta, \phi)\) due to companion star of mass \(m_1\) located at \((R, \Theta, \Phi)\), at a point \((r, \theta, \phi)\) fixed in the non-rotating frame, is given by:

\[
\phi_T = -\frac{4\pi Gm_1}{R} \sum_{l=2}^{\infty} \sum_{m=-l}^{l} \frac{1}{2l+1} \left(\frac{r}{R}\right)^l Y_{lm}(\Theta, \Phi) Y_{lm}^*(\theta, \phi)
\]

(49)

and the potential energy due to the distortion of \(m_2\) in this field is:

\[
U_T = \int \phi_T \rho_0 d^3r = \int \rho_0 \xi \cdot \nabla \phi_T d^3r
\]

(50)

(see Paper I). Assume that the axis of rotation of \(m_2\) is perpendicular to the orbital plane defined by \(\Theta = \pi/2\), that is, in the direction \(\theta = 0\). With respect to this non-rotating frame, the spherical basis functions, equation (7), have the form:

\[
\xi_{nlm}(r, \theta, \phi, t) = (\xi_{nl}^{R} \xi_{r}^{R} + \xi_{nl}^{S} \xi_{r}^{S} \nabla) Y_{lm}(\theta, \phi - \Omega t)
\]

(51)

and similarly for the toroidal modes. Then, by a small modification of the analysis in Paper I, it may be shown that the perturbation Hamiltonian \(H_1 = U_T\) assumes the form:
where, in the notation of Paper I:

\[ U_{nml}^{(e,o)} = -\sqrt{2} W_{pm} \frac{G m_1 m_2}{R} \left( \frac{R_*}{R} \right) Q_{nl} \cos m(\Phi - \Omega t), \sin m(\Phi - \Omega t) \].

The Hamiltonian will now contain explicit time-dependence, which, however, may be removed by an appropriate canonical transformation (Section 4).

In order to employ action-angle formalism in the study of resonances between normal modes of the rotating star and orbital motion, the component \( H_\alpha \) of the zero-order Hamiltonian must first be brought into normal form (equation (24) by an appropriate canonical transformation (Section 2). Then, the \( q_{nlm}^{(e)} \) appearing in equation (52) will become linear combinations of the canonical variables \( Q_k, P_k \), as expressed by the inverse of the symplectic transformation equation (37). The Hamilton–Jacobi equation for \( H_\alpha(Q,P) \) has the separable form:

\[
\frac{1}{2} \sum_k v_k \left( \frac{\partial \psi}{\partial Q_k} \right)^2 + Q_k^2 = E_\alpha
\]

(which follows from equation (24), and solution in the action-angle variables \( J_k, \theta_k \) given by:

\[ Q_k = \sqrt{2J_k} \cos \theta_k \quad P_k = -\sqrt{2J_k} \sin \theta_k \]

\[ \theta_k = v_k t + \text{const.} \]

in terms of which \( H_\alpha \) becomes:

\[ H_\alpha = \sum_k v_k J_k \]

Substituting (54) into (52) yields an expression for \( H_\alpha \) consisting of periodic terms, similar to that given by equation (38) of Paper I, except that now the rotation \( \Omega \) is involved so that \( \tilde{g} \) must be replaced by \( (\tilde{g} - \Omega t) \), as may be verified by comparing the expressions for \( U_{nml}^{(e,o)} \) (Paper I, equation 21) and equation (53) above.

4 Tidal resonance in a two-mode system

When the degeneracy of the normal-mode oscillations (of frequency \( \omega_{nl} \)) of the non-rotating star is lifted by the rotation \( \Omega \), resonances with orbital motion can in general occur with only one of the components of the rotationally split modes, with frequency \( \omega = \omega_{nl} + \omega' \). Thus, only a single mode characterized by fixed \( nlm \) will be involved in this resonance. (The possibility exists of resonance between rotationally perturbed normal modes with different indices \( nl, n'l' \), but this case will not be considered here.) The simplest case to consider is a 2-mode system defined by fixed \( nlm \), and \( \sigma = e, o \). This will be a good approximation if one ignores interactions with other, non-resonant modes of the star which occur through \( O(\Omega) \) coupling due to non-zero elements of the \( B \)-matrix of Section 2.

Denoting the canonical variables by \( (q_\sigma, p_\sigma, q_o, p_o) \), and the degenerate normal mode frequency of the non-rotating star by \( \omega_0 \), one finds from equation (12), using the selection rules (19) for the matrix elements of \( B \):

\[
H_\sigma = \frac{3}{2} [p_\sigma^2 + p_o^2 + \omega_0^2 (q_e^2 + q_o^2)] + 2 \beta_{m}(p_{q_e} - p_o q_o) + \beta_{m}^2(q_e^2 + q_o^2)
\]
\[ \beta_m = m\Omega I \]  
\[ I = \int \rho_0 r^2 \left( 2 \xi_w^S + \left( \xi_w^S \right)^2 \right) dr \quad \text{(spheroidal mode)} \]  
\[ I = \frac{1}{l(l+1)} \left( \xi_{lm}^T \right)^2 \quad \text{(toroidal mode)} \]

where, in the latter, the scaling condition equation (20) has been used. It is convenient at this stage to apply a canonical transformation \((q, p, q_o, p_o) \rightarrow (q', p', q'_o, p'_o)\) defined by
\[ q'_o = \sqrt{\omega} q_o, \quad p'_o = \frac{1}{\sqrt{\omega}} p_o, \quad \text{for } \sigma = e, o \]  
where
\[ \omega = \sqrt{\omega_0^2 + \beta_m^2} \]

Then, \(H_e\) reduces to:
\[ H_e = \frac{1}{2} \omega \left[ p_e^2 + p_o^2 + q_e^2 + q_o^2 + 2 \beta_m \omega (q_e q_o - p_e p_o) \right]. \]

The symmetric matrix defining the quadratic form (61) is then:
\[ A = \begin{bmatrix} \omega & 0 & 0 & -\beta_m \\ 0 & \omega & \beta_m & 0 \\ 0 & \beta_m & \omega & 0 \\ -\beta_m & 0 & 0 & \omega \end{bmatrix}. \]

The eigenvalues of \(K = EA\) are readily found to be \(\pm i \nu_1 = \pm i(\omega + \beta_m)\) and \(\pm i \nu_2 = \pm i(\omega - \beta_m)\). (It should be noted that the first-order perturbation theory given in Section 3 yields the same values.) The corresponding eigenvectors are \(x_k = u_k + iv_k\) where \(u_k, v_k\) are given by:
\[ X = [u_1, v_1, u_2, v_2] = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 \end{bmatrix}. \]

The scaling factor \((1/\sqrt{2})\) ensures that \(X\) is symplectic, satisfying conditions (35) and (36). Therefore, using equations (37), where \(x = (q_e, p_e, q_o, p_o)\) and \(y = (Q_1, P_1, Q_2, P_2)\), and (59), one obtains the canonical transformation:
\[ q_e = \frac{1}{\sqrt{2} \omega} (Q_1 + Q_2), \quad q_o = \frac{1}{\sqrt{2} \omega} (P_1 - P_2) \]
\[ p_c = \frac{\omega}{\sqrt{2}} (P_1 + P_2), \quad p_o = \frac{\omega}{\sqrt{2}} (Q_2 - Q_1). \]  \tag{63}

In these variables, \( H_\ast \) assumes the normal form (equation 24):
\[ H_\ast = \frac{1}{2} [(\omega + \beta_m)(P_1^2 + Q_1^2) + (\omega - \beta_m)(P_2^2 + Q_2^2)]. \] \tag{64}

It should be noted that, for toroidal modes, \( \omega_0 = 0 \), so from (56), (58) and (60)
\[ v_1 = \frac{2m\Omega}{l(l+1)}, \quad v_2 = 0 \quad \text{(toroidal modes)} \tag{65} \]

and \( H_\ast = H_\ast(Q_1, P_1) \) only, to first-order in \( \Omega \). The result (65) is well-known (see, for example, Papaloizou & Pringle 1978; Provost, Berthomieu & Rocca 1981) but was previously derived using different methods. However, to \( O(1) \), toroidal displacements do not produce density variations:
\[ \delta \rho = - \text{div}(\rho_0 \xi) = 0 \] (see equation 8), so that by equation (50):
\[ H_1 = U_t = 0. \]

Thus, the tidal potential can excite only spheroidal modes of oscillation. Toroidal modes may, however, be excited indirectly, by coupling through the Coriolis terms – see equations (15) and (19). From now on, only spheroidal modes will be considered.

The perturbation Hamiltonian (52) can now be expressed in terms of action-angle variables. Using the definition of the Hansen coefficients \( X_{k,n}^{l,m}(e) \) (Paper I, equation 37) to express \( R^{-(l+1)}[\cos(mv), \sin(mv)] \) in (53) in terms of harmonics of the mean anomaly \( \bar{t} \), and substituting the transformations (63) and (54) into (52), one obtains after some algebra:
\[ H_1 = -\sqrt{2} \sum_{n=1}^{\infty} \frac{m_1 m_2}{a} \left( \frac{R_*}{a} \right)^l \frac{Q_{ml}}{\sqrt{\Omega R_*}} X_{k,-l-k}^{l+1,m} \times [\sqrt{J_1} \cos(k\bar{t} + m\bar{g} - m\Omega t + \theta_1) + \sqrt{J_2} \cos(k\bar{t} + m\bar{g} - m\Omega t - \theta_2)]. \] \tag{66}

This expression corresponds to equation (38) of Paper I for the non-rotating case. The orbital variables \( a, e \) appearing in (66) are defined in terms of the Delaunay action variables \( L, G \) by equations (31) and (32) of Paper I.

The periodic terms in equation (66) have frequencies:
\[ k\bar{t} - m\Omega + \theta_1 = [k\omega_k - (1 - l)m\Omega] + \omega \]
\[ k\bar{t} - m\Omega - \theta_2 = [k\omega_k - (1 - l)m\Omega] - \omega \] \tag{67}

where definitions (54) for \( \theta_{1,2} \) and (56) for \( \beta_m \) have been used, and \( \omega \) is given by equation (60). The Keplerian frequency \( \omega_k \) is given by:
\[ \omega_k = \frac{\partial H_{\text{orb}}}{\partial L} = \mu \left( \frac{Gm_1 m_2}{L^3} \right) \] \tag{68}

(Paper I, equation 30). Note that in (67), \( l < 1 \) (Unno et al. 1979) and \( m \geq 0 \). Resonance can occur when either of the frequencies in (67) is nearly zero. The dominant perturbation is due to
the \( l = 2 \) tide, in which case \( m = 0 \) or 2. On the other hand, \( k \) may assume any positive or negative integer value. However, for small orbital eccentricity \( e \), the Hansen coefficients appearing in equation (66) are of order \( e^{\left| k - m \right|} \), so that the dominant terms will be those with \( k \) positive. For the sake of definiteness, assume that the term in brackets on the right-hand sides of (67) is positive. Then, since \( \omega > 0 \), the second mode can become resonant with some harmonic \( k \) of the orbit. A time average of \( H_1 \) near such a resonance will then eliminate mode 1, leaving:

\[
\langle H_1 \rangle = -\sqrt{2} W_{lm} \frac{G m_1 m_2}{a} \left( \frac{R_*}{a} \right)^l \frac{Q_{nl}}{\sqrt{\omega R_*}} X_k^{-(l+1) m} \\
\times \sqrt{J_2} \cos(k \bar{t} + m \bar{g} - m \Omega t - \theta_2)
\]

(69)

In the expression (69) for \( \langle H_1 \rangle \), explicit time-dependence may be removed, and the arguments of the periodic terms simplified, by a transformation to new action-angle variables \((J', \theta', L', G', g')\) using the generating function:

\[
S = -(k \bar{t} + m \bar{g} - m \Omega t - \theta_2) J' + \bar{t} L' + \bar{g} G'
\]

Then,

\[
l' = \frac{\partial S}{\partial L} = \bar{t} \; ; \quad L = \frac{\partial S}{\partial l} = L' - k J'
\]

\[
g' = \frac{\partial S}{\partial G} = \bar{g} \; ; \quad G = \frac{\partial S}{\partial g} = G' - m J'
\]

\[
\theta' = \frac{\partial S}{\partial \theta} = \theta_2 - (k \bar{t} + m \bar{g} - m \Omega t) \; ; \quad J_2 = \frac{\partial S}{\partial \theta_2} = J'
\]

(70)

and

\[
\langle H_1 \rangle' = \langle H_1 \rangle + \frac{\partial S}{\partial t} = \langle H_1 \rangle + m \Omega J'
\]

In these new variables, the perturbation Hamiltonian becomes:

\[
\langle H_1 \rangle' = -\sqrt{2} W_{lm} \frac{G m_1 m_2}{a} \left( \frac{R_*}{a} \right)^l \frac{Q_{nl}}{\sqrt{\omega R_*}} X_k^{-(l+1) m} \sqrt{J'} \cos \theta' + m \Omega J'
\]

(71)

To simplify notation, the variables may be normalized as in Paper I by expressing:

\[
L', G', J' \text{ in units of } \mu \sqrt{G(m_1 + m_2)} R_*
\]

\[
\text{Frequencies in units of } \sqrt{G(m_1 + m_2)/R_*^3}
\]

\[
\text{Energy in units of } G m_1 m_2 / R_*
\]

Denoting normalized variables by an overbar, the total Hamiltonian may now be written:

\[
\langle \tilde{H} \rangle = \tilde{H}_{\text{orb}} + \tilde{H}_* + \langle \tilde{H}_1 \rangle
\]

\[
= -\frac{1}{\left( L - k J \right)^2} + [\tilde{\omega} + (1 - l) m \Omega] J + \tilde{f} J^{1/2} \cos \theta'
\]

(72)
where $H_{\text{orb}}$ is given by normalizing equation (30) of Paper I, $H_\ast$ by normalizing (55), and $\tilde{F}$ by:

$$\tilde{F} = -\sqrt{2} W_m \left( \frac{R_\ast}{a} \right)^{l+1} \frac{Q_m}{\sqrt{\omega}} \frac{m_1}{m_1 + m_2} X_{k}^{-\left(l+1\right)} m(e).$$  \hfill (73)

Note that:

$$\frac{R_\ast}{a} = \left( L - k \tilde{J} \right)^{-2}; \quad e = \sqrt{1 - \left( \frac{\dot{\tilde{G}} - m \tilde{J}}{L - k \tilde{J}} \right)^2};$$

$$\dot{\tilde{w}}_K = \left( L - k \tilde{J} \right)^{-3}. \hfill (74)$$

Equation (72) is of the form:

$$\langle \tilde{H} \rangle = H_0(L, \tilde{J}) + \langle \tilde{H}_1 \rangle \langle L, \tilde{G}, \tilde{J}, \theta' \rangle$$

where $H_0 = H_{\text{orb}} + H_\ast$ is the zero-order Hamiltonian. From this, one deduces that $L'$ and $G'$ are constants of motion, which implies that

$$L + kJ_2 = \text{const.}, \quad \dot{G} + mJ_2 = \text{const.}$$

and that

$$\langle \tilde{H} \rangle = H_0(\tilde{J}) + \langle \tilde{H}_1 \rangle (\tilde{J}, \theta'). \hfill (75)$$

Thus, near resonance, the motion can be described by the Hamiltonian of a one-dimensional system.

Using the analysis of Paper I, Section 3, one may investigate motion near primary resonance, defined by:

$$\frac{\partial H_0}{\partial \tilde{J}} = -k \dot{\tilde{w}}_K + \dot{\tilde{w}} + (1 - i) m \Omega = 0. \hfill (76)$$

The (unique) solution to this equation is:

$$\tilde{J} = \tilde{J}_0 = \frac{1}{k} \left[ L - \left( \frac{k}{\dot{\tilde{w}} + (1 - i) m \Omega} \right)^{1/3} \right].$$

Expanding $H_0$ about $J = J_0$ and neglecting the redundant constant $H_0(J_0)$, the Hamiltonian becomes, in lowest order:

$$\langle \Delta \tilde{H} \rangle = -\frac{1}{2} A_0 (\Delta \tilde{J})^2 + B_0 \cos \theta' \hfill (77)$$

where

$$\Delta J = J - J_0; \quad A_0 = \frac{3 k^2}{(L - k J_0)^{\frac{1}{3}}}; \quad B_0 = \tilde{F}(J_0)^{\frac{1}{2}} \hfill (78)$$

Thus, as in Paper I, one recovers the equation of a pendulum in the $\Delta J - \theta'$ plane. Note that $|B_0| \ll |A_0|$. The following properties hold (Lichtenberg & Lieberman 1983, section 2.4). For $A_0 B_0 > 0$ (that is, $\tilde{F}(J_0) > 0$), $\theta' = 0$ is a stable equilibrium and $\theta' = \pm \pi$ is unstable. Libration about resonance will occur provided:
Tidal resonances in binary star systems – II

\[ \Delta J_{\text{max}} < 2 \left( \frac{B_0}{A_0} \right)^{1/2} = \frac{2}{k^{1/3}} \frac{[4 \dot{J}(J_0) J_0^{1/2} \dot{\omega} + (1 - I) \dot{m} \Omega]^2}{\omega + (1 - I) m \Omega}^{1/2} \]  \hspace{1cm} (79)

The libration frequency for small-amplitude librations is given by:

\[ \dot{\omega}_{\text{Lib}} = (A_0 B_0)^{1/2} = \left| 3 k^2 \dot{J}(J_0) J_0^{1/2} \dot{\omega}_k - 2/3 \right| ^{1/2} \dot{\omega}_k. \]  \hspace{1cm} (80)

The maximum possible energy of libration is:

\[ \langle \Delta \dot{H} \rangle_{\text{max}} = | \dot{J}(J_0) J_0^{1/2} |. \]  \hspace{1cm} (81)

As a numerical example, consider a polytropic star \( m_2 \) of polytropic index 3, corotating \((\omega_K = \Omega)\) with a companion of mass \( m_1 = m_2 \), in which the \( g_{20} \)-mode for \( l = 2 \) is resonant with the \( k = 3 \) orbital harmonic. From Table I of Lee & Ostriker (1986), one finds:

\[ \frac{R_*^3}{Gm_2} \omega_{nl}^2 = 0.3830, \quad Q_{nl} = 1.894 \times 10^{-3}. \]

In normalized units of the present paper, \( \omega = 0.4376 \). By (74), the condition for primary resonance (76) implies:

\[ \dot{\omega}_K = 0.2528 \]

\[ \frac{R_*}{a} = 0.40 \]

Now, \( J \) can be computed from the energy of oscillation \( \dot{E}_{\text{mode}} \) using:

\[ \dot{E}_{\text{mode}} = \dot{\omega}_{\text{mode}} \dot{J} \]

(see equation 55), where \( \dot{\omega}_{\text{mode}} \) is the frequency of oscillation modified by rotation. If \( \dot{E}_{\text{mode}} \) is assumed to be \( 10^{-3} |\dot{E}_*| \), where \( \dot{E}_* \) is the total energy of the star \((= -1.5 \text{ for a polytrope of index 3})\), then \( J_0 = 0.0028 \). From equation (73) one calculates:

\[ \dot{J}(J_0) = -2.18 \times 10^{-4} X_3^{-3.2} = -7.62 \times 10^{-4} e. \]

Hence, from (80) the libration frequency for small-amplitude librations is:

\[ \dot{\omega}_{\text{Lib}} / \dot{\omega}_K = 0.052 \ e^{1/2} \]

and from (81) the maximum possible energy of libration is:

\[ \langle \Delta \dot{H} \rangle_{\text{max}} = 0.403 \times 10^{-4} e = 0.027 \ e \dot{E}_{\text{mode}}. \]

5 Discussion

The reduction of the Hamiltonian to normal form for the normal mode oscillations of a uniformly rotating star (Section 2), has enabled the investigation of resonance with orbital motion to be carried out in a self-consistent way, free from the assumption of a fixed orbit, as was done in Paper I for non-rotating stars. The effects of rotation are (i) to lift the \((2l + 1)\)-fold degeneracy of modes of the non-rotating star, thus simplifying the discussion of tidal resonances; and (ii) to allow excitation of toroidal modes which have zero frequency in a non-rotating star. However, toroidal modes do not couple (to lowest order in \( \Omega \)) with the tidal potential, as they represent zero density-perturbations. The coupling occurs indirectly with spheroidal modes via Coriolis effects – see discussion following equation (65).
For simplicity, results have been presented only for the special case in which
(i) the rotation axis is perpendicular to the orbital plane; (ii) terms of \(O(\Omega^2)\) in the Hamiltonian \(H_a\) for the
normal mode oscillations are neglected. Neither of these assumptions is necessary, within
the framework of the methods described in this paper. Removing assumption (i) merely introduces
more terms into the perturbation Hamiltonian \(H_1\), in that all modes \((m, \sigma)\), \(0 \leq m \leq l\), \(\sigma = e, o\),
instead of just \(m = l, l-2, \ldots\), couple to the tidal potential.

Removing assumption (ii) requires that one take into account the distortion of the
equilibrium shape of the star from spherical form due to rotation, as well as include the
centrifugal term:

\[
\rho \left( \Omega \times \mathbf{\xi} \right)^2 \, d^3r,
\]

when calculating kinetic and potential energies \(T\) and \(V\) in equations (4) and (5), respectively.
The equilibrium distortion may be expressed in terms of a (non-orthogonal) coordinate system
\((\psi, \theta, \phi)\), in which \(\psi\) = constant on equipotential surfaces:

\[
r = \psi[1 + \Omega^2 f_2(\psi, \theta)]
\]

(see, for example, Martens & Smeyers 1982). The equipotential coordinate \(\psi\) may be chosen in
a number of ways, for example so that \(\psi = R_\ast\) represents the surface of the \textit{distorted}\ star,
where \(R_\ast\) is the radius of the \textit{undistorted} star in the absence of rotation. The pressure \(P_0(\psi)\),
density \(\rho_0(\psi)\), and coefficient \(f_2\) (equation 82) for the distorted star may be found, for example,
by a Chandrasekhar–Milne expansion (Tassoul 1978). In terms of the coordinate system
\((\psi, \theta, \phi)\), the operator \(\mathcal{L}\) appearing in equation (1) has the form:

\[
\mathcal{L} = \mathcal{L}_0 + \Omega^2 \mathcal{L}_2
\]

where, \(\mathcal{L}_0(\psi, \theta, \phi)\) has the same functional form as \(\mathcal{L}\) expressed in \((r, \theta, \phi)\) coordinates, and the
 correction \(\mathcal{L}_2\) arises from \(O(\Omega^2)\) terms in the expansion of derivative operators appearing
in \(\mathcal{L}\): for example:

\[
\frac{\partial}{\partial r}\left[1 - \Omega^2 \frac{\partial f_2(\psi)}{\partial \psi}\right] \frac{\partial}{\partial \psi}.
\]

Similarly, the integrations in equations (4) and (5) over the volume of the star assume the form:

\[
\int_0^{\infty} \int_0^\pi \int_0^{2\pi} \ldots \, dr \, \psi \, d\theta \, d\phi
\]

\[
= \int_0^{\infty} \int_0^\pi \int_0^{2\pi} \ldots \, \psi^2 \sin \theta \, d\psi \, d\theta \, d\phi + \Omega^2 \int_0^{\infty} \int_0^\pi \int_0^{2\pi} \ldots \, \left(3f_2 + \psi \frac{\partial f_2}{\partial \psi}\right) \psi^2 \sin \theta \, d\psi \, d\theta \, d\phi.
\]

The basis functions \(\{\xi_n\}\) may be chosen to have the same functional form in \((\psi, \theta, \phi)\)
variables as in equations (7)–(8) or (18). Then, if \(\{\xi_n\}\) represents the complete set of normal
modes of the \textit{non-rotating} star, the orthogonality property (equation 3) of the \(\xi_n\)'s is preserved
only in zeroth-order, and off-diagonal terms will appear, for example, in the evaluation of:

\[
\rho_0 \left( \frac{\partial \xi}{\partial t} \right)^2 \, d^3r
\]
in equation (4) for the kinetic energy $T$, due to the $O(\Omega^2)$ term in (84). Correct to $O(\Omega^2)$, the expressions for the matrix elements of $B$ will be the same as given in this paper (equations 14 or 19). Similarly, the centrifugal term:

$$\int \rho_0 |\Omega \times \xi|^2 d^3r$$

may be evaluated as for the undistorted configuration.

Since the rotational distortion, as expressed by equation (82), is independent of azimuthal angle $\phi$, the $O(\Omega^2)$ corrections – as in the $O(\Omega)$ case treated in this paper – will not couple modes with different values of azimuthal index $m$. However, modes with different indices $l$, $l \pm 1$, $l \pm 2$ will be coupled.

The dynamical system represented by the stellar oscillations and orbital motion coupled by tidal interaction, will evolve slowly by means of tidal dissipation of oscillations as well as structural changes in the components. The constants of motion $L'$ and $G'$, as well as the normal mode frequencies, will undergo variations on a time-scale much longer than the periods of oscillation or orbital motion. Once the functional form of these slow variations has been specified, an extension of the theory of adiabatic invariants (Henrard 1982) will enable the conditions for capture into, or escape from tidal resonance to be determined.

Finally, the rotation of the star should also be included in the dynamics and allowed to undergo slow variations, instead of being regarded as just a parameter, as was done in the present paper. The rotation then becomes a term in the kinetic energy function $T$ (equation 4), the expression for which has been given by Kopal (1959). The inclusion of these extra degrees of freedom should allow the investigation, for example, of the effects of tidal resonance on the orientation of the axis of rotation.

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**References**


