Centrifugally exhausting discs: an inverse process of disc-like accretion

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Summary. A disc-like mass-loss process from rapidly rotating, highly magnetized objects is investigated. Such a disc may be considered as an inverse-type of magnetized accretion discs. The disc plasma flows out radially with the Alfvén velocity while it rotates, in the main part of the disc, with Keplerian velocity. The magnetic stress transfers angular momentum from the central spinner to the disc. A considerable fraction of the rotational energy extracted in association with the angular momentum is liberated in the disc through the (effective) Joule dissipation.

An almost self-consistent set of analytic expressions is proposed as a solution to the steady-state, resistive MHD equations which are fairly simplified by the assumption of thin disc. The possibility of finding such discs around young neutron stars is briefly discussed with reference to SN 1987A.

1 Introduction

In previous papers (Kaburaki 1986, 1987), we have discussed disc-like accretion processes in a magnetic field. In those considerations, the effects of rotation of a central object are neglected for simplicity. However, such effects can be the cause of various interesting phenomena in actual cases, especially in the presence of strong magnetic fields.

For example, neutron stars in X-ray binaries may have considerably rapid rotation as a result of accretion. Ghosh & Lamb (1979a,b) suggested that an interplay of the central spinner and the accretion disc could explain reduced spin-up rates and period fluctuations observed in X-ray pulsars like Her X-1. Also, in the beat model of the QPOs in X-ray bursters (e.g. Alper & Shaham 1985), the rotation of the magnetosphere which is located inside of the accretion disc plays an essential role (for a review of X-ray binaries, see Henrichs 1983).

For the accretion discs around supermassive black holes in AGNs, the situation is essentially not very different from the above examples. Actually, central spinners, as well as accretion discs encircling them, are proposed to be the promising energy sources in AGNs (e.g. Begelman, Blandford & Rees 1984). Fundamental concepts concerning the energy and angular momentum extractions from Kerr holes immersed in magnetic fields have been discussed extensively by Macdonald & Thorne (1982).
It seems very useful to study some idealized situations, however, before going directly into the discussion of detailed features of the electrodynamical coupling between a central spinner and an accretion disc. Since we have examined the behaviours of accretion discs neglecting the rotation of central objects, the next step may be to consider the effects of the central spinner on the surrounding plasma, neglecting mass supply from outer systems. Thus, we are led to reconsider the problem of centrifugal wind from an electrodynamical viewpoint. In order to make the discussion definite, we restrict ourselves in the following to the cases in which the central objects are rapidly rotating, highly magnetized, isolated neutron stars. However, the essence of the results may not be altered by replacing them by white dwarfs or even by normal stars.

Section 2 discusses some underlying physics from an electrodynamical viewpoint. The basic equations for the present problem are derived, in Section 3, from the full set of resistive MHD equations by retaining only lowest order terms in the smallness parameter $\Delta$ (the half opening-angle of a thin disc). The separation of variables for relevant physical quantities are also performed in this section to make the equations soluble. A set of analytic expressions is derived in the next section as a solution to these equations. Section 5 is devoted to get, from the above mentioned solution, some integrated quantities which characterize the present problem, and to discuss the global properties of the disc. A possible application is also discussed briefly. The outline of the present work has been reported in a previous paper (Kaburaki 1988).

2 Electrodynamical considerations

The pulsar magnetospheres are usually thought of as filled with charge-separated plasmas as first suggested by Goldreich & Julian (1969). If one starts his consideration from a vacuum state surrounding a neutron star and assumes that charged particles are supplied only from the surface of the neutron star, then it seems likely that the density of charged particles is determined so as just to cancel the electric field along the magnetic field lines. This picture, however, suffers from a serious inconsistency. Namely, on an open field line along which the sign of space charge changes, how is it possible to supply, from the surface, the particles of escaping species across the region of oppositely charged species? The detailed arguments can be seen in a comprehensive review by Michel (1982) on the theory of pulsar magnetospheres.

If the neutron star is very young, however, there is another possibility for the origin of the plasma as suggested by Henriksen & Rayburn (1974) and Michel & Dessler (1981). It seems very likely for a newborn neutron star to have trapped a considerable amount of plasma by its strong gravity, even if most of the envelope matter has been blown away by a supernova explosion. We adopt this standpoint in the following discussion and assume that the plasma density is large enough to assure the quasi-neutral condition. There are two reasons for such an atmosphere of remnant plasma to take an extended structure. One is the possibility of enhanced thermal pressure due to a large effective Joule heating expected in the atmosphere (Henriksen & Rayburn, 1974; see also Section 5). The other is the strong centrifugal force due to a rapid rotation of the central star. It tries to reduce the effective gravity most efficiently in the equatorial plane. Therefore, the magnetosphere of such a neutron star is, qualitatively speaking, quite similar to that of the Jupiter.

It is well known that the Jovian magnetosphere contains a hot plasma disc which is formed by the action of centrifugal force on the magnetospheric plasma. The structure of the disc has been examined by many authors from various standpoints (e.g. Gledhill 1967; Melrose 1967; Gleeson & Axford 1976; Carbary & Hill 1978; Hill 1979). A similar idea was applied by Roberts & Sturrock (1973) also to radio pulsars. By assuming a spin-aligned dipole field, it has been shown that the steepest gradient of the plasma density which can be supported by the
magnetic field is proportional to $r^{-9/2}$ (Hill 1976), where $r$ is the radial distance in the equatorial plane. In deriving this conclusion, the deformation of the original field has been neglected. However, when the initial gradient is steeper than this, the configuration is unstable to the interchange motion and the plasma begins to flow out, distorting the magnetic field lines. The problem which we want to solve in the present paper is what the self-consistent steady-state configuration of the field and flow is.

For a better understanding of underlying physics, the description of the situation from an electrodynamical point of view is very useful. A rotating, magnetized neutron star acts as a unipolar inductor and causes a potential difference between different latitudes on its surface. If the plasma outside the equatorial disc is tenuous enough to suppress the perpendicular electrical conductivity, the potential difference is projected onto the disc along the magnetic field lines penetrating the disc. Thus, a radial electric field is induced in the disc. This field, in turn, drives a radial electric current (the conductivity in the disc is assumed to be isotropic since the density is sufficiently high) which eventually returns to the central dynamo along the magnetic field lines (see Fig. 1). The development of such a system of poloidal current results in a toroidal magnetic field which is responsible for the backward bending of the poloidal field lines.

In this configuration, angular momentum is transferred by the magnetic stress from the central flywheel to the equatorial disc along the field lines. Thereby, the disc plasma is brought into a rotation of the same sense as the flywheel and, as a consequence, it is driven outward by the centrifugal force. The radial flow across the vertical magnetic field generates an electromotive force which drives an azimuthal current. The latter causes a poloidal deformation of

![Figure 1](https://example.com/fig1.png)

**Figure 1.** Illustration of the electrodynamical aspect of a star–disc system. The central star, which is shown rather enlarged, drives a poloidally circulating current system. The latter causes a toroidal component in the magnetic field which is responsible both for the extraction of angular momentum from the star and for the formation of a collimated jet.
the magnetic field lines. This is the electrodynamical explanation for the field-line deformation due to the radial flow. Unless this current just cancels the vertical component of the original dipole field, the field lines remain closed. In order to see that this cancellation is unrealistic, suppose that this happens to occur. Then the toroidal electromotive force disappears. In other words, the driving force of the toroidal current vanishes, corresponding to the cancellation of the vertical field. Therefore, the toroidal current which is necessary to open the field lines cannot be maintained against the resistivity in the disc. Thus, it turns out that an open-field configuration is inconsistent with the presence of a resistive disc. On the other hand, if we completely neglect the plasma resistivity, the field lines should taken an open structure in order to allow a radial flow. Here, we can see a good example which shows how the assumption of infinitely conducting plasma imposes a misleading restriction on the problem.

It is also of great interest to see that the centrifugal disc configuration described hitherto is equally favourable for developing a well-collimated bipolar jet, as in the case of magnetically controlled accretion discs. As discussed above, the return current from the disc seems to converge toward a pair of conical regions around the polar axis. The self-pinching force on these current cones guarantees the collimation and accretion of the bipolar jet. Such a mechanism has been proposed for a magnetized accretion disc by Kaburaki & Itoh (1987). Although, in the present case, the sense of circulation of the poloidal current is different from the accretion case (compare Fig. 1 with Fig. 2 of Kaburaki & Itoh 1987), the pinching force, \( J_p \times B_\phi \), does not change sign, since the sign of \( B_\phi \) is also reversed. Therefore, the essential thing for the formation of a bipolar jet is the development of a quadrupole-type current system in the meridional plane. In the present case, such a system is driven by the central flywheel, whereas it is driven by the accretion disc in accretion cases. It should be emphasized, however, that the situation just mentioned is no more than a necessary condition for the formation of jets. Whether a jet is actually formed or not depends also on other conditions, e.g. on the possibility of matter supply to the polar regions.

The presence of a finite resistivity in the disc plasma has an essential meaning in the theory of electrodynamical accretion discs. It is because the conversion of gravitational energy to thermal energy in the disc is guaranteed by a non-zero resistivity. If a plasma has no resistivity (i.e. infinity conducting), the disc can liberate gravitational energy only through the hydrodynamical friction. Then the disc is essentially nothing but a viscous disc whose behaviour can be described by the standard model of an accretion disc, and the electromagnetic field plays only a secondary role. While the presence of sufficiently large viscosity is now a matter of controversy, the appearance of some anomalously large electrical resistivities has more firm ground both experimentally and theoretically (e.g. Alfvén 1977). In the context of X-ray stars and pulsars, the possibility of developing anomalous resistivity has been discussed, e.g. by Coppi & Treves (1971) (see also Ichimaru 1976; Kato 1984). In the present paper, however, we shall treat the electrical conductivity as a parameter, for simplicity. Detailed discussions of the electrical resistivity is left for future investigations.

3 The disc equations

The basic equations which are most suitable to deal with the present problem are the set of resistive MHD equations. For simplicity, we consider only axisymmetric, steady-state configurations and the viscosity of the fluid is neglected. Further, the disc is assumed to have a constant opening angle which is sufficiently small, i.e. \( \Delta = \text{constant} \ll 1 \) when measured in radian (spherical polar coordinates will be used throughout the paper). If we are satisfied with lowest-order discussion in the smallness parameter \( \Delta \), our basic equations take fairly simple forms. The method is quite similar to that used in discussing magnetized accretion discs (Kaburaki 1986, 1987).
The magnetic field consists of two parts:

$$\mathbf{B} = \mathbf{B}_0 + \mathbf{b},$$

(1)

where $\mathbf{B}_0$ is the original dipole field and $\mathbf{b}$ is the perturbed field. Within the equatorial disc (i.e. $\theta = \pi/2$), $\mathbf{B}_0$ has only the $\theta$-component:

$$\mathbf{B}_0 = \frac{\mu}{r^3} \mathbf{e}_\theta \equiv \tilde{\mathbf{B}}_0(r) \mathbf{e}_\theta,$$

(2)

where $\mu$ is the magnetic dipole moment and $\mathbf{e}_\theta$ is the unit vector in the $\theta$-direction. Since $\mathbf{B}_0$ is divergence-free, the flux conservation law reduces to

$$b_\theta(r, \xi) = -\frac{\Delta}{r} \frac{\partial}{\partial r} (r^2 b_r) \, d\xi,$$

(3)

with $\xi = (\theta - \pi/2)/\Delta$. Here, we assume that the perturbed field can be of the same order of magnitude as the original field, i.e. that $b_r$ and $b_s$ are of order unity. Then, the above equation shows that $b_\theta$ is a quantity of order $\Delta$ and hence negligible compared with the original field $\tilde{B}_0$.

The electric current is related to $\mathbf{b}$ by Ampère’s law:

$$j_r(r, \xi) = -\frac{c}{4\pi r \Delta} \frac{\partial b_\theta}{\partial \xi},$$

(4)

$$j_\theta(r, \xi) = -\frac{c}{4\pi r} \frac{\partial}{\partial r} (rb_\theta),$$

(5)

$$j_\phi(r, \xi) = -\frac{c}{4\pi r \Delta} \frac{\partial b_r}{\partial \xi}$$

(6)

In deriving these equations, we have used the facts that $\theta = \pi/2$ in the disc and $b_\theta/b_r \sim O(\Delta)$.

The mass conservation equation can be written in the integrated form,

$$\nu_\theta(r, \xi) = -\frac{\Delta}{\rho r} \frac{\partial}{\partial r} (r^2 \rho \nu_r) \, d\xi,$$

(7)

where $\nu$ and $\rho$ denote the velocity and density of the plasma. It turns out from this equation that $\nu_\theta \sim O(\Delta)$ as far as $\nu_r$ (the wind velocity) is regarded as of order unity. The azimuthal velocity $\nu_\phi$ will also be regarded as of order unity.

With the above specifications of the orders of magnitude for $b$ and $\nu$, we can derive the magnitudes of other quantities from the MHD equations. For example, we find that $\rho \sim O(\Delta^{-1})$ and $p \sim O(1)$ in order for the equation of motion to hold consistently. The former relation can be interpreted as reflecting the enhancement of the density due to the formation of a disc. The resulting relations are, up to lowest order,

$$\left( \frac{\partial}{\partial r} + \frac{\nu_\theta}{r} \frac{\partial}{\partial \xi} \right) \nu_r - \frac{\nu_r^2}{r} = \frac{GM}{r^2} + \frac{\tilde{B}_0}{4\pi \rho r \Delta} \frac{\partial b_r}{\partial \xi}.$$

(8)

$$\frac{\partial p}{\partial \xi} + \frac{1}{8\pi} \frac{\partial}{\partial \xi} (b_r^2 + b_\phi^2) = 0,$$

(9)
\[
\left( v_r \frac{\partial}{\partial r} + v_\theta \frac{\partial}{\partial \theta} \right) v_\theta + \frac{v_\phi v_r}{r} = \frac{\tilde{B}_0}{4 \pi \rho r \Delta} \frac{\partial b_\phi}{\partial \xi},
\]
where \( p \) is the pressure, \( M \) the mass of the central object and \( G \) the gravitational constant. In the above equations, the \( \theta \)-component can easily be integrated to give
\[
p + \frac{1}{8 \pi} (b_r^2 + b_\phi^2) = \tilde{p}(r).
\]

It is to be noted that the gravity does not appear in this equation since it has no \( \theta \)-component.

The electric potential induced in the disc is expanded like
\[
U(r, \xi) = U^{(0)} + U^{(1)} + \ldots
\]
Here, \( U^{(0)} \sim O(1) \) represents the potential that is projected into the disc along the unperturbed magnetic field lines and hence is a given quantity, while \( U^{(1)} \sim O(\Delta) \) is the correction to it and hence an unknown. The electric field is derived from this potential by partial differentiations:
\[
E_r(r, \xi) = -\frac{\partial U}{\partial r} = E_r^{(0)} + E_r^{(1)} + \ldots,
\]
\[
E_\theta(r, \xi) = -\frac{1}{\Delta r} \frac{\partial U}{\partial \xi} = E_\theta^{(-1)} + E_\theta^{(0)} + \ldots,
\]
\[
E_\phi(r, \xi) = 0.
\]
The last equation is a consequence of axisymmetry.

As clear from equation (15), the toroidal current should be maintained not by the electric field but by the electromotive force. This is the implication of the \( \phi \)-component of Ohm's law which is reduced to
\[
v_r \tilde{B}_0 = -\frac{c^2}{4 \pi \sigma \Delta} \frac{1}{r} \frac{\partial b_\phi}{\partial \xi}
\]
by neglecting the term of order \( \Delta \), where \( \sigma \) denotes the electrical conductivity and is assumed to be constant. In order for this equation to be self-consistent, the relation \( c^2/\sigma \sim O(\Delta) \) (and hence \( \Delta \approx \sigma^{-1} \)) should hold. Namely, the disc should adjust its thickness according to the size of the electrical conductivity. We will see later (in Section 4) a justification of this relation. The other two components of Ohm's law are
\[
E_r^{(0)} = -\frac{1}{c} v_\phi \tilde{B}_0 = \frac{c}{4 \pi \sigma \Delta} \frac{1}{r} \frac{\partial b_\phi}{\partial \xi},
\]
and
\[
E_\theta^{(-1)} = 0,
\]
\[
E_\phi^{(0)} + \frac{1}{c} (v_\phi b_r - v_r b_\phi) = 0.
\]
We can immediately conclude from equation (18) that
\[
U^{(0)}(r, \xi) = \tilde{U}_0(r) \quad \text{or} \quad E_r^{(0)}(r, \xi) = E_0(r).
\]
In this situation, the derivation of $E_r$ in the disc shown by Kaburaki (1986) is directly applicable to the present problem, and we have

$$E_o(r) = \frac{\Omega \mu}{c r^2},$$  \hspace{1cm} (21)

where $\Omega$ is the angular velocity of the central star.

We have thus derived a set of reduced equations, the disc equations, which describe the disc system under consideration. In this set of equations, $B_\theta(r)$ and $E_o(r)$ are the externally given field which control the dynamics of the disc. The former serves as a seed field for the disc magnetic fields and the presence of the latter is a fundamental difference from the idealized case of accretion discs.* The equations also contain a few differences arising from the different ordering of the relevant quantities.

We seek a solution to these equations by the method of variable separation. However, it does not seem to be essential to carry out this method in a strict sense, since the equations are already approximate. From this standpoint, we adopt the following expressions for the unknown quantities:

$$b_r(r, \xi) = -\tilde{B}_r(r) \text{sech}^2 \xi \tanh \xi,$$ \hspace{1cm} (22)

$$b_\theta(r, \xi) = \tilde{B}_\theta(r) \tanh \xi,$$ \hspace{1cm} (23)

$$v_r(r, \xi) = \tilde{v}_r(r) \text{sech}^2 \xi,$$ \hspace{1cm} (24)

$$v_\theta(r, \xi) = \tilde{v}_\theta(r) \tanh \xi,$$ \hspace{1cm} (25)

$$v_\phi(r, \xi) = \tilde{v}_\phi(r),$$ \hspace{1cm} (26)

$$\rho(r, \xi) = \tilde{\rho}(r) \text{sech}^2 \xi,$$ \hspace{1cm} (27)

$$\rho(r, \xi) = \tilde{\rho}(r) \text{sech}^2 \xi,$$ \hspace{1cm} (28)

$$E_r^{(0)}(r, \xi) = \tilde{E}_o(r) \text{sech}^2 \xi \tanh \xi,$$ \hspace{1cm} (29)

$$j_r(r, \xi) = \tilde{j}_r(r) \text{sech}^2 \xi,$$ \hspace{1cm} (30)

$$j_\theta(r, \xi) = -\tilde{j}_\theta(r) \tanh \xi,$$ \hspace{1cm} (31)

$$j_\phi(r, \xi) = \tilde{j}_\phi(r) \text{sech}^4 \xi.$$ \hspace{1cm} (32)

The choice of these forms for the angular parts are strongly suggested from experience in the study of accretion discs. These expressions are consistent with the disc equations provided that the approximations like $\tanh^2 \xi = 0$ and $\text{sech}^2 \xi = \text{sech}^4 \xi = 1$ can be used whenever it is needed. Therefore, our solution is exact only in the core part (i.e. $|\xi| \ll 1$) of the disc.

The equations which should be satisfied by the radial parts of the unknowns now become as follows:

mass continuity,

$$\tilde{v}_\theta = -\Delta \left[ \frac{d}{dr} \ln(r^2 \tilde{\rho} \tilde{v}_r) \right] \tilde{v}_r,$$ \hspace{1cm} (33)

equation of motion,

$$\tilde{v}_r \frac{d\tilde{v}_\theta}{dr} - \frac{\tilde{v}_\theta^2}{r} = -\frac{GM}{r^2} - \frac{1}{4 \pi \Delta} \frac{\tilde{B}_\theta \tilde{B}_r}{r \tilde{\rho}}.$$ \hspace{1cm} (34)

*Erratum in the previous paper (Kaburaki 1987). For the last term on the right-hand side of equation (12), $B_\theta \partial \phi / \partial r$, read $B_\phi \partial B_\theta / \partial \theta$. 

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\[ \ddot{\phi} = \frac{b_\phi^2}{8 \pi}, \]  

(35)

\[ \frac{d}{dr} (r \dot{b}_\phi) = \frac{1}{4 \pi \Delta} \frac{B_\phi \dot{b}_\phi}{\dot{\rho} \tilde{v}}, \]  

(36)

Ohm's law,

\[ \dot{v}_r = -\frac{c^2}{4 \pi \sigma \Delta} \frac{b_r}{r B_0}, \]  

(37)

\[ c E_\phi - \dot{v}_\phi B_0 = \frac{c^2}{4 \pi \sigma \Delta} \frac{b_\phi}{r}, \]  

(38)

\[ c E_\phi - \dot{v}_\phi \dot{b}_r - \dot{v}_r \dot{b}_\phi = 0, \]  

(39)

Ampère's law,

\[ \dot{j}_r(r) = \frac{c}{4 \pi \Delta} \frac{b_\phi}{r}, \]  

(40)

\[ \dot{j}_\phi(r) = \frac{c}{4 \pi} \frac{1}{r \Delta} \frac{d}{dr} (r \dot{b}_\phi), \]  

(41)

\[ \dot{j}_\phi(r) = \frac{c}{4 \pi \Delta} \frac{b_\phi}{r}. \]  

(42)

It should be emphasized again that the above expressions are not derived uniquely from the disc equations. Instead, the equations hold within equal accuracy even if, for example, the expression (26) is replaced by \( \nu_\phi(r, \xi) = \ddot{v}_\phi(r) \) \( \text{sech}^2 \xi \) which introduces a vertical shear in the toroidal velocity. In other words, the inclusion of vertical shear in \( \nu_\phi \) does not affect the forms of other quantities, especially of the toroidal magnetic field, within our approximation. Therefore, the vertical shear in the toroidal velocity does not seem to be essential in causing a toroidal bending of the magnetic field lines. Instead, the radial gradient in the specific angular moment is essential for the appearance of \( b_\phi \) as seen from equation (36).* From the electrodynamical viewpoint, the development of a poloidally circulating current system is the cause of the bending. Another remark on the above set of equations is that equation (35) implies that the disc is maintained by the magnetic pressure due to \( b_\phi \) outside the disc, i.e. by the pinch effect acting on the radial current in the disc.

4 Exhaustion discs

One of the characteristic distances in the present problem is the corotation radius,

\[ r_c = \left( \frac{GM}{\Omega^2} \right)^{1/3}, \]  

(43)

* Although Campbell (1987) insists that a vertical shear in \( r_\phi \) is essential to cause a toroidal bending of the magnetic field lines, as shown above, it does not seem to be correct, at least for the main body of a thin disc.
at which the gravity balances with the centrifugal force if the plasma keeps corotation up to this distance. Beyond this radius, the effective gravity (gravity minus centrifugal force) in the equatorial plane becomes negative and hence drives the plasma outward. Although actual plasmas cannot maintain strict corotation up to this radius, \( r_c \) can be used as a representative value of such a distance around which the centrifugal effect becomes important. Therefore, the inner edge of a centrifugal disc is at around \( r = r_c \). Within this distance, the plasma takes a more or less spherical distribution.

The other characteristic distance is the so-called Alfvén radius, \( r_A \), at which the deformation of the poloidal magnetic field lines become noticeable, owing to a well-developed poloidal flow of the plasma. Concrete expressions for this distance will be given in the next section. It is evident from this definition of \( r_A \) that \( r_A > r_c \) as far as centrifugal winds are realized. The disc can therefore be divided into two parts: the inner disc \( (r_A > r > r_c) \) and the outer disc \( (r > r_A) \). The inner disc corresponds to the acceleration region of the wind, whereas the outer disc is the wind zone.

In solving the disc equations we need boundary conditions at \( r = r_c \). Unfortunately, however, there is no simple way of specifying these boundary values. As for the value of \( \tilde{v}_r \) at \( r_c \), for example, it is not realistic to set \( \tilde{v}_r = \Omega r_c \) since there must certainly be a lag behind corotation even at this inner edge. This circumstance is closely related to the fact that the acceleration region of a poloidal flow is not restricted only to the inner disc. In fact, the plasma is gradually accelerated from smaller radii. This may be understood as follows. In the region \( r < r_c \), the centrifugal force is small compared with the gravity and hence negligible in zero-order considerations. Therefore, the gravity is approximately balanced by the radial pressure gradient. As a perturbation from this state, however, the presence of a small outward force may drive a small poloidal flow. Thus we are led to replace the boundary values for the velocity components at \( r_c \) by the asymptotic conditions,

\[
\tilde{v}_r \propto r^\alpha \quad \text{and} \quad \tilde{v}_\phi \rightarrow \Omega r \quad (r \rightarrow 0),
\]

where \( \alpha \) is a positive constant to be fixed by a later discussion. The latter condition represents the tendency of the plasma around a magnetized object to corotate with it at small radii.

In the inner disc (or in the region \( r < r_c \)), the equation of radial force balance reduces to

\[
\tilde{v}_r^2 = \frac{1}{4\pi \Delta} \frac{\tilde{B}_r \tilde{B}_r}{\tilde{\rho}}, \quad (45)
\]

since the centrifugal force exceeds the gravity (or since the gravity is balanced by the pressure gradient, in \( r < r_c \)). It can be seen from this equation that the centrifugal force is balanced by the magnetic restoring force due to the stretching (i.e. the appearance of \( \tilde{B}_r \)) of the dipolar field lines. The inertial term, \( \tilde{v}_r \frac{d\tilde{v}_r}{dr} \), has been neglected, expecting that \( \tilde{v}_r \) does not exceed the Alfvén velocity \( \sim B_r / (4\pi \Delta \tilde{\rho}) \) in this acceleration region.

In the outer disc, equation (34) reduces to

\[
\tilde{v}_r^2 = \frac{GM}{r} \quad \text{or} \quad \tilde{v}_r = V_h(r) = \left( \frac{GM}{r} \right)^{1/2}, \quad (46a)
\]

since the magnetic force decreases more rapidly with distance than the gravity. This fact can be confirmed later by using the resulting solution. In this region, the wind reaches and then keeps a constant speed, the terminal velocity,

\[
\tilde{v}_r \rightarrow V_\infty = \text{constant}, \quad (46b)
\]
since the centrifugal force is balanced by the gravity and furthermore the magnetic force diminishes. The inertial term, $\dot{\mathbf{v}}_s\cdot\dot{\mathbf{v}}_r/dr$, is negligible also in this region since $\mathbf{v}_r$ is almost constant. Equation (46a and b) serves as asymptotic conditions for the velocity components at large distances.

Since we have already made a number of approximations, there is no need to seek an exact solution to the disc equations. It is rather practical to find a pair of compact analytic expressions for $\dot{v}_r$ and $\dot{v}_\phi$ which satisfy the asymptotic conditions (44) and (46). One such expression for $\dot{v}_\phi$ is

$$\dot{v}_\phi(r) = V_c \left( \frac{r_c}{r} \right)^{1/2} \left[ 1 - \exp \left[ - \left( \frac{r}{r_C} \right)^3 \right] \right]^{1/2},$$

(47)

where $V_c = \Omega r_c = (GM\Omega)^{1/3}$. Of course, some other choice of functional form may be possible for $\dot{v}_\phi$. For example,

$$\dot{v}_\phi(r) = V_c \left( \frac{r_c}{r} \right)^{1/2} \left[ 1 - \exp \left[ - \left( \frac{r}{r_C} \right)^{3/2} \right] \right],$$

(47')

satisfies the same asymptotic conditions, and its form is rather simpler than equation (47). However, the expression (47) has a certain advantage over (47'). This will be made clear later in this section. Similarly, we can write $\dot{v}_r$ in the form

$$\dot{v}_r(r) = V_\infty \left[ 1 - \exp \left[ - \left( \frac{r}{r_\Lambda} \right)^a \right] \right].$$

(48)

The appearance of $r_\Lambda$ in this expression, instead of $r_c$, corresponds to the fact that the radius which separates the acceleration region from the constant wind region is the Alfvén radius.

Once the functional forms of the velocity components are specified, the forms of other quantities follow from the disc equations rather straightforwardly. The deformation of the poloidal magnetic field, $\vec{b}_\phi$, is obtained from equation (37) as

$$\vec{b}_\phi(r) = R_\Lambda B_\Lambda \left( \frac{r_\Lambda}{r} \right)^2 \left[ 1 - \exp \left[ - \left( \frac{r}{r_\Lambda} \right)^a \right] \right].$$

(49)

Here, $B_\Lambda = \mu/r_\Lambda^3$ (therefore $\vec{b}_0 = B_\Lambda (r_\Lambda/r)^3$) and

$$R_\Lambda = \frac{4\pi \Delta \sigma r_\Lambda V_\infty}{c^2}.$$  

(50)

The latter is the magnetic Reynolds number which is characteristic of the present problem. The exact value of this quantity cannot be determined from our rather crude scheme. However, this turns out to always have a value of order unity. This can be shown based on the definition of the Alfvén radius. At $r = r_\Lambda$, equation (49) gives $\vec{b}_\phi \sim R_\Lambda B_\Lambda$ as a strength of perturbed magnetic field. This should be just comparable with the original field $B_\Lambda$ reflecting the fact that the deformation becomes noticeable at this radius. Thus we have $R_\Lambda \sim 1$. This universality of the value of $R_\Lambda$ is consistent with the $\sigma$-dependence of the disc thickness $\Delta$, i.e. $\Delta \propto \sigma^{-1}$, which has been suggested in the previous section. Interestingly, this relation also suggests a self-regulation mechanism for the electrical conductivity and hence for the disc thickness. If the electrical conductivity is very large, the thickness $\Delta$ becomes very small.
Under such configurations, the electric current density in the disc becomes very large as seen from equations (40) and (42). Therefore, a disc with sufficiently large conductivity is very likely to develop a strong current-driven plasma instability, which in turn results in a drastic decrease in the conductivity, and consequently, a thickening of the disc.

The toroidal magnetic field which is generated by the poloidally circulating current system is calculated from equation (38) as

$$b_\phi(r) = R_A B_A \left( \frac{V_c}{V_\infty} \right) \left( \frac{r_A}{r_c} \right)^{3/2} \left( 1 - \left( \frac{r}{r_c} \right)^3 \right)^{1/2} \left[ 1 - \exp \left( - \left( \frac{r}{r_c} \right)^{3} \right) \right].$$  \hspace{1cm} (51)

Equation (36) is used to obtain

$$\dot{\rho}(r) = \rho_A \left( \frac{r_A}{r} \right)^{7/2} F(r),$$  \hspace{1cm} (52)

where

$$\rho_A = \frac{R_A B_A^2}{2 \pi \Delta V_\infty^2} \left( \frac{r_A}{r_c} \right)^{3/2},$$  \hspace{1cm} (53)

$$F(r) = \frac{[1 - (r/r_c)^{3/2}]^2 [1 - \exp[-(r/r_c)^3]]^{1/2}}{[1 - \exp[-(r/r_A)^3]] [1 - 3(r/r_c)^3] \exp[-(r/r_c)^3]].$$  \hspace{1cm} (54)

Equation (53) shows that the terminal velocity is of the order of the Alfvén velocity, $B_A/\sqrt{3\pi\Delta\rho_A}$, as long as $(r_A/r_c)$ is not too large. The asymptotic behaviours of $F(r)$ at small and large radii are

$$F(r) = \begin{cases} \frac{1}{16} \left( \frac{r_A}{r_c} \right)^{3/2} \left( \frac{r_A}{r} \right)^{a - 3/2} & \text{as } r \to 0, \\ 1 & \text{as } r \to \infty. \end{cases}$$  \hspace{1cm} (55a, 55b)

In order to fix the value of $\alpha$, we derive another expression for $\dot{\rho}$ in the inner disc from equation (45):

$$\dot{\rho}(r) = \frac{R_A B_A^2}{4 \pi \Delta} \left( \frac{r_A}{r_c} \right)^4 \left( \frac{r_A}{r} \right) \left[ 1 - \exp[-(r/r_A)^3] \right]$$  \hspace{1cm} (56a)

$$\rightarrow \frac{R_A B_A^2}{4 \pi \Delta} \left( \frac{r_c}{r_A} \right)^2 \left( \frac{r_A}{r} \right)^{7 - a} \text{ (as } r \to 0).$$  \hspace{1cm} (56b)

Comparing the above two expressions for $\dot{\rho}$ at small radii, equations (55a) and (56b), we find

$$\alpha = \frac{5}{2}$$  \hspace{1cm} (57)

$$\rho_A = \frac{4 R_A B_A^2}{\pi \Delta V_c^2} \left( \frac{r_c}{r_A} \right)^{7/2}. \hspace{1cm} (58)$$
The latter equation is further combined with equation (53) to give

\[
\frac{V_\infty}{V_C} = \frac{1}{\sqrt{8}} \left( \frac{r_A}{r_C} \right)^{5/2}.
\]  

(59)

The asymptotic forms of the density is now written as

\[
\tilde{\rho}(r) \rightarrow \begin{cases} 
\frac{\rho_A \left( \frac{r_A}{r_c} \right)^{3/2} \left( \frac{r_A}{r} \right)^{9/2}}{16} = \rho_c \left( \frac{r_c}{r} \right)^{9/2} & \text{for } r \leq r_c, \\
\rho_A \left( \frac{r_A}{r} \right)^{7/2} & \text{for } r > r_A,
\end{cases}
\]  

(60a)

(60b)

where \(\rho_c\) denotes the value of \(\tilde{\rho}\) at \(r_c\). The density in the inner disc therefore tends, toward its inner edge, to the marginally stable distribution to the interchange instability. This is the very feature that makes the expression (47) for \(\tilde{v}_\theta\) more preferable than (47'). For, in the latter case, the distribution near the inner edge becomes somewhat steeper than this marginal one. As seen from the expression (60b), the radial gradient of \(\tilde{\rho}\) decreases (in its absolute value) in the outer disc corresponding to the development of \(v_r\). The \(\theta\)-component of the plasma velocity can be derived from equation (33) by taking into account the expressions (48) and (52) with \(\alpha = 5/2\). However, we cite its asymptotic behaviour to avoid unnecessary complications:

\[
\tilde{v}(r) \rightarrow \begin{cases} 
0 & \text{for } r < r_c, \\
\frac{3A}{2} V_\infty & \text{for } r > r_A.
\end{cases}
\]  

(61a)

(61b)

It can be seen from the lower expression that, strictly speaking, the wind does not blow in the radial direction but slowly diverges in the \(\theta\)-direction. Owing to this effect, the disc structure would eventually become obscure beyond a certain distance which may be called the outer edge. At present, however, we have no practical means to evaluate this radius.

The expression for the pressure in the disc is derived from equations (35) and (51). The result is

\[
\tilde{p}(r) = \frac{R_A B_A}{\pi} \left( \frac{r_A}{r_c} \right)^3 \left( \frac{r}{r_c} \right)^2 \left[ 1 - \left( \frac{r}{r_c} \right)^{3/2} \right] \left[ 1 - \exp \left[ - \left( \frac{r}{r_c} \right)^3 \right] \right]^{1/2}.
\]  

(62)

If this pressure in the disc is dominated by the gas pressure,

\[
p = K \rho T,
\]  

(63)

where \(K = \mathcal{C}/\lambda\), with \(\mathcal{C}\) and \(\lambda\) being the gas constant and the mean molecular weight, respectively, then the temperature becomes constant in the \(\theta\)-direction, i.e. \(T(r, \xi) = \tilde{T}(r)\), and its radial part is given by

\[
\tilde{T}(r) = \frac{R_A B_A}{\pi K \rho} \left( \frac{r_A}{r_c} \right)^3 \left( \frac{r}{r_A} \right)^{3/2} G(r),
\]  

(64)

where

\[
G(r) = \left\{ 1 - \exp \left[ - \left( \frac{r}{r_A} \right)^{5/2} \right] \right\} \left\{ 1 - 3 \left( \frac{r}{r_c} \right)^3 \right\} \exp \left[ - \left( \frac{r}{r_c} \right)^3 \right].
\]
Centrifugally exhausting discs

\[
\times \left[ 1 - \left( \frac{r_c}{r} \right)^3 \right] \left[ 1 - \exp \left[ - \left( \frac{r}{r_c} \right)^3 \right] \right]^{1/2} \left[ 1 - \exp \left[ - \left( \frac{r}{r_c} \right)^3 \right] \right]^{-1}.
\]  

(65)

Since \( G(r) \to 1 \) as \( r \to \infty \), the temperature in the outer disc gradually rises toward its outer edge.

The electric currents in the disc are calculated from equations (40)-(42) as

\[
\dot{j}_d(r) = \frac{R_A}{\sqrt{2} \pi \Delta} \left( \frac{c B_A}{r_A} \right) \left( \frac{r_c}{r} \right)^{3/2} \left[ 1 - \left( \frac{r_c}{r} \right)^{3/2} \right] \left[ 1 - \exp \left[ - \left( \frac{r}{r_c} \right)^3 \right] \right]^{1/2},
\]

(66)

\[
\dot{j}_{\phi}(r) = \frac{3 \sqrt{2} R_A}{4 \pi} \left( \frac{c B_A}{r_A} \right) \left( \frac{r_c}{r} \right)^{3/2} \left[ 1 - \left( \frac{r_c}{r} \right)^3 \right] \exp \left[ - \left( \frac{r}{r_c} \right)^3 \right] \left[ 1 - \exp \left[ - \left( \frac{r}{r_c} \right)^3 \right] \right]^{1/2},
\]

(67)

\[
\dot{j}_d(r) = \frac{R_A}{4 \pi \Delta} \left( \frac{c B_A}{r_A} \right) \left( \frac{r_c}{r} \right)^3 \left[ 1 - \exp \left[ - \left( \frac{r}{r_A} \right)^{5/2} \right] \right].
\]

(68)

The correction to the externally given electric field, \( \dot{E}_{\theta} \), first appears in the \( \theta \)-direction owing to the dynamo action of the disc. As derived from equation (39), this field can be written as

\[
\dot{E}_{\theta}(r) = R_A \left( \frac{\Omega r_A B_A}{c} \right) \left( \frac{r_c}{r} \right) \left[ 1 - \exp \left[ - \left( \frac{r}{r_A} \right)^{5/2} \right] \right].
\]

(69)

5 Discussion and application of the solution

In this section, we focus our attention on some global properties of a centrifugally exhausting disc by using the solution obtained in the previous section. In doing so, the physical consistency of the solution will also be checked.

We begin with the mass-loss rate of the disc due to the centrifugal wind. The integrated mass flux across a sphere of arbitrary radius in the outer disc is

\[
\dot{m} = \int_0^\pi 2 \pi r^2 \rho v \, \sin \theta \, d\theta = \frac{8 \pi}{3} \Delta \rho_c V_c r_c^2,
\]

(70)

where the asymptotic forms of the relevant quantities at large radii have been used. If the plasma is not supplied at all from the surface of the central neutron star, this rate is identical to the decreasing rate of the total atmospheric mass. The above result is independent of \( r \), but may depend on the time. Actually, a time dependence enters into equation (70) through the decreases in the angular velocity of the star and in the density at a fixed level which are, respectively, due to the losses of angular momentum and mass. The lifetime of the disc depends on the total mass trapped by a newborn neutron star.

The angular momentum budget for the outer disc is the next to be considered. Although we have no definite way of identifying the disc surfaces in our solution, suppose that we can set them at some suitable latitudes, \( \hat{x} = \pm \hat{x}_0 \) (the upper sign corresponds to the upper surface).
Then the angular momentum carried by the wind in the $r$-direction across a disc's cross-section at arbitrary $r$ is (provided that $r > r_A$)

$$L'_w = 2\pi \int_{-\xi_0}^{\xi_0} \rho v_\phi v_r r^2 d\theta$$

$$= 2R_A \tanh \xi_0 \left( 1 - \frac{1}{3} \tanh^2 \xi_0 \right) \left( \frac{v_C}{V_\infty} \right) \left( \frac{r_A}{r_C} \right) B^2 \frac{r^3}{\Delta r^2}.$$  \hspace{1cm} (71)

This flux becomes negligible at the disc's outer edge ($r = b$) as far as $b \gg r_A$. Therefore, this flux contributes to the angular momentum budget of the outer disc only at $r = r_A$. The angular momentum carried vertically across the surfaces of the outer disc ($r \geq r_A$) by the wind is

$$L''_w = 4\pi \int_{r_A}^{\infty} \rho(r, -\xi_0) v_\phi(r, -\xi_0) b_\phi(r) r^2 dr$$

$$= -3R_A \tanh \xi_0 \text{sech}^2 \xi_0 \left( \frac{v_C}{V_\infty} \right) \left( \frac{r_A}{r_C} \right) B^2 \frac{r^3}{\Delta r^2}.$$ \hspace{1cm} (72)

where the minus sign indicates the extraction of angular momentum from the disc.

The magnetic stress also transfers angular momentum. The amount which is carried through the upper and lower surfaces of the outer disc is

$$L''_b = -\int_{r_A}^{\infty} b_\phi(r, -\xi_0) \tilde{B}_\phi(r) r^2 dr$$

$$= R_A \tanh \xi_0 \left( \frac{v_C}{V_\infty} \right) \left( \frac{r_A}{r_C} \right) B^2 \frac{r^3}{\Delta r^2}.$$ \hspace{1cm} (73)

whereas that through a vertical cross-section at arbitrary $r \geq r_A$ is negligible since it contains $\Delta$:

$$L''_b = -\frac{\Delta}{2} \int_{-\xi_0}^{\xi_0} b_\phi b_r r^2 d\xi$$

$$= \frac{\Delta}{3} R_A \tanh^3 \xi_0 \left( \frac{v_C}{V_\infty} \right) \left( \frac{r_A}{r_C} \right) B^2 \frac{r^3}{\Delta r^2}.$$ \hspace{1cm} (74)

It should be emphasized here that the magnetic field in the outer disc actually contributes to the angular momentum transport, whereas it cannot contribute to the radial force balance.

In the steady state, the income and outcome should balance. It is easy to confirm that this is indeed the case, i.e. $L'_w + L''_w + L''_b = 0$, within the accuracy of our solution (we can use the
approximations, \( \tanh^2 \xi_0 = 0 \) and \( \text{sech}^2 \xi_0 = 1 \). The balance holds irrespective of the choice of the disc surface. All the angular momentum passing through the disc comes ultimately from the central star, so that the star spins down at a rate

\[
\frac{d\Omega}{dt} = \frac{3 R_A}{I} \tanh \xi_0 \left( \frac{V_c}{V_{\infty}} \right) \left( \frac{r_A}{r_c} \right) B_A^2 r_A^3 \frac{\mu^2 \Omega^2}{GM},
\]

(75)

where \( I \) is the moment of inertia of the star and is assumed to be constant. The explicit dependence of \( \Omega \) on \( \Omega \) indicates that the braking index is 2. Actually, however, a more complex \( \Omega \)-dependence is included in the ratio \( r_c/r_A \).

Associated with the spin down described above, rotational energy is extracted from the star at a rate

\[
\frac{dE}{dt} = I \Omega \left( \frac{d\Omega}{dt} \right) = 3 R_A \tanh \xi_0 \left( \frac{V_c}{V_{\infty}} \right) \left( \frac{r_A}{r_c} \right) B_A^2 r_A^3 \Omega
\]

\[
= 6 \sqrt{2} R_A \tanh \xi_0 \left( \frac{r_c}{r_A} \right)^{9/2} \left( \frac{r_A}{r_c} \right) \frac{\mu^2 \Omega^3}{GM},
\]

(76)

It is interesting to see that the above formula is essentially identical to that given by Davis & Pringle (1981) for the case of subsonic rotation of a neutron star in a binary system.

A considerable fraction of this amount is liberated in the disc through the Joule dissipation. In order to see this, we calculate the heating rate in the outer disc:

\[
H = \int \frac{1}{\sigma} dV = 4 \pi \Delta \int_0^{\infty} dr \int_0^{\infty} \frac{d\xi}{\sigma} \left( \frac{j_x^2}{\sigma} + \frac{j_y^2}{\sigma} \right) r^2
\]

\[
= \frac{16}{3} R_A \left( \frac{r_c}{r_A} \right)^3 B_A^2 r_A^3 V_{\infty} \left[ 1 + \frac{1}{105} \left( \frac{r_A}{r_c} \right)^2 \right],
\]

(77)

where \( j_\theta \) has been neglected since \( j_\theta/j_i \sim O(\Delta) \). The Joule dissipation occurs also in the inner disc. However, the contribution from this region does not alter the above result drastically, as seen from the asymptotic forms of \( j_i \) and \( j_\theta \) at small radii. Therefore, we have the ratio

\[
H/(-dE/dt) \sim \frac{16}{9} \left( \frac{r_c}{r_A} \right)^5 \left( \frac{V_{\infty}}{V_c} \right)^2
\]

(78)

by assuming \( \tanh \xi_0 \sim 1/\sqrt{2} \), and this reduces to \( 16/2/9 \) if we use the relation (59). The rest of the energy extracted from the central star is carried away by the wind.

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Figure 2. A few examples of magnetic field-line configurations. The distinctions arising from different disc thickness are compared at smaller distances (a) and at larger distances (b). The unit of the distance is the Alfvén radius, and the toroidal component is suppressed.
We derive here the explicit expression for the Alfvén radius \( r_A \). As defined in equation (60a), the density at \( r_C \) is written in terms of \( \rho_A \) as \( \rho_C = (\rho_A/16)(r_A/r_C)^6 \). Combining this with the expression (58), we obtain

\[
r_A = \left( \frac{R_A}{4 \pi \Delta} \right)^{2/7} \left[ \frac{\mu^4 \Omega^2}{\rho_C^2 (GM)^3} \right]^{1/7}.
\]  

(79)

This expression, however, may not be so useful since \( \rho_C \) seems difficult to determine from observations. Another point to be noted is that, as the time proceeds, \( \rho_C \) decreases owing both to the mass loss and to the increase in \( r_C \) caused by a spin down. If we eliminate \( \rho_C \) by using equation (70), we have a more useful expression,

\[
r_A = \left( \frac{R_A}{3 \sqrt[7]{2}} \right)^{2/7} \left( \frac{\mu^4}{GM \Delta^2} \right)^{1/7}.
\]  

(80)

This expression is identical with that for the Alfvén radius in the accretion case, apart from a numerical factor and the sign of \( \Delta \). The region \( r \lesssim r_A \) may therefore be called the magnetosphere, with the inner disc being a boundary layer between the magnetosphere and the outer magneto-disc. A few examples of field-line configurations for different disc thickness are shown in Fig. 2.

Finally, we discuss the possibility of observing exhaustion discs around young neutron stars. Since a typical value of \( V_C \) is given by the formula

\[
V_C = 0.94 \times 10^8 \left( \frac{M}{M_\odot} \right)^{1/3} \left( \frac{P}{\text{m s}} \right)^{1/3} \text{ cm s}^{-1},
\]  

(81)

where \( P \) is the rotational period and \( M_\odot \) is the solar mass, the temperature at the Alfvén radius is roughly given by

\[
T_A = 5.3 \times 10^7 \left( \frac{M}{M_\odot} \right)^{2/3} \left( \frac{P}{\text{m s}} \right)^{-2/3} \text{ K}.
\]  

(82)

Here, a fully ionized hydrogen plasma has been assumed. The temperature increases gradually toward the outer edge of a disc and hence the emission of multi-temperature black-body spectrum ranging mainly from UV to soft X-ray is expected, provided that the disc is optically thick. Probably the best candidate in which such a disc may be observed is at present SN 1987A. In this regard, the existence of a persistent thermal component (\( T \sim 4 \text{ kev} \)) in X-ray spectrum from SN 1987A (Dotani et al. 1987) is of great interest.

The interpretation of the bright spot near SN 1987A proposed by Rees (1987) is also interesting from the standpoint of our model. Such a phenomenon is very likely to occur as a consequence of the interaction between a jet from SN 1987A and the expanding shell or interstellar medium. Anyway, a newborn neutron star would exhaust its remnant atmosphere in terms of a disc-like wind and, if the condition is favourable, of a bipolar jet. The lifetime of this disc phase depends on the total mass trapped by the neutron star as well as on its rotation and magnetic field. The radio pulsar phase would come after this phase when the plasma density around the neutron star decreases to the Goldreich–Julian density having exhausted its atmosphere.
References