The redshift projection – I. Caustics and correlation functions

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SUMMARY
The relationships between structures in real and redshift space are discussed in the context of the hierarchical scenario. In particular, the formation and properties of caustics in redshift space are investigated. Redshift caustics are shown to be a generic feature of the gravitational instability picture of galaxy formation. The anisotropy of the galaxy distribution in redshift space is also considered. It is shown that the elongation along the line-of-sight of the galaxy–galaxy redshift space correlation function, as measured by Davis & Peebles and Bean et al., indicates that this function is dominated by galaxies in dynamically non-linear regions. The elongation also suggests that the clustering measured by \( \xi_{gg}(r) \) is dynamical in origin, rather than just due to biasing; it too is likely to be dominated by galaxies in non-linear regions on scales up to 10 \( h^{-1} \) Mpc.*

1 INTRODUCTION
One of the most important goals of modern cosmology is to account for the large-scale structure of the Universe. Until recently, however, theorists were mainly concerned with the spatial distribution of matter. With the advent of extended galaxy redshift surveys (e.g. de Lapparent, Geller & Huchra 1986; Giovanelli, Haynes & Chincarini 1986) and information about large-scale streaming motions (e.g. Collins, Joseph & Robertson 1986; Dressler et al. 1987) the position is beginning to change. There has been the realization that galaxies exist not just in three-dimensional physical space, but rather populate a six-dimensional phase space. Galaxies move and this motion, too, must be explained.

The purpose of this paper, the first of two, is to consider some of the consequences of large-scale streaming motions. In particular, because of Hubble’s law, recession velocities are frequently used as a distance measure; actual distances to galaxies are still extremely hard to determine (but see Dressler et al. 1987). If galaxies were to move solely with the general expansion of the Universe then redshift would be an exact measure of distance. Since galaxies do move with respect to the general expansion – they have peculiar (or streaming) velocities – this measure is distorted. Redshift surveys locate galaxies in redshift space, not real space, and there are significant differences between the two. This is the motivation behind the first part of the paper: to examine the relationships between structures in real and redshift space. In this context both spaces are best seen as three-dimensional projections of phase space. Work in this area has already been done by Kaiser (1987). Amongst other things, Kaiser investigated the structures which occur in redshift space close to clusters with a \( 1/r^{1.75} \) density profile. He showed that there are structures which appear in redshift space which do not exist in real space. These ideas are extended in this paper within the context of the hierarchical scenario. It will be shown that caustics occur in redshift space, exactly analogous to Zel’Dovich pancakes in real space (Zel’Dovich 1970), while the density contrast in real space is still close to unity. These concepts are employed in the second paper in this series as the basis for a model of the Ly \( \alpha \) clouds seen towards high redshift quasars (e.g. Sargent et al. 1980).

One other, perhaps less obvious, consequence of large-scale motion is that redshift space is not isotropic. Peculiar velocities only alter galaxy positions along the line-of-sight. By considering the anisotropy of the galaxy distribution it is possible to obtain information about galaxy motions. This idea is taken up in Section 4, using various correlation functions in redshift space. Both Kaiser (1987) and Lilje & Efstathiou (1989) have also considered these functions. However, it will be argued that both authors have somewhat misinterpreted their results. In particular it is shown that the complete galaxy–galaxy correlation function in redshift space, determined by Davis & Peebles (1983) and Bean et al. (1983), is inconsistent with linear theory, no matter what the initial power spectrum or the cosmology. This correlation function must be dominated by galaxies in dynamically non-linear regions.

The remainder of the paper is as follows. Section 2 sets out the background and basic equations. In Section 3 the formation and properties of redshift caustics are discussed. The following section calculates a selection of correlation functions in redshift space and compares the results to observations. A problem with a model of streaming velocities

* \( h \) is the Hubble constant measured in units of 100 km s\(^{-1}\) Mpc\(^{-1}\).

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due to Peebles (1980) is also mentioned. The conclusions and a discussion of the results are given in Section 5. Appendix A covers the calculation of angular cross-correlation functions and derives the main redshift correlation function.

2 BACKGROUND AND BASIC EQUATIONS

The positions of cosmologically distributed objects are most appropriately specified in terms of comoving coordinates. If the expansion factor of the Universe is \( a(t) \) at time \( t \) then the position vector of a particle \( r \) is given in terms of its corresponding comoving coordinate \( x \) by

\[
r = a(t)x.
\]

(2.1)

Differentiating with respect to time yields the velocity

\[
\dot{r} = \dot{a}(t)x + a(t)\dot{x}.
\]

(2.2)

The first term in this equation is the velocity due to the uniform expansion and is responsible for Hubble’s law. The second represents motion with respect to the comoving reference frame – the peculiar velocity of the object.

The density at a point \( x \) can usefully be specified in terms of the fractional overdensity, defined by

\[
\delta(x, t) = \frac{\rho(x, t) - \bar{\rho}(t)}{\bar{\rho}(t)} = \frac{\Delta(x, t)}{\bar{\rho}(t)},
\]

(2.3)

where \( \rho(x, t) \) is the density at \( x \) and \( t \) and \( \bar{\rho}(t) \) is the average density. It is straightforward to linearize the equations of motion to obtain the equations of first order perturbation theory (e.g. Peebles 1980). While \( \delta \ll 1 \), the results of linear theory may be expressed as

\[
\delta(x, t) = D(t)\Delta(x).
\]

(2.4)

Here \( D(t) \) is the growth factor for perturbations and depends only on the cosmology, not the form of the initial perturbations, specified by \( \Delta(x) \). The peculiar velocity field may also be determined in the linear approximation. This is given by

\[
\dot{x}(x) = -D(t)\nabla\Psi(x),
\]

(2.5)

where

\[
\nabla^2\Psi(x) = \Delta(x).
\]

(2.6)

\( \Psi(x) \) represents the gravitational potential due to the overdensity \( \Delta(x) \).

While linear theory is adequate for some purposes, it is straightforward to extend the results into the non-linear regime using the Zel’dovich approximation (Zel’dovich 1970). This assumes that a particle placed initially at \( q \) always has the peculiar velocity given by equation (2.5). This is easily integrated to give

\[
x(q, t) = q - D(t)\nabla\Psi(q).
\]

(2.7)

The results of this approximation have been found to agree well with the results of numerical simulations until the first structures have collapsed (e.g. Efstathiou 1987). In this paper we are interested in a period before spatial collapse has occurred. The Zel’dovich approximation will more than suffice for our purposes.

The initial density field is most easily described in terms of its Fourier components. We shall assume that the Universe is periodic in some volume so that the wavevectors, \( k \), take on discrete values only. \( \Delta(x) \) is given by

\[
\Delta(x) = \sum_{k \neq 0} \Delta_k e^{ik \cdot x} \tag{2.8}
\]

The \( k = 0 \) value has been excluded since the mean value of \( \Delta(x) \) is zero. The initial field is also usually assumed to be Gaussian random. In that case the phases of the components are taken to be random, while the amplitudes are drawn from a Rayleigh distribution (e.g. Bardeen et al. 1986; Couchman 1987). The statistical properties of the field are completely specified by the power spectrum,

\[
P(k) = \langle |\Delta(k)|^2 \rangle \tag{2.9}
\]

where \( \langle ... \rangle \) represents taking an ensemble average. \( P \) has been written as a function of \( k = |k| \) only because of isotropy.

As noted in the introduction, galaxies populate a six-dimensional phase space, which we shall denote as \( \Gamma \). In general not all of \( \Gamma \) will be occupied. For example, if linear theory were thought to be valid there would be a well-defined velocity at each point. Galaxies would then lie on a three-dimensional hypersurface in \( \Gamma \). In reality, of course, galaxies will also have random motions superimposed on any large-scale streaming motion – they will probably lie only close to such a surface. The position of an object in \( \Gamma \) is completely specified by its position vector and velocity. From the point of view of this paper, real space, \( X \), and redshift space, \( R \), are best thought of as different but related three-dimensional projections of \( \Gamma \). The projection on to \( X \) is obvious – the velocity information should be discarded. It just remains to construct an expression for the projection on to \( R \).

First, however, a brief digression about notation. For many real-space quantities there is an analogous redshift-space quantity. We shall denote this by the accent ‘∗’. For example, if the real space quantity is \( y \) then the associated redshift quantity is \( \hat{y} \). The viewing direction should also be mentioned here. This is one of the most important parameters specifying the projection of \( \Gamma \) on to \( R \). We shall only consider the effects of the projection within a limited volume. If the volume is sufficiently distant, all the lines-of-sight through the volume will be approximately parallel. The viewing direction is then characterized by \( n \), the unit vector along the line-of-sight.

Consider now the position of a particle in redshift space, \( \hat{r} \), given its location in \( \Gamma \), i.e., its position vector and velocity, \( r \) and \( \dot{r} \), respectively. The vector \( \hat{r} \) may be constructed as the sum of a component parallel to the line-of-sight, just dependent on \( r \), and a component perpendicular to \( n \), depending only on \( r \)

\[
\hat{r} = (r - (r \cdot n)n) + \frac{a}{d}(\hat{r} \cdot n)n. \tag{2.10}
\]

The second term is scaled by \( a/d \), representing Hubble’s law. On using equations (2.1) and (2.2), this becomes

\[
\hat{x} = x + \frac{a}{d}(\dot{x} \cdot n)n. \tag{2.11}
\]

Equation (2.11) lies at the heart of this paper. It shows that in the absence of peculiar velocities the transformation \( \hat{x} \to x \) is
an identity. This is the fundamental connection between $X$ and $R$ and is the embodiment of Hubble’s law. Unfortunately this happy state of affairs is upset by the second term in this equation. It represents the additional displacement down the line-of-sight due to the peculiar velocity of the particle. The remainder of the paper will be concerned with the consequences of this additional term.

3 REDSHIFT CAUSTICS

Having set out the basic equations in the preceding section, we now wish to calculate the position of a particle in $R$ according to the Zel’dovich approximation. Eliminating $\dot{x}$ and $x$ from (2.11) using equations (2.5) and (2.7), respectively,

$$\ddot{x}(\mathbf{q}, t) = q - D(t)\nabla\Psi(q) + f[\nabla\Psi(q) \cdot \mathbf{n}]\mathbf{n},$$  

(3.1)

where $f = (a\dot{a}/D)/D$, $f$ is a function only of the cosmological density parameter, $Ω$. $Ω$ is the ratio of the actual mass density of the Universe to the density needed just to halt the general expansion. To a good approximation, $f(Ω) = Ω^{0.6}$ (Peebles 1980). In particular, $f(1) = 1$.

Let us examine (3.1) more closely. The first term in the brackets, $\nabla\Psi$, represents the displacement of the particle from $\mathbf{q}$ in real space. The second term is the component of this displacement along $\mathbf{n}$, multiplied by $f$. This is an additional displacement in redshift space due to the peculiar velocity of the particle. Note that this term always acts to enhance the real space displacement. Further, the ratio of the magnitude of the two terms is

$$\frac{f \left| \nabla\Psi \right| \cos θ}{\left| \nabla\Psi \right|} = f \cos θ,$$

(3.2)

where $θ$ is the angle between $\mathbf{n}$ and $\nabla\Psi$. Since $Ω \sim 1$, the two terms have approximately the same magnitude. The effects of the second term cannot be neglected. It should also be noted that the ratio depends on $θ$. The extra displacement will have greater effect when aligned with the viewing direction. This tends to emphasize structures which are collapsing along the line-of-sight, creating structures perpendicular to $\mathbf{n}$.

It is possible to demonstrate all these effects in two dimensions. In particular, the viewing direction may be altered; there is no new physics in three dimensions. Consider the evolution of particle positions in real and redshift space given an initially Gaussian random field with power spectrum

$$P(k) \propto k^\alpha \exp \left( \frac{k/k_c - 1}{α} \right) + 1^{-1},$$

(3.3)

The function in square parentheses is based on the Fermi–Dirac distribution of electron energies. It acts as a high-frequency filter, allowing both the positions and sharpness of the cut-off to be altered. For the realizations shown $n = -1$, $k_c = 12π/L$, where $L$ is the size of the box and $α = 0.3$. The same realization was used to produce all the figures and $Ω$ was taken to be unity throughout. The power spectrum was normalized so that $\langle |Δx|^2 \rangle = 1$. With this choice the growth factor, $D(t)$, would be the rms value of the evolved density field if only linear growth had occurred.

Fig. 1(a)–(d) show the evolved particle positions in red space for various values of the growth factor. Fig. 2(a)–(d) show the equivalent particle positions in redshift space, evolved using equation (3.1) and projected along the direction indicated. Fig. 3(a)–(d) show the positions in redshift space for $D = 0.4$, projected along different lines-of-sight.

It is apparent from these figures that structures are significantly more pronounced in redshift space than in real space. The additional displacement along the line-of-sight creates structures perpendicular to $\mathbf{n}$. Fig. 3 shows that exactly which structures appear depends on the viewing angle, confirming the dependence on $θ$ discussed above. Comparing Figs 1 and 3 it is also apparent that caustics appear in redshift space while the density in real space is relatively low. Let us call these structures redshift caustics. Notice too that underdense regions in real space have even lower densities in redshift space.

We may estimate the real-space density at which redshift caustics occur as follows. Consider two particles in a one-dimensional system, initially separated by a small distance in comoving coordinates $Δq$. Now allow the system to evolve such that the $\nabla\Psi$ term in equation (2.7) has decreased the real-space separation by $Δx$. The redshift separation will have decreased by a further amount $fΔx$. The remaining separations in real and redshift space are given by $Δs = Δq - Δx$ and $Δs = Δq - (1 + f)Δx$, respectively. The condition for a redshift caustic to form is that $Δs = 0$, giving $Δs = Δq/f(1 + f)$. The corresponding fractional overdensity

![Figure 1](image-url)

**Figure 1.** Evolved particle positions in real space determined using the Zel’dovich (1970) approximation (2.7). The initial density field is discussed in the text. The positions are shown for various values of the parameter $D(t)$. © Royal Astronomical Society • Provided by the NASA Astrophysics Data System
in real space is given by
\[ \delta = \frac{\Delta q}{\Delta s} - 1 = \frac{1}{f}. \] (3.4)

Thus in an \( \Omega = 1 \) universe, redshift caustics should form for \( \delta \rho/\rho \sim 1 \). Since it is expected that perturbations should break away from the general expansion and turn around for \( \delta \rho/\rho \sim 1 \), perhaps this is not surprising.

Although this argument is based on a one-dimensional system, the result should hold for any number of dimensions. Gravitational collapse is dominated by motion in one direction (Lin, Mestel & Shu 1965; Zel'dovich 1970). Moreover it is just those structures whose collapse is along the line-of-sight which will give rise to redshift caustics. It is possible to check this assertion using the simulations shown above. Since particles are initially distributed uniformly in comoving space, \( \delta(x, t) \) may be expressed as
\[ \delta(x, t) = \frac{1}{|\partial x_\alpha/\partial q_\beta|} - 1, \] (3.5)
where \( |\partial x_\alpha/\partial q_\beta| \) is the Jacobian of the transformation from \( q \) to \( x \). A similar expression may be written down for \( \delta(x, t) \).

The condition for a redshift caustic to occur is that \( \delta(x, t) = \infty \), or alternatively that \( |\partial x_\alpha/\partial q_\beta| = 0 \). This expression immediately shows that caustics will in general be sheet-like in three-dimensional space; this equation represents one constraint and lowers the dimensionality by one. It also allows us to prove the assertion above. Figs 4(a)–(d) show the frequency distribution of the value of the Jacobian, \( |\partial x_\alpha/\partial q_\beta| \),

Figure 2. The evolved particle positions for the simulation shown in Fig. 1, but this time in redshift space. The line-of-sight used to determine the redshift projection is indicated. Note that structures are much more enhanced in R and in X.

Figure 3. As for Fig. 2, this shows the particle in redshift space, but now projected down different lines-of-sight. All the figures correspond to the same value of \( D(t) = 0.4 \).

Figure 4. The frequency distribution of the Jacobian, \( |\partial x_\alpha/\partial q_\beta| \), for those points lying on a caustic in R, defined by the constraint \( |\partial x_\alpha/\partial q_\beta| = 0 \). The distributions were determined from the same realization that produced Figs 1–3. If redshift caustics are expected for \( \delta \rho/\rho \sim 1/f \) in real space, the Jacobian for these points should take on a value of \( \sqrt{f/(1 + f)} = 0.5 \) in this case, since \( \Omega = 1 \).
for those points lying on a redshift caustic. If equation (3.4) is correct, then for those points on a caustic in redshift space, $|\delta x_j/\partial q_j| = f(1+f) = 0.5$ for $\Omega = 1$. These figures clearly show that redshift caustics occur while $\delta \rho/\rho < f$ in real space. It should be emphasized that these results hold independently of the initial power spectrum.

The results presented so far indicate that redshift caustics occur for relatively low values of $\delta \rho/\rho$. What does this imply for structures in redshift space within the hierarchical scenario? For the moment, let us ignore the effects of scales which have already entered the non-linear regime, $\delta > 1$. In this case, the above results indicate that caustics will inevitably occur on the scale just entering the non-linear regime.

The effects of smaller scales must now be accounted for. These scales will already have collapsed, or will be in the process of collapsing. We can regard them as causing noise in the peculiar velocity field; their action will be to broaden the caustics. Unfortunately it is not simple to estimate the magnitude of the broadening. It is entirely possible that the caustics will be so smeared out as to be totally unrecognizable. However, if this were true then no sharp features would exist in redshift space. Since galaxy redshift surveys do show sharp features, broadening must be unimportant if these are to be explained by a hierarchical model. Moreover, Sandage & Tammann (1985) found that the galaxy flow is remarkably cold. Their results indicate an rms velocity dispersion of only 50 km s$^{-1}$ around the local large-scale velocity flow. This should be compared with bulk flows of around 500 km s$^{-1}$ (Dressler et al. 1987).

Given that broadening is unimportant, we may conclude that redshift caustics should occur on the scale entering the non-linear regime. These structures should be sheet-like and will tend to form at 90° to the line-of-sight.

### 4 Correlation Functions in Redshift Space

As the simulations in the previous section showed, redshift space is not isotropic. Even given that the distribution of galaxies in real space is isotropic, specifying a viewing direction introduces anisotropy. Moreover the anisotropy is a direct result of galaxy motion; studying the anisotropy will yield dynamical information.

The galaxy–galaxy correlation function, $\xi_{gg}(r)$, is frequently used to characterize the galaxy distribution in real space (e.g. Peebles 1980 and references therein). Because of the anisotropy discussed above, a variety of two-point correlation functions can be defined in redshift space. It will be argued that one of these, $\xi_{g}(\vec{x}_1, \vec{x}_2)$, introduced by Peebles (1980 section 76) is a good measure of this anisotropy. It contains important information about galaxy motions.

#### 4.1 Overdensities

As a prelude to calculating these correlation functions, the fractional overdensity in redshift space must be evaluated. In Section 3 it was shown that, according to the Zel'dovich (1970) approximation, $\delta(x, \ell)$ is given by

$$\delta_{\ell}(x, \ell) = \frac{1}{|\delta \chi_j/\partial q_j|} - 1,$$

where $|\delta \chi_j/\partial q_j|$ is the Jacobian of the transformation (3.1) from initial (Lagrangian) coordinates to final (Eulerian) coordinates. In component form, equation (3.1) can be written

$$\delta_{\ell}(x, \ell) = \delta_{x_j} - D(t)[\Psi_{x_j} + f\Psi_{x_k} n_k n_j],$$

where $\delta_{x_j}$ is the Kronecker delta. In general it is not possible to simplify this expression any further. It is possible, however, to calculate the Jacobian in the linear regime. Using the approximation

$$\det \begin{pmatrix} 1 & N_{ij} \end{pmatrix} = 1 + N_{ij} \quad \text{if} \quad |N_{ij}| \ll 1,$$

with equation (4.3) in (4.1) yields

$$\delta_{\ell}(x, \ell, n_k) = D(t)[\Psi_{x_j} + f\Psi_{x_k} n_k n_j].$$

The dependence of $\delta_{\ell}$ on $n_k$ has been acknowledged explicitly in this equation. Strictly speaking we have only evaluated $\delta$ at $q$ rather than $\chi$. However, it is straightforward to show that to first order $\delta(x, \ell, n) = \delta(q, \ell, n)$. Using equation (2.6) we find

$$\Psi_{x_j} = \sum_{k=0}^{\infty} \Delta_k e^{i k x},$$

and

$$\Psi_{x_j} \cdot n_k e^{i k x} = \sum_{k=0}^{\infty} \Delta_k \cos^2 \phi_k e^{i k x}.$$
over the directions of $\hat{x}$ and $\mathbf{n}$ independently. The result is

$$\xi_\perp(\mathbf{x}) = (1 + 2/3 f + 1/5 f^2) D(\mathbf{r})^2 \sum_{k \neq 0} |\Delta_k|^2 \frac{\sin k\hat{x}}{k\hat{x}} \xi(\hat{x}),$$

(4.10)

where $\xi(\hat{x})$ is the spatial correlation function according to linear theory. The function $\xi_\perp$ was first calculated by Kaiser (1987).

One point which Kaiser raised but did not clear up is related to a model of the streaming velocities due to Peebles (1978, section 76). Peebles calculated the mean streaming velocity towards a galaxy, at a distance $r$ from it, $\bar{v}(\mathbf{r})$. He modelled the velocity towards a galaxy, at a point $\mathbf{r}$ from a galaxy as $(\bar{v}(\mathbf{r})\hat{r})$, convolved with a separation independent velocity dispersion to represent random velocities. $\hat{r}$ is the unit vector in the direction $\mathbf{r}$. When Kaiser used this model for the velocity field to calculate $\xi_\perp$ he reproduced the first two terms in equation (4.10) but not the factor $1/5 f^2$.

The Peebles model is incorrect for the following reason. At any particular distance $r$ from a galaxy, there will be a range of streaming velocities towards it, even if there is no component due to random velocities. There will be a dispersion about the mean just due to the non-uniform nature of the velocity field. The effective velocity dispersion will actually increase with separation because the velocity field is correlated over small distances.

To illustrate this, let us calculate the mean-square dispersion of the component of the streaming velocity along the line-of-sight at a distance $r$ from a galaxy. Using the formalism already set out above, this is straightforward in the linear approximation. We may write this as

$$\Sigma(r) = \frac{\langle \rho(\mathbf{x}) \rho(\mathbf{x} + r) \rangle \langle [v(\mathbf{x} + r) - v(\mathbf{x}) - \bar{v}(\mathbf{r}) \hat{r}] \cdot \hat{r}^2 \rangle}{\langle \rho(\mathbf{x}) \rho(\mathbf{x} + r) \rangle},$$

(4.11)

where $v(\mathbf{x})$ is the velocity at coordinate $\mathbf{x}$. Retaining only second order terms, this becomes

$$\Sigma(r) = 2 \langle [v(\mathbf{x}) \cdot \hat{r}]^2 \rangle - 2 \langle v(\mathbf{x}) \cdot \hat{r} [v(\mathbf{x} + r) \cdot \hat{r}] \rangle.$$

(4.12)

We may calculate the second term using equation (2.5). This gives

$$\langle [v(\mathbf{x}) \cdot \hat{r}] [v(\mathbf{x} + r) \cdot \hat{r}] \rangle = D^2(t) \sum_{k \neq 0} |\Delta_k|^2 \frac{\cos^2 \phi_k e^{-ik \cdot \hat{r}}}{|k|^2}. $$

(4.13)

Here $\phi_k$ is the angle between $\hat{r}$ and $k$.

Averaging over the direction of $\hat{r}$ results in

$$\langle [v(\mathbf{x}) \cdot \hat{r}] [v(\mathbf{x} + r) \cdot \hat{r}] \rangle = D^2(t) \sum_{k \neq 0} |\Delta_k|^2 |L_0(kr)|,$$

(4.14)

where

$$L_0(\chi) = \frac{\sin \chi}{\chi} + \frac{2v\cos \chi}{\chi^2} - \frac{2v(2v-1)}{\chi^2} L_{-1}(\chi) \quad \text{with}$$

$$L_0(\chi) = \frac{\sin \chi}{\chi} .$$

(4.15)

The first term in equation (4.12) may be obtained from equation (4.14) by noting that $L_1(0) = 1/3$:

$$\Sigma_r = \frac{1}{3} D^2(t) \sum_{k \neq 0} |\Delta_k|^2 |k|^2 .$$

(4.17)

$\Sigma_r$ represents the mean-square dispersion of one component of the peculiar velocity field – perhaps unsurprisingly it is one-third of the mean-square amplitude of the entire field.

Fig. 5a–(d) shows the calculated velocity dispersions for power spectra of the form

$$P(k) \propto k^n \exp(-2k^2/k_q^2).$$

(4.18)

The position of the high-frequency cut-off sets the scale of the structure in the field: $\chi_q \propto 2\pi/k_q$. These figures clearly show that the velocity dispersion increases with $r$. The reason is that the velocity field is correlated on scales smaller than $\chi_q$. The amplitude eventually settles down to $2 \Sigma_r$ since regions far from each other are completely uncorrelated.

4.4 Line-of-sight correlation function

In some instances, the lag separation $\mathbf{x}$ may be constrained to lie along the line-of-sight. This is the case for ‘pencil beam’ galaxy surveys (e.g. Kirshner et al. 1983) and for studies of quasar absorption line systems (e.g. Sargent et al. 1980). To compare the predictions of linear theory with these results, we should set $\hat{x}$ parallel to $\mathbf{n}$ in equation (4.9) and average over the direction. It is straightforward to show that

$$\xi_\parallel(\mathbf{x}) = D^2(t) \sum_{k \neq 0} |\Delta_k|^2 [L_0(k\hat{x}) + 2fL_1(k\hat{x}) + f^2L_2(k\hat{x})].$$

(4.19)
The $L_n$ were defined above. The expression may be reduced to the form obtained by Kaiser (1987).

In his paper, Kaiser calculated $\xi_i(\delta)$ for a series of power spectra of the form (4.18). It would be pointless to repeat the exercise here.† However, Kaiser continues and argues that for $P(k) \propto k^n$, both $\xi_i(\delta)$ and $\xi_i(r)$ should scale as $r^{-(n+3)}$. Thus, he argued that the ratio $\xi_i/\xi$ should be a constant, just depending on the value of $n$. The problem is that to calculate the amplitude ratio it is necessary to introduce a high-frequency cut-off into the power spectrum. It is clear from the graphs in Kaiser’s paper that $\xi_i$ is not simply proportional to $\xi$. However, Kaiser maintains (private communication) that this should be the case for very large scales. His calculations show that the ratio is not always positive. For $n = -1.4$, it is in fact zero. Kaiser suggested that this may be the reason that Ly-z systems appear uncorrelated.

It should be pointed out that Kaiser’s results were determined only for one specific form of the cut-off. Previous studies (e.g. Lumsden, Heavens & Peacock 1989) have shown that the details of the behaviour of the correlation function do depend on the exact nature of the low-pass filter used; we have no idea about the shape of the cut-off in nature, or even if such a cut-off is a good model for the formation of any class of object. It would be most unwise to trust any results which have not been shown to be independent of the form of the cut-off. This is particularly important in this case since there are a number of bumps and wiggles evident in Kaiser’s fig. 2 – this is no more than ‘ringing’. Moreover, Kaiser does not make it clear exactly how he calculated the amplitude ratio. Kaiser’s suggestion about the lack of correlation in the Ly-z systems is somewhat of an overinterpretation. Again it would be most unwise to take this suggestion too seriously.

### 4.5 The complete redshift correlation function

This function, $\xi_i^{\nu}$, was introduced by Davis & Peebles (1983) and applied to the CfA redshift survey. It acknowledges the anisotropy of redshift space explicitly by separating $\xi_i$ into the components $\xi_i$ and $\xi_i$ parallel and perpendicular to $\nu$, respectively. Furthermore, since $\xi$ is symmetric around the viewing direction, it is possible to average over the direction of $\xi_i$ without loss of information. We may calculate $\xi_i^{\nu}(\hat{\nu}, x_i)$ from equation (4.9) by allowing the directions of $\nu$ and $\xi_i$ to vary, but in such a way as to keep the angle between them constant. In Peebles’ notation, $\xi_i$ and $\xi_i$ are $\sigma$ and $\pi$, respectively.

For a particular value of $k$, expressions of the form

$$I_i(k\delta k, k\delta x_i) = \langle e^{ik\cdot\delta}\cos^2\phi_i \rangle$$

(4.20)

must be calculated. Here $\langle \ldots \rangle$ implies taking the appropriate average, which is equivalent to calculating the angular cross-correlation of $e^{ik\cdot\delta}$ and $\cos^2\phi_i$. The details are quite complicated and the calculation of the $I_i$ has been relegated to Appendix A.

†Except to point out that Kaiser’s fig. 2 is not quite right. Since $\xi_0(0) = \xi_i(0)$, the value at the origin for $\xi_i$ is easily calculated to be $28/15 \xi_0(0)$ for $\Omega = 1$. Three of his plots show this behaviour at the origin, but one does not.

With the above definition, we have the result:

$$\xi_i^{\nu}(\hat{\nu}, x_i) = D_i^{\nu}(t) \sum_{k=0}^{\infty} \Delta_i^{\nu}(k\delta x_i, k\delta x_i) + f_1^{\nu}(k\delta x_i, k\delta x_i),$$

(4.21)

It is possible to reduce this expression to the result obtained by Lilje & Efstathiou (1989).

Fig. 6(a)–(d) shows a selection of $\xi_i^{\nu}$ calculated for power spectra of the form (4.18). $\Omega$ was taken to be one for all these calculations. The feature most obvious from these figures is that $\xi_i^{\nu}$ is compressed along the line-of-sight, independent of the power spectra. This result holds no matter what the value of $\Omega$, which just affects the degree of compression. This is hardly surprising, given that Fig. 3(a)–(d) shows that structures tend to form perpendicularly to $\nu$. This result contrasts strongly with the observed form of $\xi_i^{\nu}$; both Davis & Peebles (1983) and Bean et al. (1983) find that $\xi_i^{\nu}$ is elongated along the line-of-sight. This is presumably due to the random motion of galaxies within dense regions – the ‘finger of God’ (Jackson 1972). Whatever the cause however, we may conclude that linear perturbation theory is quite inappropriate for analysing the distribution of galaxies in redshift space.

This argument is extremely interesting. It calls into question a result obtained by Lilje & Efstathiou (1989). These authors calculated $\xi_0^{\nu}(r)$ for the CDM model using numerical simulations. They found that even though $\xi_0^{\nu}$ was elongated along the line-of-sight out to large radii in excess of $10 \ h^{-1} \ Mpc$, $\xi_0^{\nu}(r)$ agreed remarkably well with the

![Figure 6. The complete redshift correlation functions according to linear theory. The power spectra were taken to be of the form $P(k) \propto k^n \exp(-2k^2/k_c^2)$, and normalized so that the field had unit variance. $\Omega$ was set to unity for all the figures. The value of each correlation function is therefore 28/15 at the origin. The contour closest to the origin is the highest, with a value of 10/6. The contours heights decrease outwards from there in units of $1/6$. $x_i$ and $x_i$ are plotted in units of $x_i = 2\pi/k_c$, which sets the scale of structure in the field. Note that all the functions are compressed along the line-of-sight, independently of the initial power spectrum.](https://academic.oup.com/mnras/article-abstract/242/3/428/1107047)
predictions of linear theory. This is somewhat surprising since \( \xi_{\phi}(r) \) is no more than \( \xi_{\phi}(\hat{x}_1, \hat{x}_i) \) averaged over the direction of \( \hat{x} \). If \( \xi_{\phi}(\hat{x}_1, \hat{x}_i) \) is not well described by linear theory, it is unlikely that \( \xi_{\phi}(r) \) is either.

This discrepancy highlights an interesting and subtle point. Consider the mechanism that leads \( \xi_{\phi}(r) \) to have a larger amplitude than \( \xi(r) \), according to linear theory. As the simulations in Section 3 showed, the enhanced displacements in redshift space lead to a compression along the line-of-sight. This compression increases the density contrast, \( \delta \), for overdense regions. Similarly, underdense regions in real space have even lower densities in R. Since the correlation functions depend on \( \delta^2 \), this leads to an enhanced amplitude for \( \xi_{\phi}(r) \) over the correlation in real space. However, the measured \( \xi_{\phi}(\hat{x}_1, \hat{x}_i) \) indicates that the distribution is elongated along \( n \). This cannot be the mechanism that accounts for the increased amplitude of \( \xi_{\phi}(r) \). Indeed, it might be expected that since \( \xi_{\phi}(\hat{x}_1, \hat{x}_i) \) indicates an elongation, densities in redshift space should be lower than in real space and hence the amplitude of \( \xi_{\phi}(r) \) reduced. This explains why the amplitude of \( \xi_{\phi}(r) \) is reduced at very small \( r \) (Lilje & Efstathiou 1989; Couchman, private communication), but clearly does not hold as \( r \) increases.

The explanation is as follows. In real space, regions of large galaxy overdensity (i.e. clusters) have very steep profiles (Seldner & Peebles 1977; Peebles 1980; Lilje & Efstathiou 1988). The velocity dispersions in these regions are also obviously large. In redshift space the profile will effectively be a ‘smoothed’ version of the real space profile. While the density might be reduced at the centre, it is increased everywhere else.

As noted above, linear theory cannot hope to account for \( \xi_{\phi}(r) \) when \( \xi_{\phi}(\hat{x}_1, \hat{x}_i) \) is elongated along the line-of-sight. Inevitably this raises the question of why Lilje & Efstathiou (1989) found agreement between the predictions of linear theory for this function and the N-body models. It should be noted that these authors only demonstrated agreement for the CDM initial power spectrum. In view of the above results it is likely that the agreement is a peculiarity of this specific model.

5 CONCLUSIONS

The primary motivation behind this paper was to examine the relationships between real and redshift space, X and R, respectively. An obvious, though important conclusion to be drawn from the work in Section 3 is that these two spaces are not identical. As Kaiser (1987) has already emphasized, structures apparent in redshift space may not be present in real space. Indeed, the arguments presented above show that caustics in R should occur while the corresponding density contrast in real space, \( \delta \rho/\rho \), is only \( \approx 1 \). Moreover it was argued that these structures are almost inevitable within the hierarchical scenario. They will occur on the scale just entering the non-linear regime, given that random motions (due to scales which have already collapsed) do not cause significant broadening. It should be emphasized that this conclusion is valid regardless of the initial power spectrum.

Although this work has concentrated on overdense regions, streaming motions will also have an effect on the appearance of underdense regions in redshift space. The same mechanism that causes overdense regions to have an enhanced density contrast, will reduce that of underdense regions. It should be remarked, however, that while this effect can easily produce sharp features in R, it is not so adept at producing voids. Streaming motions may lower the density of a region in redshift space, but they cannot evacuate it totally. We may conclude that while this effect may aid in the production of voids, it cannot fully account for them.

As Kaiser has already noted, these effects may be responsible for some of the forms seen in galaxy redshift surveys. In particular he cites the example of a feature in the Giovanelli et al. (1986) 21-cm survey of the Perseus-Pisces chain. This structure is very sharp in redshift space and is perpendicular to the line-of-sight, two of the qualities predicted for redshift caustics in Section 3. It is, however, a linear feature. The considerations presented above show that redshift caustics should be sheet-like. While nothing would actually exclude a one-dimensional caustic, it is not likely to happen often.

The situation is almost the reverse for structures in the Cib ’slice’ redshift survey (de Lapparent et al. 1986). Here there are sharp structures which must be two-dimensional to appear in a redshift slice. The problem is in judging whether or not there is a tendency for these structures to lie at 90° to the line-of-sight. Some structures definitely do lie in this direction, and there are some which lie exactly along the line-of-sight – undoubtedly examples of the ‘Finger of God’. However, there are also structures which are not even close to lying perpendicularly to the viewing direction. Again, we must conclude that while streaming motions may be partly responsible for the structures seen in redshift surveys, they cannot totally account for them either.

It has been tacitly assumed throughout that \( \Omega = 1 \). However, it is apparent from equation (3.1) that the effect of streaming velocities on the position of particles in redshift space increases with \( \Omega \). If \( \Omega \) is less than one, the effect is less, more if \( \Omega \) is bigger than unity. If galaxy formation is biased then that effectively lowers \( \Omega \). The clustering we see is statistical rather than dynamic. Since it is galaxy motion that causes the differences between real and redshift space, decreasing the importance of dynamical clustering decreases the importance of projecting into redshift space.

The most important result of the paper concerns \( \xi_{\phi}(\hat{x}_1, \hat{x}_i) \), the complete redshift space correlation function. It was shown that in the linear approximation, this function is always compressed along the line-of-sight, \( n \). This result is independent of both the initial power spectrum and the value of \( \Omega \). Varying \( \Omega \) just alters the amount of compression. Since the measured \( \xi_{\phi}(\hat{x}_1, \hat{x}_i) \) is elongated along the line-of-sight, this shows that this statistic must be dominated by galaxies in dynamically non-linear regions. If the fractional galaxy overdensity was greater than one because of, say, biasing – statistically non-linear – then \( \xi_{\phi} \) would still be compressed along \( n \).

It was then argued that \( \xi_{\phi}(\hat{x}) \), the average of \( \xi_{\phi} \) over the direction of \( \hat{x} \), could not therefore be accounted for within linear theory. In this approximation, the enhancement of \( \xi_{\phi}(\hat{x}) \) over \( \xi(r) \), the density correlation function in real space, is due to the enhanced (positive or negative) density contrast in redshift space over real space. Since \( \xi_{\phi} \) is

\[ \Delta \text{In Peebles' notation, } \hat{x}_1 \text{ and } \hat{x}_i \text{ are } \sigma \text{ and } x, \text{ respectively.} \]
elongated along $n$, this mechanism cannot be responsible for the observed enhancement of $\xi_{\parallel}(\hat{n})$, even though Lilje & Efstathiou (1989) found that the predictions of linear theory agreed with the results of their N-body simulations. It was suggested that the reason for the enhancement was more mundane. A steep density profile in real space would effectively be smoothed in redshift space because of large galaxy motions within the region. Although this would lower the contrast at the centre, it would increase the contrast elsewhere.

This result has important implications about $\xi_{\parallel}(r)$, the two-point galaxy–galaxy correlation function. $\xi_{\parallel}(r)$ may be obtained from $\xi_{\parallel}(\hat{n})$ through a suitable projection (Davis & Peebles 1983). If $\xi_{\parallel}$ is dominated by galaxies in non-linear regions, this should also be the case for $\xi_{\parallel}(r)$. One cannot appeal to the random motions of galaxies in clusters to elongate $\xi_{\parallel}$ and then deny that these galaxies have a significant effect on $\xi_{\parallel}(r)$. This is important because only when $\xi_{\parallel}(r)$ represents the results of linear theory is it directly related to the initial power spectrum. This result suggests that even on scales of up to $10^{-1}$ Mpc, $\xi_{\parallel}(r)$ may still not directly reflect the initial $P(k)$.

Further, this result suggests that the observed correlation is fundamentally dynamical in origin, rather than statistical, i.e. not due to biasing. It should be noted that the effects of biasing would still alter the magnitude of the correlation function. Introducing a threshold density for galaxy formation will effectively reduce the galaxy density outside the overdense regions. The fact that applying a threshold in N-body simulations increases the amplitude of $\xi_{\parallel}(r)$ is no guarantee that the clustering measured by this statistic is statistical.

On the face of it, there appears to be a contradiction between the elongation of $\xi_{\parallel}(\hat{x}_1, \hat{x}_i)$ along the line-of-sight and the fact that sharp structures exist in redshift space. How can random motions elongate the correlation function, but at the same time not overly broaden caustics? Perhaps the answer lies in the definition of $\xi_{\parallel}$. Since this function depends on $\langle \delta \rho / \rho \rangle^2$, a small fraction of galaxies in a region with a large overdensity can have a significant effect. This is true particularly in clusters where the density contrast can reach several hundred. $\xi_{\parallel}(\hat{x}_1, \hat{x}_i)$ may only be probing the galaxy distribution in extremely overdense regions. Indeed, McGill (1990) has shown that the known distribution of galaxies in clusters can account for the observed $\xi_{\parallel}(r)$ within observational uncertainties. The same is perhaps true for $\xi_{\parallel}(\hat{x}_1, \hat{x}_i)$.

The result that in linear theory $\xi_{\parallel}$ is compressed along the line-of-sight is particularly important because it is independent of the value of $\Omega$ and the initial power spectrum. It is frequently assumed that, because $\xi_{\parallel}(r)$ on some scale is less than one, linear theory must apply on those scales since it implies that $\delta \rho / \rho \leq 1$. However, as we have seen this is not necessarily the case. $\xi_{\parallel}$ is elongated along the line-of-sight on scales of $10^{-1}$ Mpc, suggesting non-linearity, while $\xi_{\parallel}(r)$ on those scales is significantly less than one. $\xi_{\parallel}(\hat{x}_1, \hat{x}_i)$ provides a test of linearity. Only when we observe compression of $\xi_{\parallel}(\hat{x}_1, \hat{x}_i)$ along the line-of-sight on some scale, can we even begin to expect linear behaviour.

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REFERENCES


APPENDIX A: ANGULAR CORRELATION FUNCTIONS

The purpose of the appendix is to determine an analytic expression for the functions $I_2$, which were defined in Section 4. As we shall see, the $I_2$ are two-point angular cross-correlation functions. As a first step the correlation, we will determine a general expression for such functions.

A1 Notation and definitions

Many of the formulae in this appendix concern rotations. We shall specify a particular rotation in terms of Euler angles (e.g. Rose 1957, p. 50). Let us define $\alpha$ to be the rotation angle about the original $z$-axis, $\beta$ the angle about the new $y$-axis and $\gamma$ the angle about the new $z$-axis. It will be convenient to refer to the Euler angles through the triplet $\omega = (\alpha, \beta, \gamma)$.
denotes the integral of $X(\omega)$ over the space of all rotations.

Continuing in the same vein, it will frequently be necessary to specify a vector in terms of its spherical polar coordinates. If the vector is $\mathbf{r}$ then we shall denote them: $(r, \theta, \phi)$. As a convenient shorthand, the angular coordinates will be referred to through the doublet $\Omega = (\theta, \phi)$.

To conclude this section, we will consider the transformation of a vector, $\mathbf{r}$ under a rotation, $\omega$. In general, $\mathbf{r}$ will be mapped to a new vector, $\mathbf{r}'$. We may write $\mathbf{r}'$ in terms of the rotation operator, $\mathbf{R}(\omega)$:

$$\mathbf{r}' = \mathbf{R}(\omega) \mathbf{r}. \quad (A1)$$

The following calculations depend on the properties of spherical harmonics, $Y_{lm}(\Omega)$. Those needed are set out below. Derivations of all of these properties can be found in The Elementary Theory of Angular Momentum by Rose (1957).

### A1.1 Property 1: value at the poles

The value of $Y_{lm}(0,0)$ is given by (Rose 1957, p. 60)

$$Y_{lm}(0,0) = \left(\frac{2l+1}{4\pi}\right)^{1/2} \delta_{m,0}, \quad (A2)$$

where $\delta_{ij}$ is the Kronecker delta.

### A1.2 Property 2: the Addition Theorem

It may be shown that (Rose 1957, p. 60)

$$\sum_{m} Y_{lm}(\Omega_q) \ Y_{lm}^{*}(\Omega_r) \left(\frac{2l+1}{4\pi}\right) P_l(\cos \psi). \quad (A3)$$

Here, $P_l$ is the Legendre polynomial of degree $l$. $\cos \psi$ is the angle between $\mathbf{q}$ and $\mathbf{r}$.

### A1.3 Property 3: behaviour under a rotation

Under a rotation, $\omega$, the spherical harmonics transform according to (Rose 1957, p. 52)

$$Y_{lm}(\Omega_\omega) = \sum_{m'} R^{l}_{mm'}(\omega) \ Y_{lm'}(\Omega_\omega), \quad (A4)$$

$R^{l}_{mm'}(\omega)$ are the rotation matrices for the rotation $\omega$.

### A1.4 Property 4: orthogonality of the rotation matrices

It may be shown that (Rose 1957, p. 75)

$$\int_{\Omega} R^{l}_{mm'}(\omega) R^{l*}_{MM'}(\omega) \ d\omega = \frac{8\pi^2}{2l+1} \delta_{l,L} \delta_{m,M} \delta_{m',M'}. \quad (A5)$$

### A2 The two-point angular cross-correlation function

We wish to calculate the two-point angular cross-correlation function, $\kappa$, of the functions $f(\mathbf{q})$ and $g(\mathbf{r})$. $\kappa$ may be defined as

$$\kappa = \langle f(\mathbf{q}) g^*(\mathbf{r}) \rangle, \quad (A6)$$

where $\langle \ldots \rangle$ implies averaging over all directions of the vectors $\mathbf{q}$ and $\mathbf{r}$ such that the angle between them, $\psi$, remains constant. We may evaluate this average in terms of an integral over rotations. Using the rotation operator, $\mathbf{R}(\omega)$,

$$\kappa = \frac{1}{8\pi^2} \int_{\Omega} \int_{0}^{2\pi} \int_{0}^{\pi} \sin \beta \ d\beta \int_{0}^{2\pi} \ d\gamma \ f(\mathbf{R}(\omega) \mathbf{q}) \ g^*(\mathbf{R}(\omega) \mathbf{r}) \ d\omega. \quad (A7)$$

In this expression, $\mathbf{q}$ and $\mathbf{r}$ represent the initial positions of the vectors, i.e., when $\omega = 0$. The normalizing term is the value of

$$\int_{\Omega} d\omega = \int_{0}^{2\pi} \ d\alpha \int_{0}^{\pi} \sin \beta \ d\beta \int_{0}^{2\pi} \ d\gamma = 8\pi^2. \quad (A8)$$

To proceed further, let us expand $f$ in terms of spherical harmonics. Defining

$$f(\mathbf{q}') = \sum_{l,m} F_{lm}(\mathbf{q}) \ Y_{lm}(\Omega_{q'}), \quad (A9)$$

We have written $F_{lm}$ as a function of $q$ since $|q| = |q'|$. Under rotations, spherical harmonics transform according to property 3. Thus we may write

$$Y_{lm}(\Omega_q) = \sum_{m} R^{\ d}_{mm'}(\omega) \ Y_{lm'}(\Omega_q). \quad (A10)$$

Substituting this expression into (A9),

$$f(\mathbf{q}') = \sum_{l,m,m'} F_{lm}(\mathbf{q}) R^{\ d}_{mm'}(\omega) Y_{lm'}(\Omega_q). \quad (A11)$$

A similar expression may be formed for $g^*(\mathbf{r}')$.

Substituting the expansions of $f$ and $g$ into equation (A7),

$$\kappa = \frac{1}{8\pi^2} \sum_{l,m,m', L,M} F_{lm}(\mathbf{q}) G_{LM}(\mathbf{r}) \ Y_{lm}(\Omega_q) \ Y_{LM}^{*}(\Omega_r) \times \int_{\Omega} R^{\ d}_{mm'}(\omega) R^{\ d*}_{MM'}(\omega) \ d\omega. \quad (A12)$$

Using property 4, this expression reduces to

$$\kappa = \sum_{l,m} F_{lm}(\mathbf{q}) G_{lm}^{*}(\mathbf{r}) \ Y_{lm}(\Omega_q) \ Y_{lm}^{*}(\Omega_r). \quad (A13)$$

We may now use the addition theorem for spherical harmonics, property 2. This leaves us with a final expression for $\kappa$:

$$\kappa(q, r, \cos \psi) = \frac{1}{4\pi} \sum_{l,m} F_{lm}(\mathbf{q}) G_{lm}^{*}(\mathbf{r}) P_l(\cos \psi), \quad (A14)$$

where $\cos \psi$ is the angle between the two vectors.

### A3 The calculation of the $I_{\nu}$

It is evident from the definitions of $\kappa$ and $I_{\nu}$ equations (A6) and (4.20), that the $I_{\nu}$ are angular cross-correlation functions.
We may identify
\[ f(q) = e^{ik \cdot q} \]  
(A15)
and
\[ g(r) = \cos^2 \theta, \]  
(A16)
where \( \theta \) is the angle between \( r \) and \( k \). To apply the above results, it only remains to determine the expansions of \( f \) and \( g \) in terms of spherical harmonics.

The expansion of \( e^{ik \cdot q} \) is given by (e.g. Rose 1957, p. 99):
\[ e^{ik \cdot q} = 4\pi \sum_{l,m} i^{-l} j_l(kq) Y^*_{lm}(\Omega) Y_{lm}(\Omega). \]  
(A17)
\( j_l \) is the spherical Bessel function of order \( l \). Since we are averaging over directions, we may choose \( k \) to lie along the \( z \)-axis: \( \Omega_k = (0,0) \). Using property 1,
\[ F_{lm}(q) = \{(2l+4\pi)^{1/2} i^{-l} j_l(kq)\} \delta_{m,0}. \]  
(A18)

The coefficients \( G_{lm} \) in the expansion of \( \cos^2 \psi \) are zero unless \( l \) is even and no greater than \( 2v \). If these conditions are satisfied,
\[ G_{lm} = \{(2l+1)\pi^{1/2} \frac{2^{l+1} \Gamma(l+1/2)\Gamma(\nu+l/2)}{(2\nu+l+1)\Gamma(\nu-l/2)}\} \delta_{m,0}. \]  
(A19)

To take account of the odd/even nature of \( G_{lm} \), it will be convenient to define \( L \) such that \( l = 2L \).

Substituting these expressions into equation (A14),
\[ \varsigma(q, r, \cos \psi) = \sum_{L=0}^{v} (4L+1)(-1)^L j_{2L}(kq) \]  
(A20)

\[ \times \frac{2^{2L}(2v)!/(v+L)!}{(2v+2L+1)(v-L)!} P_{2L}(\cos \psi). \]

It should be noted that this expression is independent of \( r \). In terms of the original variables and replacing \( L \) and \( l \), we may write
\[ I_{l}(k\tilde{r}_l, k\tilde{q}_l) = \sum_{L=0}^{v} (4L+1)(-1)^L j_{2L}(k\tilde{q}_l) \]  
(A21)

\[ \times \frac{2^{2L}(2v)!/(v+L)!}{(2v+2L+1)(v-L)!} P_{2L}(\cos \psi). \]