Models of radio galaxies with tangled magnetic fields – I. Calculation of magnetic field transport, Stokes parameters and synchrotron losses

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SUMMARY
This is the first of two papers describing and discussing numerical simulations of radio sources in which a randomly tangled magnetic field is introduced in the jet, and is then convected passively with the fluid flow. The intrinsic polarization distribution of the source model is therefore determined solely by the distortion of the field by compression and shear in the flow. Such models show up a number of important features which do not appear in models with ordered poloidal, toroidal or helical fields. Our models incorporate synchrotron loss; apart from its intrinsic interest, a small amount of synchrotron loss has the effect of removing the radio emission from small regions where the distortion produces unrealistically high fields. This paper is devoted to calculating the distortion of a frozen-in field by the fluid flow and the synchrotron losses suffered by the relativistic electrons injected with the fluid, and showing how the calculation of the resulting radio emission can be made manageable. The second paper will exhibit and discuss simulations made with various input parameters.

1 INTRODUCTION
The hotspots and lobes of extragalactic radio sources are due to the action of a pair of jets (of uncertain composition and unknown but probably relativistic speed) emerging from the active nucleus in opposite directions. Many attempts have been made to model the radio sources, from crude but analytically tractable models in the early 1970s to more ambitious numerical simulations in recent years. The modelling may usefully be divided into two stages:

(i) the simulation of the fluid dynamics of a (given) jet interacting with the ambient gas, and
(ii) the derivation of the synchrotron emissivity, and hence the appearance of the source as observed in projection on the sky.

It cannot be too strongly emphasized that the results of such modelling depend at least as strongly on the physical assumptions which are inevitably made as on the sophistication of the numerical methods used. In stage (i), most simulations assume ordinary non-relativistic hydrodynamics and a jet ejected with constant speed and direction into ambient gas of uniform density, producing a flow axisymmetric about the axis of the jet. Furthermore, the coarseness of the grid in even the most elaborate simulations causes the effective Reynolds number to be fairly small, and allows a good deal of numerical diffusion of both mass and momentum, as well as suppressing instabilities on small scales. All of these restrictions apply to the fluid simulations used in the present paper, which were made with the flux-corrected FLIC code previously used by Williams & Gull (1984) and Williams (1985). Some simulations have been made which are more realistic in one of these respects: for example, relativistic jets (Wilson 1987), jets of varying power (Wilson 1984), ambient gas whose density decreases outwards (Williams 1985) in a way resembling that indicated by X-ray observations of giant elliptical galaxies and associated clusters, jets with dynamically significant magnetic fields (Clarke, Norman & Burns 1986), and a first attempt at three-dimensional simulations of sources lacking axial symmetry (Williams & Gull 1984, 1985). Mixing of fluid initially in the jet with initially ambient fluid can be prevented by LeBlanc interface tracking, which in effect inserts a frictionless plastic film between the two fluids (see Norman et al. 1982).

Stage (ii) itself consists of several distinct pieces of physics: we must decide how to estimate the magnetic field and the distribution of relativistic electrons, and must then compute the radio image due to the synchrotron radiation when the source is viewed from some direction.

The principal innovation in the present work lies in the development of the magnetic field. We assume that an isotropically tangled magnetic field is injected and, from that
point on, the field is convected passively with the fluid, as a frozen-in flux. The original motive for these computations was to investigate whether one can account for the polarization properties of sources without any particular assumptions about the seed field, simply by the distortion of any initial field by the fluid flow. It is fairly obvious that quite moderate compression or shear will align the field enough to cause highly polarized synchrotron emissivity from a small volume element; what is less obvious (and therefore called for a simulation) is whether the polarization averaged along a line-of-sight is reduced too little, too much, or by about the right factor, by the rather complicated geometry of the flow pattern. It will become clear in Paper II that the simulations drew attention to other important processes. The mechanics of computing the distortion of an initially 'random' field are straightforward in principle and are set out in Section 2, together with ways we used to deal with some problems that arise in practice.

Regarding the fast particles, we assume that they are generated near the active nucleus or at a low level in the jet, that there is no diffusion of particles, and that the particle momentum distribution remains isotropic. Thus the fast particles are convected passively with the 'ordinary' fluid, and the particle energies change like those in an adiabatically expanding or contracting relativistic gas, i.e. \( \propto \) (volume of fluid element)\(^{-1/3}\). The momentum distribution is probably kept isotropic by resonant pitch angle scattering off Alfvén waves; this is also a process likely to inhibit particle diffusion. The location of the particle acceleration processes, and indeed that of magnetic field injection is not crucial for our purposes, and we have generally injected both at the base of the jet, i.e. with the jet material as it enters our computing grid. If there is particle acceleration at the strong shock(s) where the jet becomes a hotspot, then the radio emission of the jet is still fainter relative to the rest of the source than is shown in our simulations. The magnetic field is distorted somewhat by the flow along the jet, but the results show that this distortion is overwhelmed by the distortion occurring further downstream.

It turns out that the straightforward application of magnetic field transport produces radio sources very different from earlier models, in which the magnetic field strength was estimated simply by assuming equipartition with fast particle energy, or, even more crudely, by taking synchrotron emissivity to be \( \propto (\text{pressure})^2 \). The reason is that there are regions of very strong shear, where the magnetic field becomes much stronger than it is elsewhere, and these few places dominate the radio emission. Evidently this is a real physical phenomenon, but in real sources its effect on field amplification will be limited by the reaction of magnetic forces on the fluid flow and also by field-line reconnection, and its effect on radio emission will be further limited by synchrotron loss in strong fields. These matters are discussed further in the second paper. In Section 3 we describe algorithms for computing synchrotron losses of the relativistic electrons as they move through the source, and the Stokes parameters of the resulting radio emission. The effect of synchrotron losses on the radio image is of interest in itself, but also has a particular function in the present work, for the exaggerated effect of a few points with extremely (and unphysically) strongly amplified fields can be eliminated by taking synchrotron loss into account.

2 TRANSPORT OF MAGNETIC FIELDS AND RELATIVISTIC ELECTRONS

This section and Section 3 set out successive stages in calculating the synchrotron emission from a model source of the kind outlined in the Introduction:

(i) following a fluid element and its distortion, essentially by introducing marker particles into the flow simulation,

(ii) calculating how the magnetic field frozen into a small element of fluid changes as the fluid element is distorted by the flow,

(iii) calculating the Stokes parameters of the radio emission from that element as seen from a given direction, and averaging them over all directions of injected field, and

(iv) computing the image as seen from the given direction by numerical integration along lines-of-sight through the source.

Stage (iii) will be done here for the relatively easy case of a distribution of particle energies \( \gamma m_e c^2 \) of the standard form

\[
dn = K \gamma^{-\alpha} dy, \tag{2.1}
\]

with \( \alpha = 1 \), where \( dn \) is the number of electrons with Lorentz factors \( \gamma \) in the range \( d\gamma \) and \( K \) is a constant per unit volume. The initial value of \( K \), where the relativistic electrons are injected, will be called \( K_0 \). A much more complex calculation is needed, e.g. when the spectrum is modified by synchrotron losses, and that is postponed until Section 3.

2.1 Tracing the distortion of fluid elements

The fluid simulations are produced by a two-dimensional FLIC code in cylindrical polar coordinates \((z, r, \varphi)\) with \( z \) in the direction of the jet, using flux-corrected transport (FCT: Boris & Book 1973; Book, Boris & Hain 1975) and time-splitting (Strang 1968). A supersonic jet of density \( \rho_j \), pressure \( p_j \), and Mach number \( M \) is injected at \( z = 0, r = 0 \) to \( r_j \) (in grid spacings), into a uniform gas of density \( \rho_a \) and pressure \( p_a \). Boundary conditions are made reflective at \( z = 0, r > r_j \) and continuous at the other two boundaries; however, in the simulations used here the bow shock never reaches either of the continuous boundaries.

We shall assume that a tangled field, uniformly distributed in direction and too weak to affect the dynamics, is injected with the jet, and is thereafter frozen into the fluid. To trace its distortion, marker particles are injected at each \( r < r_j \) once every eight time-steps. Their trajectories are estimated using velocities interpolated to the position of the marker, say \((\Delta z, \Delta r)\), by bilinear interpolation

\[
v(\Delta z, \Delta r) = v_0(1 - \Delta z)(1 - \Delta r) + v_1\Delta z(1 - \Delta r) + v_2\Delta z\Delta r + v_3(1 - \Delta z)\Delta r \tag{2.2}
\]

in terms of the velocities \( v_0, v_1, v_2, v_3 \) at the grid points \((0, 0), (1, 0), (1, 1), (0, 1)\). The position of the marker used in formula (2.2) is taken to be that at the mid-point of its first-order step (the mid-point method of Press et al. 1986). Adaptive step sizes are used, meaning that the time subinterval is halved and the number of substeps doubled until the difference between successive estimates of a test parameter at the end of the step is less than a specified error \( \epsilon \).

This algorithm for displacing marker particles was tested...
on a sheared circular flow of constant speed 5, using the final position of the marker particle as a test parameter. Its performance on a marker placed initially at (5.5, 0) is shown in Fig. 1.

The marker particles trace the location of a particular fluid element, but the marker particles at the corners of an (initially rectangular) fluid element are not suitable for tracing the distortion of a fluid element, because sometimes a fluid 'box' is strongly sheared that its faces cross over, and the volume goes through a zero, with disastrous consequences for the estimates of field and particle densities. Therefore, distortion was computed by the following method, which depends only on velocity gradients near the marker particle — even when initially adjacent marker particles have moved far away.

The computer simulation attaches three vectors \( \mathbf{a}, \mathbf{b}, \) and \( \mathbf{c} \) to each marker particle. Initially these are a set of right-handed orthogonal unit vectors, \( \mathbf{a} \), \( \mathbf{b} \), and \( \mathbf{c} \) in the \( z \)-direction; they distort like the sides of an infinitesimal fluid element at the position of the marker particle, i.e.

\[
\mathbf{da}_i = a_i \frac{d\mathbf{v}_i}{dt} d\mathbf{x}_i dt.
\]

(2.3)

The velocity gradients are bilinearly interpolated from the (Eulerian) grid to the marker particle position at the spatial mid-point of its step

\[
\frac{d\mathbf{v}_i}{dz} = [v_i + (v_2 - v_1) \Delta r] - [v_0 + (v_3 - v_0) \Delta r]
\]  
\[
\frac{d\mathbf{v}_i}{d\mathbf{r}} = [v_3 + (v_2 - v_3) \Delta z] - [v_0 + (v_1 - v_0) \Delta z]
\]

(2.4)

in the notation of (2.2), where \( v_i, v_0, \ldots \) refer to either the \( v_i \) or the \( v_0 \) component. Again, adaptive step size was used, with error limit \( \varepsilon = 0.001 \) in the two parameters which, as we shall find in Section 3, control the magnetic field strength

Shear factor = \( (a^2 + b^2 + c^2)/3 \)

Volume = \( \mathbf{c} \cdot \mathbf{a} \times \mathbf{b} \).

(2.5)

This algorithm was also tested on a number of curved and sheared flows and on a shock front, and found to be limited only by the precision of the interpolated velocity gradient.

2.2 Transport of a uniform magnetic field and of relativistic electrons

Consider a cube of plasma with frozen-in field \( \mathbf{B}_0 = B_{0i} \mathbf{i} + B_{0j} \mathbf{j} + B_{0k} \mathbf{k} \), which distorts into the parallelepiped with edges \( \mathbf{a}, \mathbf{b}, \) and \( \mathbf{c} \) containing field \( \mathbf{B} = b_0 \mathbf{a} + b_1 \mathbf{b} + b_2 \mathbf{c} \). Conservation of flux through each face requires that

\[
B_{0i} = B_{0} \cdot \mathbf{j} \times \mathbf{k} = B_{0} \cdot \mathbf{b} \times \mathbf{c} = b_1 a_2 b_3 c_1 + b_2 a_1 b_3 c_2 + b_3 a_1 b_2 c_3
\]

etc., and therefore

\[
\mathbf{B} = (a \cdot b \times c)^{-1} (B_{0i} \mathbf{a} + B_{0j} \mathbf{b} + B_{0k} \mathbf{c})
\]

(2.6)

At the same time the volume containing the relativistic particles expands by a factor of \( R^3 = a \cdot b \times c \). Under our assumption of continuing isotropy of the particle momentum distribution, and an energy distribution of the form (2.1), the \( \mathbf{d} \mathbf{n} \) particles with Lorentz factors \( \gamma \) to \( \gamma_0 + d\gamma_0 \) become

\[
\mathbf{R} \cdot d\mathbf{n} = \mathbf{d} \mathbf{n} \text{ particles with Lorentz factors between } \gamma = \gamma_0/R \text{ and } \gamma + d\gamma = (\gamma_0 + d\gamma_0)/R,
\]

and we therefore have the standard result

\[
\mathbf{d} \mathbf{n} = R^{-3} \mathbf{d} \mathbf{n}_0 = R^{-3} K_0 (R \gamma)^{-2a-1} d(R \gamma)
\]

i.e. \( K = R^{-2a-3} K_0 \).

(2.7)

2.3 Synchrotron radiation from a grid cell

It is now convenient to set up a Cartesian coordinate system with orthogonal unit base vectors \( \mathbf{i}, \mathbf{m}, \) and \( \mathbf{n} \), where \( \mathbf{n} \) points at the observer and the other two are fixed directions in the sky (e.g. RA and Dec). Standard synchrotron radiation theory (e.g. Rybicki & Lightman 1979) shows that the Stokes parameters per unit volume of emitter are

\[
I = CK \left( B_i^2 + B_m^2 \right)^{(\alpha - 1)/2}
\]

(2.8a)

\[
Q = \Delta_0 CK \left( B_i^2 + B_m^2 \right)^{(\alpha - 1)/2} \left( B_i^2 - B_m^2 \right)
\]

(2.8b)

\[
U = \Delta_0 CK \left( B_i^2 + B_m^2 \right)^{(\alpha - 1)/2} 2B_i B_m
\]

(2.8c)

where \( C \) is a universal constant [times (distance of source)]^{-2} if we define Stokes parameters properly in jansky and \( \Delta_0 = (3a + 3)(3a + 5)/0.75 \) for \( a = 1 \).

We wish to average these with equal weight over all possible directions of the injected magnetic field \( B_0 \). That is now easy if we assume a spectral index \( \alpha = 1 \), since then (2.8) reduces to

\[
I = CK \left( B_i^2 + B_m^2 \right)
\]

(2.9a)

\[
Q = \Delta_0 CK \left( B_i^2 - B_m^2 \right)
\]

(2.9b)

\[
U = \Delta_0 CK 2B_i B_m
\]

(2.9c)

Substituting the expression (2.6) for \( \mathbf{B} \), and remembering that mean cross-products \( \langle B_{0i} B_{0j} \rangle = 0 \) while \( \langle B_{0i}^2 \rangle = \langle B_{0j}^2 \rangle = \langle B_{0k}^2 \rangle = 3B_0^2/3 \), we find that

\[
\langle I \rangle = CK \left( B_0^2/3 \right) (a_i^2 + a_m^2 + b_i^2 + b_m^2 + c_i^2 + c_m^2)
\]

(2.10a)

\[
\langle Q \rangle = \Delta_0 CK \left( B_0^2/3 \right) (a_i^2 - a_m^2 + b_i^2 - b_m^2 + c_i^2 - c_m^2)
\]

(2.10b)

\[
\langle U \rangle = \Delta_0 CK \left( B_0^2/3 \right) 2(a_i a_m + b_i b_m + c_i c_m)
\]

(2.10c)

where \( a_i = \mathbf{a} \cdot \mathbf{i}/(a \cdot b \times c) \), etc.

All that remains is to integrate these quantities along each line-of-sight.
2.4 Simulating the ratio image

In our axisymmetric model, each marker particle really represents a ring about the z-axis. Therefore, the first step is to make a three-dimensional model of the radio emissivity by distributing continual emitting points around this ring at equal intervals $\Delta \varphi$ in azimuth, $\Delta \varphi$ being chosen to make the gap between points no more than one grid cell. Each emitter is allotted Stokes parameters corresponding to the emissivities (2.10) and its current volume, i.e. a $b \times c$ times its initial volume $v_i \Delta t_{i0} r_0 \Delta \varphi$, where $\Delta t_{i0}$ is the interval between injecting marker particles. The three-dimensional array of points is projected on to an image plane normal to $n$, which is divided into square pixels equal in size to the grid cells. An image can now be made by simple binning, i.e. making each Stokes parameter of the pixel the sum of the corresponding Stokes parameters for the emitter points projected into that pixel. Such simple binning produced rather noisy images, owing to the accidents of emitter points falling just on one side or another of a bin boundary; the binned images were therefore smoothed with a Gaussian beam. A convolving beam with full width between half-maximum points of two pixels was found to reduce the spurious structure at high spatial frequencies to a satisfactorily low level. This part of the program was tested separately on a uniform cylindrical source.

3 SOURCES WITH SYNCHROTRON LOSSES

The effects of synchrotron loss on the distribution of radio emission is of obvious interest in itself, but it is also useful in the context of the results to be presented in this paper for suppressing, in a way that is not arbitrary, the otherwise very powerful emission from a few small regions of very strong field, which would often be inhibited from forming at all by the back-reaction of magnetic forces if the magnetic forces had been included in the fluid dynamics.

As already adumbrated in the introductory remarks of Section 2, and as became evident in Section 2.3, the computation of the synchrotron emissivity and its averaging over initial field directions becomes vastly more complex if the electron energy distribution is anything but a power law with index $-(2\alpha + 1) = -3$. In the presence of synchrotron losses we are forced to grasp this nettle, and we now propose to make the reader do likewise.

3.1 The particle energy distribution after a history of synchrotron and adiabatic losses

Since injection — presumably as a power-law spectrum — the relativistic electrons currently in a fluid element have suffered synchrotron loss in a magnetic field of varying strength as well as adiabatic losses (or gains), but the flow simulations containing marker particles allow us to trace the history of the magnetic field experienced by a given fluid element.

The rate of change of a given particle's Lorentz factor is given by

$$\frac{d\gamma}{dt} = \frac{2}{3} \frac{\sigma_T}{mc^2} \frac{B^2}{R} \gamma \left( \frac{dR}{dt} \right),$$

(3.1)

where $\sigma_T$ is the Thomson scattering cross-section and, as in Section (2.2), $R^3$ is the volume expansion factor. The first term on the right-hand side represents the synchrotron loss and the second term the adiabatic loss (or gain).

Dividing by $\gamma^2 R$ and rearranging we obtain

$$\frac{d\gamma}{\gamma R} = \frac{2}{3} \frac{\sigma_T}{mc^2} \left[ \frac{B^2}{R} \right] \frac{dR}{dt} dt,$$

which integrates immediately to become

$$\frac{1}{\gamma R} - \frac{1}{\gamma_0 R_0} = \frac{2}{3} \frac{\sigma_T}{mc^2} \left[ \frac{B^2}{R} \right] \frac{dR}{dt} dt = b.$$  

(3.2)

This loss parameter $b$ takes both synchrotron and adiabatic loss into account.

From (3.2) we find, since $R_0 = 1$,

$$\gamma_0 = \frac{\gamma R}{1 - byR}$$

(3.3)

and

$$d\gamma_0 = \frac{R}{(1 - byR)^2} dy,$$

hence, since the number of particles is conserved,

$$R^3 dn = R^2 \gamma_0 d\gamma_0$$

$$dn = K_0 R^{-2\alpha - 2} \gamma^{-2(\alpha + 1)/3} (1 - byR)^{2\alpha - 1} d\gamma$$

(3.4a)

at Lorentz factors up to $\gamma_{\max} = 1/bR$, and zero for $\gamma > \gamma_{\max}$. Thus the energy distribution in a given volume element is determined by the input $a$ and a single loss-parameter $b$, which is accumulated according to the formula (3.2) for each marker particle as the flow simulation proceeds. We shall find that we can reduce the computations to manageable proportions partly because, in terms of the scaled energy $Y = \gamma/\gamma_{\max}$ and a suitably scaled $K$, the energy distribution (3.4) is the same for all loss parameters,

$$dn = K_0 R^{-3} b^{2a} Y^{-2(\alpha + 1)/3} (1 - Y)^{2\alpha - 1} dY$$

(3.4b)

and hence the synchrotron spectrum is also the same in terms of a suitably scaled frequency.

3.2 The probability distribution of magnetic field in a volume element

We shall need the probability distribution function (pdf) of the magnetic field $B$, not just its mean square (which sufficed in the special case $b = 0$, $a = 1$ treated in Section 2), and to find it we must make some assumption about the pdf of the magnitude of the injected field. Giving $B_0$ fixed magnitude and random direction makes the pdf of $B$ an ellipsoidal shell, and the pdf of the projection of $B$ on the plane of the sky (over which we shall have to integrate) is a thoroughly nasty function. Therefore we take $B_0$ to be normally distributed, i.e. the probability of finding it in a given element of $B_0$-space is

$$dp = \frac{1}{2\pi B_0^2} \exp(-B_0^2/2\langle B_0^2 \rangle) dB_0 dB_0 dB_0.$$  

(3.5)

From the linearity of the relation (2.6) between the com-
ponents of $\mathbf{B}$ and the components of $\mathbf{B}_{\parallel}$, it follows that $dp$ is also normally (but of course anisotropically) distributed in $\mathbf{B}$-space, i.e. it is of the form

$$dp \propto \exp[\text{homogeneous quadratic in } B_x, B_y, B_z] \, dB_x dB_y dB_z.$$  \hspace{1cm} (3.6)

The synchrotron radiation in the $n$ direction depends only on the part $\mathbf{B}_{\parallel} = B_x \mathbf{i} + B_y \mathbf{j} + B_z \mathbf{k}$ of the field that is perpendicular to the line-of-sight $n$; evidently the pdf of $\mathbf{B}_{\parallel}$ is just (3.6) integrated over all $B_z$, and is therefore a two-dimensional normal distribution.

It is advisable to calculate the synchrotron radiation in a frame with base vectors $\hat{x}$, $\hat{y}$, $\hat{n}$, such that $\hat{x}$ and $\hat{y}$ are along the major and minor axes of the pdf of $\mathbf{B}_{\parallel}$. Clearly, $\hat{x}$ and $\hat{y}$ are different for each volume element, and we shall have to transform our answers for $Q$ and $U$ back to our original $l$, $m$, $n$ frame before adding Stokes parameters for different volume elements. We can compute the orientation of the new frame relative to the $l$, $m$, $n$ frame without explicitly following (3.6), for it must be independent of any particle distribution, and the angle $\chi$ between the frames is therefore given by the ratio of $U$ to $Q$ in our previous calculation. Thus, from (2.10),

$$\tan 2\chi = \frac{2a_x a_m + 2b_x b_m + 2c_x c_m}{a_l - a_x + b_l - b_x + c_l - c_x},$$  \hspace{1cm} (3.7)

and by applying a rotation through $\chi$ we obtain the components of $a$, $b$, and $c$ in the $\hat{x}$, $\hat{y}$, $\hat{n}$ system. The variances of $B_x$ and $B_y$ are given by taking the mean squares of the corresponding components of (2.6) and averaging over the pdf of $\mathbf{B}_0$.

$$\langle B_x^2 \rangle = \frac{\langle B_x^2 \rangle}{\langle B_y^2 \rangle} / (a_x^2 + b_x^2 + c_x^2),$$

$$\langle B_y^2 \rangle = \frac{\langle B_y^2 \rangle}{\langle B_y^2 \rangle} / (a_y^2 + b_y^2 + c_y^2),$$  \hspace{1cm} (3.8)

where $a_x = a_x / \sqrt{a_x \cdot a_x}$ etc.; cf. equations (2.10).

Thus we have all the numbers needed to specify the joint pdf of $B_x$ and $B_y$:

$$dp = (4\pi^2 \langle B_x^2 \rangle \langle B_y^2 \rangle)^{-1/2} \exp \left( -\frac{1}{2} \langle B_x^2 \rangle \langle B_y^2 \rangle / \langle B_x^2 \rangle + \langle B_y^2 \rangle / \langle B_y^2 \rangle \right) \, dB_x dB_y.$$  \hspace{1cm} (3.9)

Next we express $\mathbf{B}_{\parallel}$ in polar coordinates,

$$B_x = r \cos \theta, \quad B_y = r \sin \theta,$$  \hspace{1cm} (3.10)

and in these terms

$$dp = (4\pi^2 \langle B_x^2 \rangle \langle B_y^2 \rangle)^{-1/2} \exp \left( -ur^2 - uv^2 \cos 2\theta \right) \, r \, dr d\theta,$$  \hspace{1cm} (3.11)

where

$$u = \frac{1}{2} \langle B_x^2 \rangle + \frac{1}{2} \langle B_y^2 \rangle,$$  \hspace{1cm} (3.11a)

and

$$v = \frac{1}{2} \langle B_x^2 \rangle - \frac{1}{2} \langle B_y^2 \rangle.$$  \hspace{1cm} (3.11b)

Integrating over $\theta$ we obtain the pdf of $r = \mathbf{B}_{\parallel}$:

$$dp = (\langle B_x^2 \rangle \langle B_y^2 \rangle)^{-1/2} \exp \left( -ur^2 \right) I_0(u r^2) \, dr.$$  \hspace{1cm} (3.12)

$I_0(\ )$ is the Bessel function of imaginary argument and order zero, as defined by Abramowitz & Stegun (1965).

### 3.3 Synchrotron radiation from a grid cell

The power per hertz which one electron with pitch angle $\Psi$ contributes to Stokes parameters $I$ and $Q$ at frequency $\nu$ are

$$I = A_\nu \sin \Psi \, F(X); \quad F(X) = X \int_{X}^{\infty} K_{5/3}(\eta) \, d\eta$$  \hspace{1cm} (3.13a)

$$Q = A_\nu \sin \Psi \, G(X); \quad G(X) = X K_{5/3}(X),$$  \hspace{1cm} (3.13b)

where

$$A = \sqrt{3} \gamma^2/(2\epsilon_0 c)$$  \hspace{1cm} (3.13c)

and $K_\nu(X)$ is a Bessel function (Abramowitz & Stegun 1965, section 9.6),

$$\nu \sin \Psi = (e/2\pi m) |\mathbf{B}_{\parallel}| = e\gamma/2\pi m$$  \hspace{1cm} (3.13d)

and

$$X = \nu [(3/2) \gamma^2 \nu \sin \Psi].$$  \hspace{1cm} (3.13e)

In these standard formulae $Q$ is referred to the direction of $\mathbf{B}_{\parallel}$, which makes an angle $\theta$ with $\hat{x}$; therefore the electron’s contribution to $Q$ (referred to the direction of $\hat{x}$) is $Q_\cos \theta$. We need not consider contributions to $U$, since they will obviously cancel out on averaging over the pdf of $\mathbf{B}_{\parallel}$.

Let us find a formula for the expectation of volume emissivity, $W$ Hz$^{-1}$ sr$^{-1}$ m$^{-3}$, in $I(\nu)$. Integrating over the particle energy distribution $n(\gamma) \, d\gamma/4\pi$ per steradian in range $d\gamma$ given by (3.4) and averaging over the pdf (3.12) of $r = |\mathbf{B}_{\parallel}|$, we exploit the scaling properties of the formulae for the particle energy distribution and the synchrotron radiation formulae by writing $I(\nu)$ in terms of the variables $Y = \gamma/\gamma_{\max}$ [already introduced in (3.4b)] and

$$Z = (3\gamma_{\max}^2 e/4\pi m) \, r = Mr,$$  \hspace{1cm} (3.15)

the quantity $M$ being a constant for any given volume element. Then

$$I(\nu) = (A_\nu/8\pi^2 m) \left[ K_0 b^{2a} R^{-3} \right. \left. Y^{-1-2a}(1-Y)^{2a-1} dY \right.$$  \hspace{1cm} (3.16)

$$\times \int_0^{\infty} \exp(-u M^2 Z^2) I_0(u M^2 Z^2) \, \left( \langle B_x^2 \rangle \langle B_y^2 \rangle \right)^{1/8} \, \times \, \left[ d(Z/M)/(Z/M) \right] F(1/ Y^2 Z)$$

$$= \frac{A_\nu}{8\pi^2 m} \frac{K_0 b^{2a} R^{-3}}{\langle B_x^2 \rangle \langle B_y^2 \rangle} \left[ \left( \langle B_x^2 \rangle \langle B_y^2 \rangle \right)^{1/8} \right]$$

$$\times \int_0^{\infty} \exp(-u \gamma Z^2) I_0(u \gamma Z^2) \, Z^2 dZ$$

$$\times \int_0^{\infty} Y^{-1-2a}(1-Y)^{2a-1} F(1/ Y^2 Z) \, dY.$$  \hspace{1cm} (3.16)

Thus the intractable parts of the formula for $I(\nu)$ are reduced to a nasty function of just two variables

$$u^1 = u/M^2$$

and

$$v^1 = v/M^2$$  \hspace{1cm} (3.17)

which together describe the distortion of the volume element.
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QROMO and MIDPT from Press et al. (1986); the Bessel functions \( I_0 \) and \( I_1 \) were computed using NAG (1987) routines S18AEF and S18AFF, respectively. In practice, they were tabulated on a \((40, 40)\) array in terms of variables

\[
s = \left[ (u^1 + v^1)^{-2} + (u^1 - v^1)^{-2} \right]^{-1/2}
\]

and

\[
e = 1 - \left[ (u^1 + v^1)/((u^1 + v^1))^{-1/2} \right]
\]

which measure a normalized field strength and anisotropy, respectively, with linearly spaced increments over \(0 < e < 1\) and logarithmic increments in \(0.01 < s < 10^6\). A double series of Chebychev polynomials was fitted to the array and the coefficients stored in function routines, using NAG (1987) routines E02CAF and E02CBF. A \((20, 20)\) array of coefficients was found sufficient to keep the errors below 1 per cent.

3.5 Discussion of assumptions made in Section 3

Two additional assumptions have been made in this section.

One is the assumption of a gaussian distribution of field strengths injected into the jet. One must make some assumption, and this is almost certainly no worse a guess at reality than to assume a delta function.

The second occurs in the computation of the loss parameter \(b\) given by formula (3.2). In this formula, \(\langle B^2 \rangle\) was used throughout as the estimate of \(B^2\). To be strictly self-consistent we should have computed a mixture of particle energy distributions corresponding to the pdf of field strengths (3.12) at each stage of the flow simulation. We could defend our procedure on physical grounds by claiming that the relativistic electrons diffuse a little through the immediate neighbourhood of the marker particle, so that each electron samples the pdf, but prefer to say simply that we believe that the uncertainty arising from this procedure is less than many others inherent in such a model. The most important effect of a strictly self-consistent calculation would be to protect a small fraction of the electrons from synchrotron losses; this could be significant for radiation at very high frequencies or in very old parts of a source.

4 CONCLUSIONS

In this paper we have shown

(i) how the distortion and amplification of an initially randomly tangled magnetic field in a fluid flow may be followed, by reasonably simple methods,

(ii) how the synchrotron losses suffered by the relativistic electron distribution can be described by a single loss parameter \(b\), accumulated during the passage of a fluid element through the flow, and

(iii) how the calculation of the radio emission and its polarization, even in the presence of the non-power-law particle energy distribution caused by synchrotron losses, may be reduced to manageable proportions by exploiting scaling laws.

Simulations which use these methods, made for a variety of input parameters, and the extent to which they may be relevant to real radio sources, will be discussed in a second paper.
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