Spheroidal galaxy models with anisotropic stresses

N. W. Evans and D. Lynden-Bell

1School of Mathematical Sciences, Queen Mary College, Mile End Road, London E1 4NS
2Institute of Astronomy, Madingley Road, Cambridge CB3 0HA

Accepted 1991 February 21. Received 1991 February 11; in original form 1990 November 20

SUMMARY
This paper applies a method for finding the kinematical properties of galaxy models without the explicit construction of distribution functions. For any axisymmetric mass density generated by a Stäckel potential, an ansatz is introduced for the stress tensor, which enables a unique solution of the equations of stellar hydrodynamics to be deduced. The methods are used to find velocity ellipsoids for oblate and prolate Kuzmin–Kutuzov models. We find that the velocity ellipsoids can be rounder for flatter systems, which illustrates the limitations of the virial equations.

1 INTRODUCTION
The self-consistent problem of stellar dynamics is to find solutions for the distribution function $F$, potential $\psi$ and density $\rho$ of a model galaxy that simultaneously satisfy the Vlasov and Poisson equations (see e.g. Binney & Tremaine 1987):

$$\int \int \int Fd^3v = \rho, \quad \nabla^2 \psi = -4\pi G\rho,$$

(1.1)

$$\frac{\partial F}{\partial t} + v_j \frac{\partial F}{\partial x_j} + \frac{\partial \psi}{\partial x_i} \frac{\partial F}{\partial v_i} = 0.$$

Jeans (1915) noted that in a steady-state ($\partial / \partial t = 0$) the distribution function depends on the isolating integrals of motion only. For axisymmetric galaxies, he suggested it has the general form

$$F = F(E, L_z),$$

(1.2)

where $E$ is the binding energy and $L_z$ the angular momentum parallel to the symmetry axis. Lindblad (1933) pointed out that these models suffer from the severe defect of possessing spheroidal, not triaxial, stress tensors. To satisfy the observed triaxiality of the velocity ellipsoid, the distribution function must depend on the phase space coordinates through an isolating integral additional to the classical ones. Eddington systems – that is star cluster models with Stäckel potentials – naturally fulfill this condition by admitting three integrals quadratic in the velocities and so have been very widely used in galactic modelling (see e.g. Eddington 1915; Kuzmin 1956; Lynden-Bell 1962b; de Zeeuw 1985). For only a very few Eddington systems, however, have exact analytic distribu-
advantages over the Vlasov system - it is much more directly related to the observable properties such as the velocity dispersions and it is much more readily solved by analytic or computational means. 

In Section 2, we use the ansatz (1.4) to derive properties of the stress tensor and show that this enables a unique solution to the stellar hydrodynamical equations to be found. There will normally be many distribution functions which have the same second moments but different higher moments. We therefore merely regard the distribution function (1.4) as a way of motivating the study of stellar systems with stress tensors of a simple form. The general procedure of prescribing a stress tensor and solving the hydrodynamical equations has been used recently by Bacon (1985). His investigations are limited to stress tensors oriented on spherical and cylindrical polar coordinate systems. In Section 3, we apply these methods to give pictures of the behaviour of the velocity ellipsoid for a simple Eddington system in which the stress tensor is oriented along a set of spheroidal coordinates. Finally, Section 4 concludes the paper with a discussion of the advantages and disadvantages of this approach.

2 KINEMATIC PROPERTIES: SECOND MOMENTS

Let (λ, φ, v) be spheroidal coordinates. In each plane of constant φ (meridional plane), λ and v are defined as the roots for r of

\[
\frac{R^2}{r + a} + \frac{z^2}{r + \gamma} = 1,
\]

where \(R = x^2 + y^2\). If \(-a \leq v \leq -\gamma \leq \lambda\), the coordinates are oblate spheroids, a diagram of which is given in Evans & Lynden-Bell (1989), hereafter EL. If \(\gamma \leq v \leq a \leq \lambda\), the coordinates are prolate spheroids. The Stäckel potential (see e.g. de Zeeuw 1985) has the structure

\[
\psi(\lambda, v) = \frac{f(\lambda) - f(v)}{\lambda - v},
\]

where \(f\) is an arbitrary function. In general, a gravitational potential separable in prolate spheroids produces an oblate density distribution, and vice versa. Stars moving in a general axisymmetric potential possess two isolating integrals of motion, which we take as \(E\) and \(I_z = I_z\), where \(E\) is minus the Hamiltonian per unit mass or the binding energy and \(I_z\) is the angular momentum component parallel to the symmetry axis. For Stäckel potentials, the Hamilton–Jacobi equation separates and there is a further isolating integral, namely

\[
I_z = \frac{1}{2} \left( (\lambda + \gamma) v_z^2 + (\lambda + \gamma) v_\phi^2 + z^2 \rho^2 \right) - \frac{(\lambda + \gamma) f(\lambda) - (\lambda + \gamma) f(v)}{\lambda - v}.
\]

Proofs of these formulae are given in Lynden-Bell (1962b), de Zeeuw (1985) and DZ, while the recent work on applications of Stäckel potentials in galactic modelling is reviewed in de Zeeuw & Franx (1991).

For the moment, let us consider distribution functions of form

\[
F = F(C + I_z + I_3),
\]

where \(C\) is a constant and \(J = C + I_z + I_3\) is quadratic in the velocities. Using this ansatz and the definition of the components of the stress tensor or second moments, it follows that

\[
\sigma^2_{\lambda} = \int \int \frac{v^2}{2} F(C + I_z + I_3) dv \phi dv_z,
\]

\[
\sigma^2_{\phi} = \int \int \frac{v^2}{2} F(C + I_z + I_3) dv_z dv_\phi,
\]

where the integration is taken over all velocity space subject to \(E > 0\). Introducing a further constant \(\kappa = \gamma - C\), and making a change of variables to

\[
f' = \frac{1}{2} (\lambda + \kappa) v_\phi^2,
\]

\[k' = \frac{1}{2} (\lambda + \kappa) v_z^2,
\]

then (2.5) becomes

\[
\frac{\sigma^2_{\lambda}}{\sigma^2_{\phi}} = \frac{(\lambda + \kappa)}{(\lambda - \kappa)} \left[ \int \frac{f'}{2} (\lambda + \kappa)^2 - c, I_z \right] dv_\phi dv_z dk
\]

\[
\sigma^2_{\phi} = \frac{1}{2} (\lambda + \kappa) \left[ \int \frac{k'}{2} F(j' + k^2 - c, I_z) dv_\phi dv_z dk \right],
\]

where

\[
c = \frac{(\lambda + \kappa)}{(\lambda - \kappa)} \left[ \frac{1}{2} (\lambda + \kappa) \left[ (\lambda + \kappa) v_\phi^2 + f(\lambda) \right] + (\lambda + \kappa) \left[ (\lambda - \kappa) v_z^2 + f(\lambda) \right] \right].
\]

As \(j\) and \(k\) are dummy variables, we obtain the simple result

\[
\sigma^2_{\lambda} = \frac{(\lambda + \kappa)}{(\lambda - \kappa)} \left[ \int \frac{f'}{2} (\lambda + \kappa)^2 - c, I_z \right] dv_\phi dv_z dk.
\]

For oblate mass models, \(-\gamma \leq v\) and so for (2.9) to be well-behaved we must choose \(\kappa > \gamma\) or equivalently \(C < 0\). (As we shall see in Section 3, this is very much a lower limit and in practice \(\kappa > \gamma\).) For prolate mass models, \(-\alpha \leq v\) and \(\kappa > a\) or \(C < \gamma - a\). The parameter \(\kappa\) controls the anisotropy of the model. As \(\kappa \to \infty\), the distribution function becomes more weakly dependent on the third integral and the velocity ellipsoids more nearly spheroidal.

We emphasize that there are many different distribution functions with the same second moments but different higher moments and so (2.9) is a much less restrictive assumption than (2.4). Even though we can prove that there are no positive definite distribution functions of form (2.4), we still regard the ansatz for the stress tensor (2.9) as physically appealing and worthy of further investigation.

The equations of stellar hydrodynamics for Eddington systems separable in spheroidal coordinates are (Lynden-Bell 1960; EL)

\[
\frac{\partial}{\partial \lambda} \rho \sigma_{\lambda}^2 + \frac{\rho (\sigma_{\lambda}^2 - \sigma_{\phi}^2)}{2(\lambda - \kappa)} = \rho \frac{\partial \psi}{\partial \lambda},
\]

\[
\frac{\partial}{\partial v} \rho \sigma_{\phi}^2 + \frac{\rho (\sigma_{\phi}^2 - \sigma_{\lambda}^2)}{2(\lambda + \kappa)} = \rho \frac{\partial \psi}{\partial v},
\]
where \( \sigma^2 = \sigma^2(\lambda, \nu) \). This is a mathematical statement of Eddington's (1915) theorem – for steady state stellar systems with integrals quadratic in the velocities, the principal axes of the stress tensor are everywhere oriented along a confocal quadric coordinate system. Taking the potential and density as given, the equations of stellar hydrodynamics can be solved to find the second moments. Analytic solutions to these equations are deduced by EL and DZ, while computational methods and solutions are given in Evans (in preparation).

For stress tensors of form (2.9), the following set of linear inhomogenous partial differential equations for \( \rho \sigma^2 \) and \( \rho \sigma^3 \) in the independent variables \( \lambda \) and \( \nu \) can be deduced:

\[
\begin{align*}
\frac{\partial \sigma_1^2}{\partial \lambda} + A_1 \sigma_1^2 &= B_1, \\
\frac{\partial \sigma_1^3}{\partial \nu} + A_2 \sigma_1^3 &= B_2, \\
\frac{\partial \sigma_2^2}{\partial \lambda} + C_1 \sigma_2^2 &= D_1, \\
\frac{\partial \sigma_2^3}{\partial \nu} + C_2 \sigma_2^3 &= D_2,
\end{align*}
\]  

(2.12)

where

\[
\begin{align*}
A_1 &= \frac{1}{2(\lambda + \kappa)} + \frac{1}{2(\lambda + \alpha)} , \\
A_2 &= \frac{3}{2(\nu + \kappa)} + \frac{1}{2(\nu + \alpha)} , \\
B_1 &= \rho \frac{\partial \psi}{\partial \lambda} + \frac{\rho \sigma_1^2}{2(\lambda + \alpha)} , \\
B_2 &= \frac{\lambda + \kappa}{\nu + \kappa} \left( \rho \frac{\partial \psi}{\partial \nu} + \frac{\rho \sigma_2^2}{2(\nu + \alpha)} \right), \\
C_1 &= \frac{3}{2(\lambda + \kappa)} + \frac{1}{2(\lambda + \alpha)} , \\
C_2 &= \frac{1}{2(\nu + \kappa)} + \frac{1}{2(\nu + \alpha)} , \\
D_1 &= \frac{\nu + \kappa}{\lambda + \kappa} \left( \rho \frac{\partial \psi}{\partial \lambda} + \frac{\rho \sigma_1^2}{2(\lambda + \alpha)} \right), \\
D_2 &= \rho \frac{\partial \psi}{\partial \nu} + \frac{\rho \sigma_2^2}{2(\nu + \alpha)} .
\end{align*}
\]  

(2.13)

For an arbitrary \( \sigma^2 \), the set of equations (2.12) does not in general possess a solution. Only if certain integrability conditions are satisfied do smooth solution submanifolds exist.

There is a very useful discussion of this problem in the book of Schutz (1980). First, we deduce the integrability conditions by an elementary method and then we show how they lead to a solution of the stellar hydrodynamical equations. The connection with the less familiar geometric method is discussed in Appendix A.

If we write \( x \) for \( \lambda \) and \( y \) for \( \nu \), then (2.12) becomes

\[
\begin{align*}
(\nabla + A_1) \sigma_1^2 &= B_1, \\
(\nabla + C_1) \sigma_2^2 &= D_1,
\end{align*}
\]  

(2.14)

where the vector \( A \) has components \( (A_1, A_2, 0) \) and similarly for \( B, C \) and \( D \). So, taking the curl immediately gives the integrability conditions as

\[
\begin{align*}
\frac{\partial A_1}{\partial \nu} &= \frac{\partial A_2}{\partial \lambda} , \\
\frac{\partial C_1}{\partial \nu} &= \frac{\partial C_2}{\partial \lambda} , \\
\frac{\partial B_1}{\partial \nu} - \frac{\partial B_2}{\partial \lambda} + A_2 B_1 - A_1 B_2 &= 0, \\
\frac{\partial D_1}{\partial \nu} - \frac{\partial D_2}{\partial \lambda} + C_2 D_1 - C_1 D_2 &= 0.
\end{align*}
\]  

(2.15)

An alternative way to find the same result is to equate

\[
\frac{\partial}{\partial \nu} \left( \frac{\partial \sigma^2}{\partial \lambda} + A_1 \sigma^2 - B_1 \right) = \frac{\partial}{\partial \lambda} \left( \frac{\partial \sigma^2}{\partial \nu} + A_2 \sigma^2 - B_2 \right),
\]  

(2.17)

and annihilate the coefficient of \( \sigma^2 \) and the independent term. The integrability conditions (2.15) are satisfied if and only if there exists a function \( h(\lambda, \nu) \) such that

\[
B_1 = \frac{1}{(\lambda + \alpha)^{\frac{1}{2}} (\nu - \alpha)^{\frac{1}{2}} (\lambda + \kappa)^{\frac{1}{2}} (\nu + \kappa)^{\frac{1}{2}}} \frac{\partial h}{\partial \lambda} ,
\]  

(2.18)

\[
B_2 = \frac{1}{(\lambda + \alpha)^{\frac{1}{2}} (\nu - \alpha)^{\frac{1}{2}} (\lambda + \kappa)^{\frac{1}{2}} (\nu + \kappa)^{\frac{1}{2}}} \frac{\partial h}{\partial \nu} ,
\]  

(2.19)

The solution of equations of stellar hydrodynamics is then given on substituting (2.18) into (2.12) as

\[
\rho \sigma^2 = \frac{h(\lambda, \nu)}{(\lambda + \alpha)^{\frac{1}{2}} (\nu - \alpha)^{\frac{1}{2}} (\lambda + \kappa)^{\frac{1}{2}} (\nu + \kappa)^{\frac{1}{2}}} ,
\]  

(2.20)

\[
\rho \sigma^3 = \frac{h(\lambda, \nu)}{(\lambda + \alpha)^{\frac{1}{2}} (\nu - \alpha)^{\frac{1}{2}} (\lambda + \kappa)^{\frac{1}{2}} (\nu + \kappa)^{\frac{1}{2}}} ,
\]  

(2.21)

where the function \( h(\lambda, \nu) \) is still to be found. Note that the solutions already obey the general theorems on the second moments given in EL.

From the condition (2.18), it follows that

\[
\rho \sigma^2 = \frac{2(\lambda + \alpha)^{\frac{1}{2}}}{(\nu - \alpha)^{\frac{1}{2}} (\lambda + \kappa)^{\frac{1}{2}} (\nu + \kappa)^{\frac{1}{2}}} \frac{\partial h}{\partial \lambda} - 2 \rho(\lambda + \alpha) \frac{\partial \psi}{\partial \lambda} ,
\]  

(2.22)

and

\[
\rho \sigma^3 = \frac{2(\nu - \alpha)^{\frac{1}{2}}}{(\lambda + \alpha)^{\frac{1}{2}} (\lambda + \kappa)^{\frac{1}{2}} (\nu + \kappa)^{\frac{1}{2}}} \frac{\partial h}{\partial \nu} - 2 \rho(\nu + \alpha) \frac{\partial \psi}{\partial \nu} ,
\]  

(2.23)

Demanding these two expressions be consistent, \( h(\lambda, \nu) \) satisfies the first-order partial differential equation

\[
\frac{(\lambda + \alpha)^{\frac{1}{2}}}{(\nu - \alpha)^{\frac{1}{2}} (\lambda + \kappa)^{\frac{1}{2}} (\nu + \kappa)^{\frac{1}{2}}} \frac{\partial h}{\partial \lambda} + \frac{(\nu - \alpha)^{\frac{1}{2}}}{(\lambda + \alpha)^{\frac{1}{2}} (\lambda + \kappa)^{\frac{1}{2}} (\nu + \kappa)^{\frac{1}{2}}} \frac{\partial h}{\partial \nu} = - k(\lambda, \nu),
\]  

where

\[
k(\lambda, \nu) = (\nu - \alpha) \rho \frac{\partial \psi}{\partial \nu} - (\lambda + \alpha) \rho \frac{\partial \psi}{\partial \lambda} .
\]  

(2.24)

A single first-order partial differential equation is equivalent to a vector field whose integral curves are conventionally known as characteristics (see e.g. Garabedian 1986). Given boundary conditions, the characteristics sweep out the solution submanifold. Thus, finding the solution of (2.23)
involves integrating the system of ordinary differential equations

\[
\frac{(-v-a)^{3/2}(\lambda+\kappa)^{1/2}(v+\kappa)^{3/2}}{(\lambda+a)^{1/2}(v+\kappa)^{1/2}} \, d\lambda = \frac{d\lambda}{(\lambda+\kappa)^{1/2}(v+\kappa)^{1/2}} = - \frac{dv}{k(\lambda, v)},
\]

with initial value information, which is simple enough numerically.

The projection of the characteristic lines on the \((\lambda, v)\) plane is given by the equation

\[
\frac{d\lambda}{(\lambda+\kappa)(\lambda+a)} = - \frac{dv}{(v+\kappa)(v+a)},
\]

which on integration yields

\[
\frac{(\lambda+a)(v+a)}{(\lambda+\kappa)(v+\kappa)} = - w,
\]

where for oblate (prolate) mass models \(w\) is a positive (negative) constant satisfying \(0 \leq |w| \leq (\gamma - \alpha)/(\kappa - \gamma)\). For a model with finite total mass, the boundary conditions must be taken as \(h(\lambda, v) \to 0\) as \(\lambda \to \infty\). By integrating (2.26) along the characteristic hyperbola with parameter \(w\), the formal solution of (2.23) is

\[
h(\lambda, v) = \frac{(\kappa - \alpha)^{3/2}(\lambda - \alpha)^{1/2}(v - \alpha)^{1/2}}{(\lambda+\kappa)^{1/2}(v+\kappa)^{1/2}} \times \int_s^{\infty} \frac{ds(s+\alpha)(s+\kappa)}{|w(s+\kappa)+s+\alpha|^2} \left( \frac{s}{w(s+\kappa)+s+\alpha} \right)^{1/2} k_{s} ds
\]

For any mass model, substituting (2.28) into (2.19), (2.20) and (2.21) generates the second moments consistent with the ansatz (2.9). In general, the quadrature must be done numerically, but this still gives a straightforward way of finding solutions to the equations of stellar hydrodynamics.

3 APPLICATIONS: KUZMIN–KUTUZOV MODEL

The methods developed above are now applied to a beautiful model introduced by Kuzmin & Kutuzov (1962) and subsequently studied by DZ. The Kuzmin–Kutuzov model has defining function \(f(\tau) = GM\tau\) and so the Stäckel potential (2.2) becomes

\[
\psi(\lambda, v) = \frac{GM}{\sqrt{\lambda + \sqrt{v}}},
\]

Setting \(\alpha = -a^2\) and \(\gamma = -c^2\), the density distribution generated through Poisson’s equation is oblate if \(c/a < 1\) and prolate if \(c/a > 1\) and given by

\[
\rho(\lambda, v) = \frac{Mc^2}{4\pi} \frac{X^2 + a^2(Y^2 + X)}{X^3 Y^3},
\]

where \(X\) and \(Y\) are symmetrized variables defined as

\[
X = \sqrt{\lambda} \sqrt{v}, \quad Y = \sqrt{\lambda + \sqrt{v}}.
\]

Using the expressions for the potential-density pair (3.1) and (3.2), the auxiliary function \(k\) (defined in 2.24) is

\[
k(\lambda, v) = \frac{\epsilon^2(\sqrt{\lambda} - \sqrt{v})[\sqrt{\lambda} v + a^2(\sqrt{\lambda} v + a^2(\lambda + 3\sqrt{\lambda} v + v)]}{8\pi\lambda^2 v(\sqrt{\lambda} + \sqrt{v})^5}.
\]

The integration in (2.28) leads to incomplete elliptic integrals and in practice is best performed by Gauss-Rational quadrature. The second moments on the axes \(\lambda = -\alpha\) or \(v = -\alpha\) reduce to elementary integrals and can be calculated explicitly. We find (with \(GM = 1\))

\[
\sigma_\lambda^2(-\alpha, v) = \frac{n^2(a^2 + \kappa^2)(a^2 + \kappa)}{12a^4(2n + a)^2(\kappa^2 + \alpha)^2} \times [12(a^2 + 2\kappa) \ln 2 - (16\kappa + 7a^2)],
\]

\[
\sigma_v^2(\lambda, -\alpha) = \frac{6a^2(2\ell + a)(\ell + \kappa)(\kappa + a))}{6a^2(\ell^2 + \kappa^2)(\kappa + a)} \times [2(11\kappa + 3\ell^2)a^2\ell^2 + (3\kappa - 7\ell^2)]
\]

\[
\times a^2 - 3(\kappa + 5\ell^2) a^2 \ell^2 - 30a^2\kappa\ell^4 - 12a^2\kappa^2 - 12a^2\kappa^2\ell^4
\]

where \(\ell = \sqrt{\lambda}\) and \(n = \sqrt{v}\). The \(\sigma_\lambda^2\) components on axis are deduced from (2.9) to be

\[
\sigma_\lambda^2(\lambda, -\alpha) = \frac{\kappa - \alpha}{\lambda + \kappa} \sigma_\lambda^2(\lambda, -\alpha),
\]

\[
\sigma_v^2(-\alpha, v) = \frac{v + \kappa}{\kappa - \alpha} \sigma_v^2(-\alpha, v).
\]

EL prove that the transverse components on the symmetry axis are the same for spheroidal Eddington systems

\[
\sigma_\lambda^2(\lambda, -\alpha) = \sigma_v^2(\lambda, -\alpha), \quad \sigma_\lambda^2(-\alpha, v) = \sigma_v^2(-\alpha, v),
\]

and hence the second moments on axis are all elementary. At the focus of the coordinate system, the stress tensor is spheroidal with

\[
\sigma_\lambda^2 = \sigma_v^2 = \frac{1}{9a^2(a^2 + \kappa)} \left[12(a^2 + 2\kappa) \ln 2 - (16\kappa + 7a^2)\right].
\]

For oblate mass models, the focus is on the symmetry axis and so the stress tensor is isotropic.
Numerical solutions of the velocity ellipsoids are given in Figs 1–3 for oblate Kuzmin–Kutuzov models with \( c/a = 0.30, 0.60 \) and 0.90 respectively. The corresponding central ellipticities (DZ, equation 4.6) are 0.67, 0.36 and 0.09, although all models become more spherical with increasing radius. The full ellipses are sections with the \((v_x, v_y)\) plane, the dotted ellipses are sections with the \((v_y, v_z)\) plane, while logarithmically spaced contours of density are plotted in broken lines. The focus of the spheroidal coordinate system is marked with a cross. To present the results, we adopt the normalization

\[ \psi(-\alpha, -\gamma) = 1, \quad GM = 1, \quad a + c = 1, \]  

introduced in DZ. The velocity ellipsoids are nearly isotropic in the core but dominated by radial orbits in the outer parts of the galaxy. The dispersion is always greatest at the focus of the spheroidal coordinate system and not in the centre of the galaxy. The components of the stress tensor are, though, always greatest in the centre. In each case, the solution of the hydrodynamical equations is given for the minimum possible value of \( \kappa \) consistent with everywhere positive stresses — that is, further radial anisotropy gives negative densities. As \( \kappa \) is increased, the velocity ellipsoids become more spheroidal while in the limit \( \kappa \to \infty \), the solution reduces to that for Jeans models given in DZ.

The obvious tool for investigating the relationship between flattening and anisotropy is the tensor virial theorem (Binney 1981), the independent components of which are

\[ 2T_{RR} + W_{RR} = 0, \quad 2T_{zz} + W_{zz} = 0, \]  

where

\[ T_{ij} = \int \rho \sigma_i^2 d^3x, \quad W_{ij} = \int \rho x_i \frac{\partial \psi}{\partial x_j} d^3x. \]  

**Figure 1.** Velocity ellipsoids for the \( c/a = 0.3 \) oblate Kuzmin–Kutuzov model with \( \kappa = 2.31 \).

**Figure 2.** Velocity ellipsoids for the \( c/a = 0.6 \) oblate Kuzmin–Kutuzov model with \( \kappa = 1.24 \).

**Figure 3.** Velocity ellipsoids for the \( c/a = 0.9 \) oblate Kuzmin–Kutuzov model with \( \kappa = 0.62 \).

Note we have made no distinction between organized streaming and random motions. For a flattened galaxy, \(|W_{zR}| < |W_{RR}|\) and so from (3.11) there must be more kinetic energy in the \( R \) than \( z \) direction, i.e. \( T_{zz} \ll T_{RR} \). Hence, it might be expected that as the Kuzmin–Kutuzov models are flattened, the stellar hydrodynamical solutions stay positive for smaller \( \kappa \) (greater radial anisotropy). In fact, the reverse is true. The rounder the density contours, the more elongated we can make the velocity ellipsoids! This is understood on
observing that $T_{ee}$ is dominated by contributions from the equatorial plane. The flatter the Kuzmin–Kutuzov model, the greater the density in the equatorial plane. So, the $T_{ee}$ component of the kinetic energy tensor can be larger for flatter models (as required by the virial theorem) even though the velocity ellipsoids are less anisotropic.

Finally, in Fig. 4 a solution for the prolate Kuzmin–Kutuzov model with $c/a = 1.6$ and central ellipticity 0.33 is given. It is much easier to construct physical solutions to the hydrodynamical equations for prolate mass models – generally, any allowable $\kappa$ in the range $0 < \kappa < \infty$ gives positive stresses. Prolate Eddington systems admit two families of orbits, inner and outer long axis tubes, whereas oblate systems admit only one, short axis tubes (de Zeeuw 1985). There is much greater freedom to reproduce the required density with two orbital families than one, and so we expect the range of prolate stellar dynamical equilibria to be much richer. This is confirmed by the investigations into thin orbit solutions of Hunter et al. (1990), who find that prolate models have many solutions whereas oblate models have only one.

4 CONCLUSIONS

There are essentially three approaches to the search for distribution functions of self-consistent stellar dynamical models. The first is to select a functional form for the distribution function and then solve the Poisson equation to find the potential (Prendergast & Toner 1970; Kashtansky 1988). Straightforward application to Eddington systems is hampered by the comparative scarcity of Stäckel potentials – that is, the zero measure of the manifold of Stäckel potentials on the manifold of triaxial potentials. An entirely different way to construct models using linear programming or Lucy’s method on a mesh of cells in phase space has been proposed (Schwarzschild 1979; Newton & Binney 1984) and successfully applied to Eddington systems (Statler 1987). This gives the distribution function as a set of occupation numbers of cells but is necessarily very coarse-grained and difficult to use. The final method is to prescribe the zeroth moment or density and construct the self-consistent (even part of the) distribution function by using integral transform methods (Lynden-Bell 1962a; Hunter 1975; Dejonghe 1986). For Eddington systems, this approach is thwarted by the complexity of the kernel of the integral equation which precludes the successful use of Laplace or Mellin transforms. So, the construction of smooth distribution functions for flattened stellar systems with triaxial stress tensors seems to have reached an impasse and we believe it is valuable to find methods of finding kinematic properties of stellar dynamical models without explicitly deriving the distribution functions. Indeed, even this very much easier problem has no analytic or computational solutions for triaxial models.

Of course, the disadvantage is that there is no guarantee that an everywhere positive distribution function $F$ exists. That is, we do not know whether it is possible to solve the integral equation

$$\rho(\lambda, \psi) = \int_0^\infty dE \int_0^1 dI_2 \int_0^{I_2} dI_3 F(E, I_2, I_3),$$ (4.1)

with

$$I_2^* = R^2(\psi - E), \quad I_3^* = f(\lambda) - (\lambda + \gamma) E - \frac{(\lambda + \gamma) I_2}{(\lambda + \alpha)},$$

$$I_3^* = f(\nu) - \frac{(\nu + \gamma) E}{(\nu + \alpha)},$$

to give the required mass density, together with a stress tensor satisfying

$$\sigma_{22} = \frac{\lambda + \kappa}{\nu + \kappa} \sigma_{33}.$$

(4.2)

In general, there is expected to be such a solution because there is enormous freedom in solving (4.1) for a function of three variables in terms of a known function of two variables, but theorems for existence and uniqueness have not been rigorously established.

Finally, it is worthwhile concluding with a few remarks on the distribution functions

$$F = F(CE + I_2 + I_3, I_3),$$

(4.3)

introduced in (2.4) which motivated our original work. By hypothesis, at fixed $I_2$, the distribution function is stratified on surfaces of constant $J = CE + I_2 + I_3$. So, we must impose the conditions that $I_2 \geq 0$ (which can be done without loss of generality by fixing $f(\tau)$ to vanish on the equatorial plane) and that $J \leq 0$ – otherwise low $I_3$ orbits that are not gravitationally bound will be incorporated in the model. Unfortunately, all such models now suffer from the defect that there exist positions in configuration space that cannot be reached by any stellar orbit (except in the limit $C \to \infty$). At any position $(\lambda, \nu)$ the minimum value of the integral $I_3$ is

Figure 4. Velocity ellipsoids for the $c/a = 1.6$ prolate Kuzmin–Kutuzov model with $\kappa = 1.24$. © Royal Astronomical Society • Provided by the NASA Astrophysics Data System
\[ I_3 = \frac{(\lambda + \gamma) f(\nu) - (\nu + \gamma) f(\lambda)}{\lambda - \nu}, \] (4.4)

and so for example this implies that no orbits with vanishing \( E \) exist at all in the models except on the equatorial plane. A system of infinite extent cannot therefore be generated by these distribution functions. It is certainly possible to construct stellar systems with a finite tidal radius, but excepting the perfect ellipsoid (de Zeeuw 1985) – these cannot be derived from a gravitational potential of Stäckel form and then the assumed form for the third integral does not hold.

**ACKNOWLEDGMENTS**

NWE thanks Kim de Zeeuw for much valuable encouragement and advice, as well as for forwarding material in advance of publication. The referee Herwig Dejonghe made several useful comments and supplied one of the elementary proofs of the integrability conditions. Financial support was received from the Science and Engineering Research Council of the United Kingdom and King’s College, Cambridge. The paper was completed at the Aspen Centre for Physics, whose generous hospitality is gratefully acknowledged.

**REFERENCES**


**Spherical galaxy models with anisotropic stresses**


**APPENDIX A: GEOMETRIC METHOD**

In this appendix, we reconstruct the integrability conditions of the set of partial differential equations

\[
\frac{\partial \rho \sigma^2}{\partial \lambda} + A_1 \rho \sigma^2 = B_1, \quad \frac{\partial \rho \sigma^2}{\partial \nu} + A_2 \rho \sigma^2 = B_2, \quad (A1)
\]

\[
\frac{\partial \rho \sigma^2}{\partial k} + C_1 \rho \sigma^2 = D_1, \quad \frac{\partial \rho \sigma^2}{\partial \nu} + C_2 \rho \sigma^2 = D_2,
\]

by a geometric method to illustrate the connection with the ideas in Schutz's (1980) book. Consider the four-dimensional manifold \( M \) with coordinates \((\rho \sigma^2, \rho \sigma^2, \lambda, \nu)\). Two one-form fields \( \tilde{a} \) and \( \tilde{\beta} \) are defined by

\[
\tilde{a} = \tilde{a} (\rho \sigma^2) + \rho \sigma^2 \tilde{A} - \tilde{B}, \quad \tilde{\beta} = \tilde{\beta} (\rho \sigma^2) + \rho \sigma^2 \tilde{C} - \tilde{D},
\]

where \( \tilde{d} \) is the exterior derivative operator, the one-form \( \tilde{A} \) is given as

\[
\tilde{A} = A_1 \tilde{d} \lambda + A_2 \tilde{d} \nu,
\]

and similarly for \( \tilde{B}, \tilde{C} \) and \( \tilde{D} \). Solving \((A1)\) is the same mathematical problem as finding a two-dimensional submanifold \( H \) in \( M \) on which the restriction of the forms \((A2)\) to \( H \) vanishes – because we have chosen the components of the one-forms to be the partial differential equations. The two one-forms \( \tilde{a} \) and \( \tilde{\beta} \) define at each point \( P \) of \( M \) a two-dimensional vector subspace \( X_P \) of the tangent space to \( M \). Each vector in \( X_P \) annihilates \( \tilde{a} \) and \( \tilde{\beta} \), or equivalently \( \tilde{a} \) and \( \tilde{\beta} \) pull back to zero on \( H \). The complete ideal of the set \([\tilde{a}, \tilde{\beta}]\) at \( P \) is all the forms whose restriction to \( X_P \) vanishes.

By Frobenius’ theorem (see Schutz 1980), if and only if \( \tilde{d} \tilde{a} \) and \( \tilde{d} \tilde{\beta} \) are in the complete ideal, then there exist functions \( U, V, W, X, Y \) and \( Z \) of \((\rho \sigma^2, \rho \sigma^2, \lambda, \nu)\) such that

\[
\tilde{a} = W \tilde{d} U + X \tilde{d} V, \quad \tilde{\beta} = Y \tilde{d} U + Z \tilde{d} V.
\]

The solution submanifold is therefore given by

\[
U(\rho \sigma^2, \rho \sigma^2, \lambda, \nu) = \text{constant}, \quad V(\rho \sigma^2, \rho \sigma^2, \lambda, \nu) = \text{constant}.
\]

This is because tangent vectors to the curves \((A5)\) annul \( \tilde{a} \) and \( \tilde{\beta} \).

It follows that the integrability conditions for \((A1)\) are deduced by requiring that \( \tilde{d} \tilde{a} \) and \( \tilde{d} \tilde{\beta} \) lie in the complete ideal of \([\tilde{a}, \tilde{\beta}]\). This is well-known (see Schutz 1980) to be

\[
\tilde{d} \tilde{a} \wedge \tilde{\beta} = 0, \quad \tilde{d} \tilde{\beta} \wedge \tilde{a} = 0,
\]

where \( \wedge \) denotes the wedge operator. Inserting \((A2)\) and simplifying gives exactly the equations \((2.15)\). Note that the more powerful geometric methods demonstrate that \((2.15)\) are both necessary and sufficient for the existence of solutions to \((A1)\). Equating \((A2)\) with \((A4)\) and working in a
one-form basis \(\{d\rho \sigma_1^2, d\rho \sigma_2^2, d\lambda, d\nu\}\), it is easily found that
\[
X = 0, \quad Y = 0, \quad \lambda = \lambda', \quad \nu = \nu'.
\]  
\[
W = \frac{1}{(\lambda + \alpha)^{1/2}(-\nu - \alpha)^{1/2}(\lambda + \kappa)^{1/2}(\nu + \kappa)^{3/2}}, \tag{A7}
\]
\[
Z = \frac{1}{(\lambda + \alpha)^{1/2}(-\nu - \alpha)^{1/2}(\lambda + \kappa)^{1/2}(\nu + \kappa)^{1/2}}, \tag{A8}
\]
\[
U = (\lambda + \alpha)^{1/2}(-\nu - \alpha)^{1/2}(\lambda + \kappa)^{1/2}(\nu + \kappa)^{1/2} \rho \sigma_0^2 \rho \sigma_0^2 - h(\lambda, \nu), \tag{A9}
\]
\[
V = (\lambda + \alpha)^{1/2}(-\nu - \alpha)^{1/2}(\lambda + \kappa)^{1/2}(\nu + \kappa)^{1/2} \rho \sigma_0^2 - h(\lambda, \nu). \tag{A9}
\]

The solution of the equations of stellar hydrodynamics is then just
\[
\rho \sigma_0^2 = \frac{h(\lambda, \nu)}{(\lambda + \alpha)^{1/2}(-\nu - \alpha)^{1/2}(\lambda + \kappa)^{1/2}(\nu + \kappa)^{3/2}}, \tag{A10}
\]
\[
\rho \sigma_0^2 = \frac{h(\lambda, \nu)}{(\lambda + \alpha)^{1/2}(-\nu - \alpha)^{1/2}(\lambda + \kappa)^{1/2}(\nu + \kappa)^{1/2}}, \tag{A11}
\]
which is in accord with (2.19) and (2.20).