Stokes parameters of radiation propagating through an aligned gaseous-dust medium

A. Z. Dolginov and V. I. Siklitsky
Joffe Physical Technical Institute, Polytechnicheskaya st. 26, Leningrad 194021, USSR

Accepted 1991 July 1. Received 1991 May 2; in original form 1990 July 30

SUMMARY
The Stokes parameters of radiation propagating through a gaseous-dust medium are obtained in explicit analytical form. The dust grains are assumed to be aligned. These formulae are applicable to any type of alignment, any shape, chemical composition, and size of grain, if the grains are not very optically thick. The axis of predominant alignment is the external magnetic field or the direction of the flux of ambient gas. The important role of the Barnett effect for the rotating grain is taken into account. The formulae obtained are applied to some specific sorts of grains (anisotropic graphite flakes, silicate spheroids, etc.) and conditions in the medium (interstellar and circumstellar clouds). Stokes parameters for the zodiacal light polarization are obtained. Interpretation of polarimetric observation of the light scattered by circumstellar dust around the red supergiant α Orionis is presented.

1 STATEMENT OF THE PROBLEM
Theory and observation show that the dust component of the circumstellar and interstellar medium determines the observed polarization of starlight in many cases. The type of polarization depends on the properties and orientation of non-spherical grains. There are a number of papers devoted to these problems [see for example, Asano & Sato (1980); Dolginov, Gnedin & Silant’ev (1979); van de Hulst (1957); Purcell & Pennypacker (1973)]. Circumstellar polarization has been observed for example by Scarrott (1988); Le Borgne & Mauron (1989); Le Borgne, Mauron & Leroy (1986, Paper I; McMillan & Tapia (1978); Scarrott, Draper & Warren-Smith (1989); Scarrott, Rolph & Mannion (1989); Gledhill & Scarrott (1989).

Only the case of scattering on particles much smaller than the wavelength has been considered in most papers or only very special shapes of particles have been taken into account. The mechanism of paramagnetic relaxation of Davis & Greenstein (1951) and the ‘propeller’ mechanism of Purcell (1973) have commonly been used to describe the dust grain orientation. These mechanisms meet serious difficulties because they need a very long time for the grains to reach a noticeable degree of orientation. The orientation of grains by interaction with gaseous fluxes in the presence of magnetic fields has been considered by Dolginov (1972) and Dolginov & Mytrophanov (1976). The orientation time in these cases is, as a rule, shorter than that for the paramagnetic relaxation. The importance of the role of the Barnett effect for the orientation process has been demonstrated by Dolginov & Mytrophanov (1978), Purcell (1973) and Dolginov (1989). However, this effect was disregarded by many authors.

In this paper we will consider the polarization of radiation scattered by dust grains oriented in interstellar or circumstellar space. Various cases of orientation and the role of the Barnett effect are taken into account.

The grain model utilized in the paper is an oblate or prolate ellipsoid of arbitrary size and anisotropic polarizability (e.g., a graphite grain) situated in a medium with a magnetic field and an anisotropic gaseous flux.

The vectors which determine the behaviour of the physical system of the dust and radiation are: \(k_0\) and \(k\), wave vectors of the incident and scattered wave respectively; \(h\), the direction of external magnetic field; \(a\), the symmetry axis of the dust grain; \(J\), angular momentum of the grain; and \(\Omega\), its angular velocity. \(\mu || \Omega\) is the grain magnetic moment arising from the Barnett effect.

Let us choose a coordinate system \(X_0\), \(Y_0\), \(Z_0\) rigidly fixed with respect to the observer. The \(OZ_0\) axis is directed along \(k_0\), and the plane \((k_0,k)\) is chosen as the plane \(X_0OZ_0\). The direction of \(h\) is determined by the polar angles \(\theta_h\), \(\varphi_h\). The angle between \(h\) and \(a\) is \(\theta_a\), and the scattering angle is \(\theta\) (see Fig. 1).
The following expressions will be used: \( E^0 \exp[i(k_0 r - \omega t)] \) is the incident plane wave, where \( k_0 = (\omega/c) n_0 \) and \( \omega \) is the wave frequency.

\[
E_a^0 = (E^0_y/R) t_{a0}(n, n_0, \omega) \exp[i(kR - \omega t)].
\]  

(1)

\( E^0 \) is the wave scattered in the \( k \) direction. \( R \) is the distance from the point of scattering to the observer. \( t_{a0}(n, n_0, \omega) \) is the scattering matrix which connects the incident and the scattered waves.

It is convenient to determine different vector quantities in different reference frames. The natural reference frame for the incident wave is the coordinate system \( X_0 Y_0 Z_0 \) with \( OZ_0 \parallel n_0 \) and for the scattering wave the coordinate system with \( OZ \parallel n \). The particle polarizability is conveniently determined in the system \( X_a Y_a Z_a \) rigidly connected to the particle body with \( OZ_a \parallel a \).

Transformation from one reference frame to another can be performed by the operation of rotation determined by the three Euler angles. The relation between the components of a vector in different reference frames is determined by

\[
E_s(h) = D_{a0}(h, n_0) E_s(n_0),
\]

(2)

where the cyclic, not the Cartesian, components of the vector are used. The cyclic components are determined as \((\alpha, \gamma = 0, \pm 1)\):

\[
E_0 = E_z, \quad E_1 = \sqrt{1/2} (E_+ + iE_-), \quad E_{-1} = -\sqrt{1/2} (E_+ - iE_-).
\]  

(3)

\( E_s(h) \) is given in the system \( OZ_0 \parallel h \) and \( E_s(n_0) \) in the system \( OZ_0 \parallel k_0 \). The function \( D_{a0}(h, n_0) \) is the Wigner matrix of rotation which effects the rotation from the reference frame \( X_0 Y_0 Z_0 \) to \( X_a Y_a Z_a \).

2 THE STOKES PARAMETERS

The state of polarization can be determined by the four Stokes parameters \( I, Q, U, \) and \( V \), as well as by the so-called density matrix

\[
\rho_{aa} = (I + aV)/2, \quad \rho_{a-a} = -(Q + iU)/2, \quad a = \pm 1.
\]  

(4)

in the reference frame with \( OZ \parallel k \). For the incident wave \( \rho_{a0}(n_0) = (c/8\pi)|E^0_y E^0_\gamma|^2 \) and for the scattered wave

\[
\rho_{a0}(n) = (c/8\pi)|E^0_y E^\phi|^2 = (c/8\pi R^2)|t_{a0}(n, n_0, \omega)| E^0_y E^\phi |t_{0a}(n, n_0, \omega)|^2,
\]

(5)

where \(|\cdot\rangle\rangle\) denotes the averaging over the characteristics of the scatterers, particularly over the particle orientations. The summation over the repeated indices is implied. The distribution function of the particle orientation is assumed to be known.

The polarization can also be determined by the linear \( P_1 \) and circular \( P_2 \) net polarization values and by the position angle \( \chi \) which determines the direction of the large axis of the ellipse of polarization according to the relations

\[
P_1 = V/I, \quad P_2 = \sqrt{(Q^2 + U^2)/I}, \quad \tan(2\chi) = U/Q.
\]  

(6)

Stokes parameters of the incident \( I^0(n_0) \) and scattered \( I^\phi(n) \) waves are related by the expression

\[
I^\phi(n) = F_{a0}(n, n_0, h, \omega) I^0(n_0),
\]

(7a)

where the following notations are introduced:

\[
I_1 = I, \quad I_2 = Q, \quad I_3 = U, \quad I_4 = V.
\]  

(7b)

Figure 1. The coordinate system used in the paper and definition of the angles.
To calculate (7a) one must begin with the explicit expression for the scattering matrix. The matrix \( t_{pq}(n, n_0, \omega) \) in equation (5) is given in the XYZ system. The transformation of \( t_{pq}(n, n_0, \omega) \) from the XYZ to the \( X_a Y_a Z_a \) coordinate system has the form

\[
t_{pq}(n, n_0, \omega) = k^2 D_{pq}^{(3)}(n, n) \beta_{pq}(n, n_0, \omega | a) D_{pq}^{(3)}(n_0, n_0, \omega | a)
\]

where \( \beta_{pq}(n, n_0, \omega | a) \) is the tensor of particle polarizability in the coordinate system \( X_a Y_a Z_a \). The \( \epsilon_{pq}(r, \omega) \) is the dielectric tensor and \( \sigma_{pq}(r, \omega) \) is the tensor of electrical conductivity of the dust grain substance. If the grain is isotropic, then \( \epsilon_{pq} = \delta_{pq} \) and \( \sigma_{pq} = \sigma_{pp} \). The integral in (9) is taken over the grain volume. The indices \( \alpha, \beta, p \) and \( q \) are equal to 0, ±1 and determine the cyclic components of the vectors and tensors.

If the grain is uniform (but not necessarily isotropic) then

\[
\beta_{pq}(n, n_0, \omega | a) = \alpha_{pq}(\omega) f(q),
\]

\[
\alpha_{pq}(\omega) = (4\pi \omega) \sigma_{pq}(\omega - \delta_{pq}) / 4\pi,
\]

\[
f(q) = \int_{V} \exp(-iqr) \, dr,
\]

where \( q = k - k_0 = k(n - n_0) \).

Let us consider the most known case when \( \alpha_{pq}(\omega) = \alpha_{pp}(\omega) \). Then the values of \( F_{ii} \) are given by the expressions:

\[
F_{ii} = \frac{1}{3} \left[ S_{00} + \frac{(3x^2 - 1)}{4\sqrt{5}} S_{22}^0 \right] R_{ii} + \frac{1}{6\sqrt{2}} \left[ S_{02}^0 - \frac{1}{\sqrt{7}} S_{22}^2 \right] \left( 3x^2 - 1 \right) + \frac{25}{22} \left[ S_{22}^2 R_{20} + \frac{1}{2\sqrt{3}} \left[ -2S_{02}^0 + \frac{1}{\sqrt{7}} S_{22}^2 \right] xyR_{21} + \left( S_{02}^0 + \frac{1}{\sqrt{7}} S_{22}^2 \right) y^2 R_{22} + \frac{1}{2\sqrt{14}} \left[ 3x^2 - 1 \right] R_{40} - 2xyR_{41} + \frac{y^2}{\sqrt{2}} R_{42} \right] \right] S_{22}^2,
\]

\[
F_{10} = \frac{y^2}{4\sqrt{5}} S_{22}^2 R_{10} - \frac{1}{\sqrt{3}} \left[ S_{02}^0 - \frac{(3x^2 - 1)}{2\sqrt{7}} S_{22}^2 \right] R_{22} + \frac{3}{2\sqrt{7}} \left[ xyR_{21} + \frac{y^2}{\sqrt{6}} R_{22} \right] S_{22}^2,
\]

\[
- \frac{1}{4} \left[ \frac{1}{\sqrt{70}} R_{10} + \left( \frac{3x^2 - 1}{\sqrt{4}} \right) R_{22} - \frac{1}{\sqrt{7}} R_{40} + \frac{y^2}{\sqrt{2}} R_{43} \right] S_{22}^2,
\]

\[
F_{11} = \frac{1}{\sqrt{3}} \left[ S_{02}^0 + \frac{(3x^2 - 1)}{2\sqrt{7}} S_{22}^2 \right] Q_{22} + \frac{3xy}{2\sqrt{7}} Q_{21} S_{22}^2 - \frac{1}{4} \left[ \frac{(3x^2 - 1)}{\sqrt{7}} Q_{42} - \frac{1}{\sqrt{7}} Q_{41} + Q_{43} \right] + y^2 Q_{44} \left[ S_{22}^2, \right.
\]

\[
F_{12} = \frac{1}{2\sqrt{3}} \left[ xyQ_{21} - y^2 Q_{22} \right] S_{12},
\]

\[
F_{20} = \frac{y^2}{4\sqrt{5}} S_{22}^2 R_{00} - \frac{y^2}{2\sqrt{2}} \left[ S_{00} - \frac{1}{\sqrt{7}} S_{22}^2 \right] R_{20} + \frac{1}{2\sqrt{3}} \left[ 2S_{02}^0 - \frac{1}{\sqrt{7}} S_{22}^2 \right] xyR_{21} + \left( S_{02}^0 + \frac{1}{\sqrt{7}} S_{22}^2 \right) \left( 1 + x^2 \right) R_{22}
\]

\[
- \frac{1}{4\sqrt{7}} \left[ \frac{6}{\sqrt{5}} y^2 R_{40} + 2\sqrt{2}xyR_{41} + (1 + x^2) R_{42} \right] S_{22}^2,
\]

\[
F_{21} = \frac{(1 + x^2)}{4\sqrt{5}} R_{00} S_{00} - \frac{3}{2\sqrt{14}} \left[ \frac{(1 + x^2)}{\sqrt{6}} R_{20} - xyR_{21} + y^2 R_{22} \right] S_{22}^2
\]

\[
- \frac{1}{4} \left[ 3y^2 R_{42} - 2\sqrt{2}xy \left( \frac{1}{\sqrt{7}} R_{41} - R_{43} \right) + (1 + x^2) \left( \frac{1}{\sqrt{70}} R_{40} + R_{44} \right) \right] S_{22}^2.
\]
\[ F_{Q_1} = \frac{3}{28} (x y Q_{21} - y^2 Q_{22}) S_{22}^2 - \frac{1}{4} \left[ \frac{3}{2} y^2 Q_{43} - \sqrt{2} x y \left( \frac{1}{\sqrt{7}} Q_{41} - Q_{43} \right) + (1 + x^2) Q_{44} \right] S_{22}^4, \]

\[ F_{Q_v} = \frac{1}{2\sqrt{3}} \left[ x y Q_{21} + (1 + x^2) Q_{22} \right] S_{12}^2, \]

\[ F_{u_1} = \frac{1}{\sqrt{3}} \left[ \left( S_{02}^0 - \frac{1}{2} S_{22}^2 \right) y Q_{21} + \left( S_{02}^0 + \frac{1}{2} S_{22}^2 \right) x Q_{22} \right] - \frac{1}{2\sqrt{7}} \left( \sqrt{3} y Q_{21} + \sqrt{3} x Q_{22} \right) S_{12}^2, \]

\[ F_{u_2} = \frac{1}{2\sqrt{5}} \left[ \left( \frac{1}{2} \frac{1}{\sqrt{7}} Q_{41} + Q_{43} \right) + x Q_{44} \right] S_{22}^2, \]

\[ F_{u_3} = \frac{x}{2\sqrt{3}} R_{00} S_{22}^2 + \frac{1}{2\sqrt{7}} \left[ \left( - \sqrt{3} y R_{21} + \sqrt{3} x R_{22} \right) S_{12}^2 + \frac{1}{2} \left( \frac{1}{\sqrt{7}} R_{01} + R_{43} \right) + \frac{1}{\sqrt{3}} R_{00} - R_{11} \right] S_{22}^2, \]

\[ F_{v_1} = \frac{1}{2\sqrt{3}} y Q_{21} S_{12}^2, \]

\[ F_{v_2} = -\frac{1}{\sqrt{3}} \left[ \frac{1}{2} Q_{21} + x Q_{22} \right] S_{12}^2, \]

\[ F_{v_3} = \frac{1}{3} \left[ \frac{1}{2} R_{21} + x R_{22} \right] S_{12}^2, \]

and

\[ F_{v_4} = -\frac{x}{2\sqrt{3}} R_{00} S_{11}^0 + \left( \frac{x}{\sqrt{3}} R_{20} - \frac{y}{2} R_{21} \right) S_{11}^2. \]

The following notations are used:

- \( x = \cos \theta \), \( y = \sin \theta \),
- \( \alpha_0 = - (\alpha_1 + 2 \alpha_2) \), \( \alpha_1 = (\alpha_1 - \alpha_2) \sqrt{2} \),
- \( S_{01}^0 = |\alpha_0|^2 + |\alpha_2|^2 \),
- \( S_{11}^0 = (|\alpha_0|^2 + |\alpha_2|^2/2) / \sqrt{3} \),
- \( S_{22}^0 = (|\alpha_0|^2 + |\alpha_2|^2/2) \sqrt{5}/3 \),
- \( S_{23}^0 = \text{Re}(\alpha_0 \alpha_2^*) \sqrt{2} + |\alpha_2|^2/14 \sqrt{14}/3 \),
- \( S_{02}^0 = \text{Re}(\alpha_0 \alpha_2^*) - |\alpha_2|^2/\sqrt{2} / 3 \),
- \( S_{11}^1 = \text{Re}(\alpha_0 \alpha_2^*) + |\alpha_2|^2/\sqrt{2} / \sqrt{3} \),
- \( S_{12}^2 = - \text{Im}(\alpha_0 \alpha_2^*) \),

and

\[ S_{23}^2 = |\alpha_2|^2 / \sqrt{18}/35. \]

The \( \alpha_0 \) and \( \alpha_1 \) values are determined by the formula (15) with \( \varepsilon_0 \), \( \sigma_1 \), \( \varepsilon_1 \), \( \sigma_2 \) directed along and perpendicular to the grain symmetry axis. This axis is assumed to be directed along the major axis of the tensors \( \varepsilon_{00} \) and \( \sigma_{00} \). For example, for the graphite grain \( \alpha_1 = 0.352 + 0.1 i \) and \( \alpha_2 = 0.32 + 0.681 i \); for the silicate grain \( \alpha_1 = \alpha_2 = 3 + 0.1 i \) (for \( \lambda = 5740 \text{ Å} \) (see Draine & Lee 1984).
The coefficients $R_{Ln}$ and $Q_{Ln}$ are related to the coefficients $P_{Ln}$ from the Appendix by

$$R_{Ln} = \Re \{ P_{Ln} + (-1)^n P_{L-n} \}$$  \hspace{1cm} (18a)

$$Q_{Ln} = \Im \{ P_{Ln} - (-1)^n P_{L-n} \}.$$  \hspace{1cm} (18b)

The quantities $R_{Ln}$ and $Q_{Ln}$ determine the grain alignment. For the randomly oriented particle [$W(h, a) = 1/(8\pi^2)$]

$$R_{Ln} = 4V^2\pi/(2L+1) \int \xi_L(q) Y_{L0}(\theta_q, 0) \, d\theta_q$$  \hspace{1cm} (19a)

$$Q_{Ln} = 0$$  \hspace{1cm} (19b)

$$\xi_L(q) = \frac{1}{2} \int_{-1}^{1} | f(q)|^2 P_L(\cos \theta_q) \, d(\cos \theta_q)$$  \hspace{1cm} (20)

$$f(q) = V^{-1} \int_{(V)} \exp(-iqr) \, dr,$$  \hspace{1cm} (21)

where $f(q)$ is the form-factor of the grain; $q = k - k_0$; $\theta_q$ and $\varphi_q$ are the polar angles of $q$ in the coordinate system $X_0 Y_0 Z_0$; $q = (2\pi/L)(1 - \cos \theta)$; $\theta$, and $\varphi = 0$ are the scattering angles, $\cos \theta = (k_k - k_0)/k^2$; $V$ is the particle volume; $a$ is the symmetry axis of the particle; $b$ is the axis in the perpendicular direction. In this case the formulae for the $F_{ki}$ are identical with those obtained by Dolginov & Silant’ev (1976).

In the extreme case of small particles one can get the simple approximate expressions

$$f(q) = 3(\sin \psi - \psi \cos \psi) \psi^{-3},$$  \hspace{1cm} (22)

where $\psi = q^2/2b^2 + a^2 - b^2 \cos^2 \theta_q$.

In the case of very small or very large $qb$ and $(b/a)$ one can get

$$\xi_L(q) = \int_{-1}^{1} dx \, P_L(x) (1 - 0.2 \psi^2 + (3/175) \psi^4)$$  \hspace{1cm} (23)

for $\psi < 1$

and

$$\xi_L(q) = 4 \int_{-1}^{1} dx \, P_L(x) \psi^{-4}$$  \hspace{1cm} (24)

for $\psi > 1$.

Neglecting the terms of the order of $\psi^2$ and smaller, one obtains

$$R_{Ln} = -Q_{Ln} \eta \frac{\tan \eta \varphi_0}{\sin \varphi_0} = V^2 \Pi_L d_{L0}^0(\theta_h) \cos(\eta \varphi_0),$$  \hspace{1cm} (25)

$$\Pi_L(q) = \int_{-1}^{1} W(\theta_0) P_L(\cos \theta_0) \, d(\cos \theta_0),$$  \hspace{1cm} (26)

and

$$D_l^0(\theta_h, 0) = d_{L0}^0(\theta_h) = \sqrt{4\pi/(2L+1)} \, Y_{L0}(\theta_h, 0),$$  \hspace{1cm} (27)

where $\theta_h$ and $\varphi_0$ are the polar angles of $h$ in the coordinate system $X_0 Y_0 Z_0$; $W(\theta_h)$ determines the distribution of the $\cos \theta_h$, where $\theta_h$ is the angle between the grain symmetry axis $a$ and the external direction $h$ of the grain ensemble alignment.

The $W(\theta_h)$ expression for the grain in gaseous stream is given by formula (35).

If the grain is not small then $R_{Ln}$ and $Q_{Ln}$ are much more complicated

$$R_{Ln} = -Q_{Ln} \cos(\eta \varphi_0) = U_{Ln} \cos(\eta \varphi_0),$$  \hspace{1cm} (28)

$$U_{Ln} = 8\pi \sqrt{(2L+1)/(2k+1)} \, C_{L0}^{\pi} L_0^{\pi} \Pi_k \xi_L(\theta_q, 0) \, Y_k(\theta_h, 0).$$  \hspace{1cm} (29)

Here $\xi_L$ is determined by (21), $\Pi_k$ is determined by (26), $C_{L0}^{\pi} L_0^{\pi}$ is the Clebsh–Gordan coefficient whose explicit form and table of values are given, for example, by Wigner (1959) and Varshalovich (1988). $Y_k(\theta, \varphi)$ are the spherical harmonics.

The above formulae are also applicable to large molecules. In this case $f(q)$ has the form $\int | \Psi(r) |^2 \exp(-iqr) \, dr$, where $\Psi(r)$ is the wave function of the molecule. One must take $\theta = 0$ in the formula for $F_{\ell}(k_w, k)$, if the radiation comes straight to the observer.

© Royal Astronomical Society • Provided by the NASA Astrophysics Data System
3 THE GRAIN ORIENTATION

To get the explicit expression of the \( \Pi \) value one must decide about the mechanism of orientation operating in the medium considered. In interstellar and circumstellar media the grains rotate with high angular velocities. The rotation is due to such processes as collisions with ambient gas atoms, hydrogen molecule formation on the grain surface, photoeffects, non-uniformity of the accommodation ability on the surface area etc. (Hunter & Watson 1978; Purcell 1979). The equilibrium angular velocity may reach \( \Omega = 10^{6} - 10^{8} \text{ rad s}^{-1} \).

Any rotating grain acquires a magnetic moment \( \mu = \chi \Omega (2m_e c / eg) \) because of the Barnett effect. One can say that any rotating object is magnetized as strongly as it would be if at rest in an external field \( B_0 = (2m_e c / \Omega g) \). Here \( m_e \) is the mass and \( e \) the charge of electron, \( V \) is the grain volume, \( \chi \) is the magnetic susceptibility of the grain substance, and \( g = 1 - 2 \) is the gyromagnetic ratio. The field \( B_0 \) for a grain with \( \Omega = 10^{6} - 10^{8} \text{ rad s}^{-1} \) is about 0.05 - 0.5 G, i.e., it is much larger than the typical interstellar field value of \( \approx 3 \times 10^{-6} \text{ G} \). For pure silicate, ice or carbon dust grains with \( V = 10^{-14} \text{ cm}^3 \) and \( \Omega = 10^{8} \text{ rad s}^{-1} \), one gets \( \mu = 10^{-20} - 10^{-21} \text{ erg G}^{-1} \), and for a similar dust grain with paramagnetic inclusions \( \mu = 10^{-18} \text{ erg G}^{-1} \). These estimates show that the magnetic moment of a cosmic grain is mainly caused by the Barnett effect (see Dolginov & Mytrophanov 1978).

In the process of the grain rotation the angular velocity vector \( \Omega \) precesses relative to the grain body. This leads to a continuous reorientation of the electron spins in the body. The reorientation takes some time and leads to the energy dissipation. In turn this leads to some lag in the orientation of \( \mu \) with the \( \Omega \) direction. The energy dissipation leads to orientation of the \( \Omega \) with respect to the direction of angular momentum \( J \) and to orientation of the angular momentum with respect to the grain body axes. If the grain is axisymmetric, with the symmetry axis \( a \), then the process results in \( a \parallel J \) for an oblate and \( a \perp J \) for a prolate grain. This process for a grain with \( V = 10^{-14} \text{ cm}^3 \) and \( \Omega = 10^{8} \text{ rad s}^{-1} \) needs a time less than a year. So we can assume that \( J \) possesses the orientation with respect to the axis \( a \) from the very beginning.

If there is an external field \( B_{\text{ext}} \), then the magnetic moment of the grain also takes part in the precessional motion around the field lines. The characteristic time-scale of precession is \( t_{\text{pr}} = 2\pi J / \mu B_{\text{ext}} \). A dust grain’s precession in the interstellar medium tends to be upset many times during \( t_{\text{pr}} \) since the time interval between collision \( t_{\text{coll}} \) with the surrounding atoms is typically less than \( t_{\text{pr}} \). Although \( t_{\text{pr}} < t_{\text{pr}} \), a sum of fragmentary precessions is sufficient to smear out any alignment of \( J \) except that which is parallel to the magnetic field \( B \).

This will lead to a random distribution of turning angles of the vector \( J \) about the direction of \( B \) for grains in the dust medium. Averaging of the directions \( J \) about the field direction leads to the field being likely to reduce the orientation axis of the grain ensemble even though the orientation mechanism is not dynamically caused by the field (Dolginov & Mytrophanov 1978).

The mechanism of Davis & Greenstein when applied to ‘standard’ interstellar conditions (diffuse H I clouds) requires a magnetic field to achieve alignment which is about one order of magnitude too high compared with other determination. The suprathermal rotation of grains which may occur because of the molecular H\textsubscript{2} formation on the grain surface (Purcell 1979) increases the effective rotational temperature of the grain and provides a better alignment. However, the time necessary to reach a sufficient alignment is too long, \( \gg 10^3 \) s.

The time of alignment in gaseous fluxes is much shorter in many cases. They are an effective mechanism of alignment, especially in stellar vicinities where the stellar wind provides a stationary, well directed flux. Various gaseous fluxes and streams are common in interstellar and circumstellar media. The alignment in a gaseous flux takes place only until the grain velocity differs from the flux velocity. The time \( t_{\text{ga}} \) necessary for the grain to get the flux velocity is longer than the alignment time in many cases. It is the time necessary to transmit the momentum \( m_v V_{\text{gas}} = m_v n_{\text{gas}} V_{\text{gas}} \) of the gas to the grain, i.e., \( t_{\text{ga}} = m_v n_{\text{gas}} V_{\text{gas}} / V_{\text{flux}} \). If \( n_{\text{gas}} = 1 \text{ cm}^{-3}, V_{\text{gas}} = 10^4 \text{ cm s}^{-1} \), the grain having mass of \( 10^{-14} \text{ cm}^3 \) the time \( t_{\text{ga}} = 10^4 \) s. If \( n_{\text{gas}} = 10^3 \text{ cm}^{-3} \) then \( t_{\text{ga}} = 10^{11} \) s.

The time \( t_{\text{ga}} \) of alignment in the gaseous flux can be estimated as follows (Dolginov, Gnedin & Silant’ev 1979)

\[
t_{\text{ga}} = \langle J_{\text{rad}}^2 \rangle \left[ 4 b^2 (\langle P_{\perp}^2 \rangle - \langle P_{\perp}^2 \rangle) + 2 (\langle P_{\perp}^2 \rangle) \right] n_{\text{gas}} V_{\text{gas}} f(q, s)^{-1}
\]

\[
f(q, s) = (1 + s + q) (1 + s \cos^2 \theta_1 + q \cos^2 \theta_2)^{-1}
\]

\[
q = (a^2 - b^2) / 2b^2
\]

\[
s = [3 \langle P_{\perp}^2 \rangle - \langle P_{\parallel}^2 \rangle] [2 \langle P_{\perp}^2 \rangle - 2 \langle P_{\parallel}^2 \rangle + 4 \langle P_{\perp}^2 \rangle]^{-1}
\]

Here \( \langle J_{\text{rad}}^2 \rangle \) is the mean square of the random component of angular momentum which is due to random collisions with atoms of the gas, to H\textsubscript{2} molecular formation on the randomly distributed short lived sites of the grain surface etc. The random collisions lead to \( \langle J_{\text{rad}}^2 \rangle = J_{\text{rad}}^2 k_B T_{\text{gas}} \), where \( J_{\text{rad}} \) is the grain moment of inertia, \( k_B \) is the Boltzman constant, and \( T_{\text{gas}} \) is the temperature of the ambient gas. The random H\textsubscript{2} formation leads to \( \langle J_{\text{rad}}^2 \rangle = 2J_{\text{rad}}^2 E_{\text{rad}} \), where \( E_{\text{rad}} \) is the recoil energy acquired by the grain per an act of H\textsubscript{2} formation. It has been estimated by Hunter & Watson (1978) as \( \approx 0.2 \) eV. The quantity \( a \) in (32) is the length of the grain symmetry axis, and \( b \) is the axis perpendicular to \( a \). \( \langle P_{\perp}^2 \rangle \) and \( \langle P_{\parallel}^2 \rangle \) are the mean square values of momentum and its projection on to the magnetic field that are acquired by a grain in a single collision with an atom of gas. The momentum can be written as \( p = p_0 + p_1 \), where \( p_0 \) is the component of momentum along the flux direction and \( p_1 \) is its random component due to the thermal motion of atoms where \( \langle p_{1,\parallel}^2 \rangle = 3 \langle p_{1,\perp}^2 \rangle, s = -0.5 \) (i.e. \( \cos \beta = 0 \)) if the flux is perpendicular to the field \( \langle p_{\parallel}^2 \rangle = 0 \) and \( \langle p_{1,\parallel}^2 \rangle = \langle p_{1,\perp}^2 \rangle \) and \( s = \langle p_{1,\parallel}^2 \rangle / \langle p_{1,\perp}^2 \rangle \) if the flux is directed along the field.

© Royal Astronomical Society • Provided by the NASA Astrophysics Data System
For example, if \( \langle J^2_{\text{ren}} \rangle = 2 J_{\text{ren}} E_{\text{r}} \), \( E_{\text{r}} = 0.2 \) eV, \( J_{\text{ren}} = 3 \times 10^{-24} \) g cm\(^2\), \( S_{\text{g}} = 2 \times 10^{-9} \) cm\(^2\), \( b = 2 \times 10^{-5} \) cm, \( m_a = 2 \times 10^{-24} \) g, \( \langle p^2 \rangle = 2 \langle p^2_{\text{y}} \rangle = 2 m_a^2 V^2_{\text{g}} \), \( V_{\text{g}} = 2 \times 10^5 \) cm s\(^{-1}\), \( n_a = 100 \) cm\(^{-3}\), and \( f(q) = 1 \) one obtains \( t_{\text{obs}} = 10^4 \) yr.

The solution of the Fokker–Plank equation for the distribution of the grain axes directions in the gaseous medium gives the following stationary distribution:

\[
W(\theta_1, \theta_2) = \text{const}(1 + s \cos^2 \theta_1 + q \cos^2 \theta_2)^{-3/2}
\]

(34)

where const can be determined from the normalization condition \( \int W(\theta_1, \theta_2) d\Omega_1 d\Omega_2 = 1 \), \( \theta_1 \) is the angle between \( J \) and the external magnetic field, and \( \theta_2 \) is the angle between \( J \) and the grain axis of symmetry \( a \). The stationary alignment of \( J \) with respect to the grain body needs a time of a few years, and one can take in all formulae that \( \cos \theta_2 = 0 \) for prolate and \( \cos \theta_2 = 1 \) for oblate grains. In this case one obtains instead of (34) the following expression for the distribution function (Dolginov 1989)

\[
W(\theta_1) = (A/2)(\sqrt{(A + s)/(A + s \cos^2 \theta_1)})^{1/2}
\]

(35)

\( A = 1 \) for prolate and \( A = [(a^2 + b^2)/2b^2] < 1 \) for oblate grains. The \( W(\theta_1) \) is normalized to 1 and is maximum if \( \theta_1 = 0 \) and \( s < 0 \) or if \( \theta_1 = \pi/2 \) and \( s > 0 \). It can be seen from (34) that the isotropic distribution of grain orientations occurs not only in a gas with an isotropic velocity distribution but also in gaseous streams that are directed at angles \( \theta_0 = 55^\circ \) or \( \theta_9 = 125^\circ \) with respect to the magnetic field. In all these cases \( s = 0 \) and \( W_n = (\cos^s \theta_1)/(n + 1)^{-1} \). For the oblate grains \( W_2 \) has the following form

\[
W_2 = A/s \left[ -1 + 0.5 \sqrt{(A + s)/s} \ln \left( \sqrt{A + s + \sqrt{s}} / \sqrt{A + s - \sqrt{s}} \right) \right] s > 0
\]

(36a)

\[
W_2 = -A/s \left[ 1 - \sqrt{(A + s)/s} \sin^{-1} \left( s/\sqrt{(A + s - s)} \right) \right] s < 0.
\]

(36b)

For extremely oblate grains \( (A = 0.5) \) which are situated in a ‘flat’ flux directed perpendicular to the field \( (s = -0.5) \) the \( W_2 = 1 \) is maximum.

For the prolate grains

\[
W_2 = 0.5/s \left[ 1 + s - 0.5 \sqrt{(1 + s)/s} \ln \left( \sqrt{1 + s + \sqrt{s}} / \sqrt{1 + s - \sqrt{s}} \right) \right] s > 0
\]

(37a)

\[
W_2 = -0.5/s \left[ -1 - s + \sqrt{(1 + s)/s} \sin^{-1} \left( s/\sqrt{(-s)} \right) \right] s < 0.
\]

(37b)

The \( \Pi_2 = 0.5(3W_2 - 1) \) and other \( \Pi_n \) are simply related to \( W_n \). The relative contributions of \( \Pi_n \) and \( W_n \) are shown in the Fig. 2.

![Figure 2. The relative value of the \( \Pi_n \) and \( W_n \).](https://academic.oup.com/mnras/article-abstract/254/3/369/1023590)
4 COMPUTED RESULTS

Data about the distribution of gaseous fluxes and magnetic fields in the interstellar and circumstellar gaseous-dust clouds are very uncertain. So is the evidence about the chemical composition, physical structure and geometrical form of the dust grains in the cloud. Nevertheless, the comparison of the predicted values of the linear and circular polarization of starlight with observations allows us to derive some important information about conditions in the clouds and the nature of the grains.

We will give below illustrative examples of the Stokes parameters calculation for some specific cases. The values of $P_L = V/I$ and $P_C = \sqrt{(Q^2 + U^2)/I}$ for various shapes and sizes of the graphite and silicate grains for various angles between the field and the flux are presented in the Figs 3–9.

4.1 Polarization of starlight scattered on dust grains

Linear polarizations $P_L$ of the light scattered in a cloud of oblate ($b/a = 3$) graphite grains aligned by a gaseous flux directed at the angle $\theta_0 = 53.7^\circ$ ($\cos \theta_0 = 0.59$, $s = 0.5$) with respect to the magnetic field is presented in Fig. 3(a). The field is directed at angles $\theta_b = \pi/4$, $\theta_n = \pi/2$ with respect to the incident light direction $k_0$. One can see that $P_L$ can reach 85 per cent if the scattering angle $\theta = 78.5^\circ$ ($\cos \theta = 0.2$). If the particles are small ($b \ll \lambda$) the max$(P_L) = 82$ per cent, if $\theta = \pi/2$.

The circular polarization $P_C$ (see Fig. 3b) decreases for large particles except for some definite angles $\theta$. One can see that the angular dependence of $P_L$ and $P_C$ for large grains is kink. The similar behaviour of polarization for the case of scattering on spherical grains has been shown by Bohren & Huffman (1983, fig. 13.7). This is due to the wave scattered on the edge of the grain. The net polarization of the wave scattered on a single grain is large (see Bohren & Huffman, fig. 5.2). The net polarization becomes much smaller being averaged over sizes and orientation of the grains. The dependence of $P_L$, $P_C$ and $\chi$ on the angles between the gaseous flux and the field $\theta_b$ is given in Fig. 4(a–c) for the case $\theta = \pi/2$, $(r_\text{eq}/\lambda) = 0.02$ [where $r_\text{eq}$ is defined as $r_\text{eq} = (ab^2)^{1/3}$].

If a gas flux propagates along the field and the momenta of atoms are either parallel or antiparallel to the field ($\cos \theta_0 = 1$) then $P_C$ is negative and tends to zero and $P_L$ tends to its maximum value.

If the flux is perpendicular to the field ($\cos \theta_0 = 0$) then $P_C$ is positive and tends to a maximum and $P_L$ tends to its minimum value.

In an isotropic flux ($\cos \theta_0 = 1/3$) $P_C = 0$ and $P_L$ is perpendicular to the plane of scattering $(k_b,k)$.

The dependence of the $P_C$ value on the degree of the grain oblateness is presented in Fig. 5(a–c).

4.2 Zodiacal light

As an example let us consider the polarization of solar light scattered in the zodiacal cloud, where both the dust grain distribution and magnetic field and non-uniform. The solar equatorial plane is the symmetry plane of the oblate zodiacal cloud. Observations show that the dust grain number density in the symmetry plane of the cloud depends on the distance $r$ from the...
Stokes parameters for radiation propagation

Figure 4. The dependence of $P_L$, $P_V$ and $\chi$ on the angle between the gaseous flux and magnetic field.

Sun as $N_R(r) = N_0(R_0/(R_0/r)^n$, here $R_0 = 1$ AU and $n = 1.52$. It is obvious from the geometry of the system that $R_0 \sin \epsilon = r \sin \theta$, where $\epsilon$ is the angle of elongation between the line-of-sight and the line from the Earth to the Sun, and $\theta$ is the scattering angle.

The flux of radiation falling on unit area perpendicular to $R$ is equal to $L_0/(4\pi R^2)$. The Stokes parameters of radiation scattered in a volume $r^2 dr d\Omega$, are equal to

$$\Delta I^0(\epsilon, \theta, \vartheta_b, \varphi_b) = [L_0/(4\pi R^2)] N_0(R_0) F_k(\vartheta_b, \varphi_b) I^0(\sin \theta/\sin \epsilon)^n \sin \epsilon d\epsilon d\varphi d\theta.$$

(38)

Here $r$ is the distance from the Earth to the point of scattering, and $d\Omega = \sin \epsilon d\epsilon d\varphi$ is the element of the solid angle of observation. $I^0(\epsilon)$ are the Stokes parameters of the incident radiation. $\vartheta_b$ and $\varphi_b$ are the local angles of the magnetic field direction in the reference frame $X_0 Y_0 Z_0$. The interplanetary field is approximately a spiral field with the two main component (see Parker 1958). The component $B_\phi$ is small.

$$B_r = B_0(R_0/r)^2 \quad B_\phi = B_0(\Omega/U) r \sin \psi,$$

(39)

where $U$ is the Solar wind velocity, $\Omega$ is the Sun's angular velocity, $r$ is the distance from the centre of the Sun and $\psi$ is the polar angle from the axis of the Sun's rotation.

$\vartheta_b = \tan^{-1}(B_r/B_\phi)$. 

© Royal Astronomical Society • Provided by the NASA Astrophysics Data System
4.3 Polarization of the starlight scattered by circumstellar dust around the star $\alpha$ Ori

Observations of linear polarization of light scattered by circumstellar dust around the red M giant $\alpha$ Orionis have been performed by Le Borgne & Mauron (1989) employing the $V (\lambda_V = 5460 \text{ Å})$ and $B (\lambda_B = 6800 \text{ Å})$ filters. The data obtained are given in Figs 8–9. To apply our results to this case we must decide on the model of circumstellar dust cloud. There are evidence that red giants are sources of silicate grains. It is possible that among dust grains there are graphite grains. The pure silicate grain is isotropic but a dirty one can have some anisotropic properties. The graphite grains are strongly anisotropic and have, as a rule, a shape of the flakes. The interstellar extinction curve suggests that there are at least two kinds of grain in the interstellar space: (a) dielectric, apparently silicate, grains with sizes ($a = \lambda$) of order of the wavelength in visual spectral range and (b) small ($a < \lambda$), apparently graphite grains (Lec & Draine 1985). Assume that both types of the grains are present in circumstellar cloud around $\alpha$ Ori. The radius of the star is about $10^3 R_\odot$ and that of the cloud is about 0.03 pc. The magnetic field in the cloud is unknown. However, only the field direction not the field magnitude is important for us. Even if the field is as weak as the interstellar field, it is strong enough to provide the averaging of the grain orientation around the field direction. The average radial velocity of the

The $\phi_h$, according to observation (Smith & Wolfe 1979), is about $10^5$. The $F_{\lambda 1}$ values are given in (16). If there are oblate ($b > a$) graphite grains then the circular polarization is negative. The dependence of circular $P_v$ polarization on the elongation angle $\epsilon$ for the case ($b/a = 3, (r_{eq}/\lambda) = 0.02, 1, 100$ [where $r_{eq}$ is defined as $r_{eq} = (ab^2)^{1/3}$] is presented in the Fig. 6.
gas outflow from the star is about 15–20 km s\(^{-1}\) (Le Borgne et al. 1986, Paper I). Fluctuations of this value are not less than 5 km s\(^{-1}\). Apparently the flow is formed by a number of streams with various velocities. It is probable that the grain, during its life within the cloud \((10^3–10^4\) yr\), falls in different streams and possesses, practically all the time, a velocity relative to the gas that is much larger than the thermal velocity. So, the flux can provide the grain orientation if the time \(t_g\) required for orientation is less than the time \(t_e\) that the grain spends within the cloud. If the star possesses a magnetic field, it must be expanded to the periphery of the cloud being carried out by the ionized gas outflow. Assume that (analogous to the field in the solar system) the field in the cloud has the geometry of a spiral Parker field (equation 39). At distances much larger than the star's radius the field is practically azimuthal, i.e. perpendicular to the radius vector from the star. In our case (see Fig. 7),

\[
\text{tg} \theta_n = \frac{B_r}{B_\phi} = \frac{Ur}{\Omega} \sin \epsilon \sin \theta \sin \psi,
\]

\(r/\lambda = 0.02\)
\(r/\lambda = 1\)
\(r/\lambda = 100\)

Figure 6. The dependence of the circular polarization of the zodiacal light on the angle of elongation.

Figure 7. The coordinate system which determines the angles for circumstellar dust cloud around the star \(\alpha\) Ori.
\cos \varphi_h \sin \psi = \cos(\varphi_* - \varphi_0) \sin \delta_* ,

(41)
\cos \psi = \cos(\theta - \epsilon) \cos \delta_* + \sin(\theta - \epsilon) \sin \delta_* \cos(\varphi_* - \varphi_0) ,

(42)

where \( \varphi_h \) and \( \varphi_* \) are the polar angles of the field direction \( h \) in coordinate system \( X_0,Y_0,Z_0 \); \( \epsilon \) is the angle between the line-of-sight and direction to the star; \( \psi \) is the polar angle between the axis of the star rotation \( n_0 = \Omega / \Omega \) and \( k_0 \); coordinate system \( X_0,Y_0,Z_0 \); \( \delta_* \) and \( \varphi_* \) are the polar angles of the vector \( n_0 \) in the reference frame \( XYZ \) of observer; and \( \varphi_0 \) is the angle between the plane of scattering and the plane \( XZ \).

The \( B_* \) and \( B_\delta \) are unknown for the \( \alpha \) Ori. All we can do is choose \( B_* \) and \( B_\delta \) in such a way that allows us to get a good agreement with observation of polarization. We will take \( \delta_* = 60^\circ \) and \( \varphi_* = -10^\circ \) that is not a unique but a possible choice.

The dust grain number density in the cloud is assumed to be \( N_r(r) = N_r(R_0) (R_0/r)^2 \), where \( r \) is distance from the star.

Employing this model of circumstellar cloud for numerical calculation of the net linear polarization we obtain, according to the formulae \( (16-27) \), the polarization value at various distances from the star. The values of \( P_g = \langle P_g I \rangle I_* \times 10^6 \sqrt{(Q^2 + U^2)} / I_* \) are given in the Fig. 8. The Stokes parameters \( Q \) and \( U \) are intensities of polarized radiation and, hence, \( P_g \) depends on the distance from the star. The \( Q \) and \( U \) depend also on the grain shape and size and on the relative directions of the gaseous flux and the field. It leads to the different dependence of \( P_g \) on the distance for different wavelengths. The good agreement with observations can be obtained if we assume that there are isotropic, oblate, silicate grains with \( a = 3b \) and \( r_{eq}/\lambda_v = 1 \) [where \( r_{eq} \) is defined as \( r_{eq} = (ab^2)^{1/3} \) in the cloud. Even a significant fraction of small \( r_{eq} < \lambda \) graphite grains do not alter the agreement. The \( P_g \) depends on the distance as \( r^{-2.3} \) for \( \lambda = 5460 \) \( \text{Å} \) (\( V \)-filter), as \( r^{-2.6} \) for \( \lambda = 4320 \) \( \text{Å} \) (\( B \)-filter), and as \( r^{-3.9} \) for \( \lambda = 3540 \) \( \text{Å} \)-

Figure 8. Ratio of the polarized light intensity of the cloud to the light intensity of the star \( \alpha \) Ori plotted as a function of distance from the star. The continuous and dotted lines correspond to decreasing of the polarized intensity for the silicate grains \( (r_{eq} = 0.5 \text{Å}) \) cloud and for the silicate grains \( (r_{eq} = 0.5 \text{Å}) \), 33 per cent) and the small graphite flakes \( (r_{eq} = 0.5 \times 10^{-2} \text{Å}) \), 67 per cent) cloud respectively. Open circles correspond to \( B \) measurement published by McMillan & Tapia (1979); filled squares to \( V \) and filled circles to \( B \), published by (Le Borgne, Maron & Leroy 1986); and open squares to \( V \), published by Borgne & Maron (1989).

Figure 9. Direction of polarization of the light from the dust envelope \( \alpha \) Ori as a function of angular distance from the star for the silicate grains \( (r_{eq} = 0.5 \text{Å}) \), 33 per cent) and the small graphite flakes \( (r_{eq} = 0.5 \times 10^{-2} \text{Å}) \), 67 per cent) cloud. Filled squares correspond to \( V \) measurement, filled circles to \( B \) and open triangles to \( U \), published by Borgne, Maron & Leroy (1986); open squares correspond to \( V \) measurement published by Le Borgne & Maron (1989).
filter). The Stokes parameter $U$ is zero, if the light is scattered on isotropic grains with the size $r_{eq} = \lambda$. The position angle ($\varpi = U/Q$) is also zero. Deviations of the directions of polarization from the perpendicular to the plane of scattering ($\chi = 0$) have been observed by Le Borgne & Mauron (1989). The circumstellar cloud was mapped from 10 to 30 arcsec from the star in four directions (north, east, south, and west). The $\chi = \theta_{in} = 3^\circ$ has been found for east–west (EW) and $\chi = \theta_{in} = -90^\circ = -8^\circ$ for north–south (NS). To explain these deviations we must assume the presence of anisotropic grains that may be either dirty silicate or graphite grains. We assume 67 per cent of small ($r_{eq} = 10^{-2} \lambda$) graphite flakes ($b = 18a$) have their axis of oblateness $a$ along the weakly bound axis of the graphite crystal lattice. 33 per cent of isotropic silicate grains ($a = 3b$ and $r_{eq} = \lambda$) are in the same region. This gives us the mass of silicate. This assumption practically does not alter Fig. 9 and leads to good agreement with observations of the position angles $\chi^{NS}_{\text{lens}} = \theta^{NS}_{\text{in}} - 90^\circ = -7^\circ$ and $\chi^{EW}_{\text{lens}} = \theta^{EW}_{\text{in}} = 2^\circ$.

REFERENCES


APPENDIX

The scattering matrix $t_{ij}$ in equation (5) has the following explicit form

$$t_{ij}(n, n_0, \alpha) = |k|^2 D_{ij}(a, n) \beta_{pq}(a) D_{pq}^{\nu} a, n_0).$$

(A.1)

(The indexes $\alpha, \gamma$ have only two values, $\pm 1$. The density matrix $\rho_{\alpha\beta}(n, n_0, h)$ of scattering on the separate particle has the form

$$\rho_{\alpha\beta}(n, n_0, h | n_1) = k^4 D_{ij}(a, n) D_{pq}^{\nu} a, n) \beta_{pq}(a) \beta_{pq}^{\nu} (a) D_{pq}^{\nu} (a, n_0) \rho_{\alpha\beta}^{(0)}(n_0).$$

(A.2)

Using the properties of the Clebsch–Gordan coefficients and the matrix of rotation one gets:

$$\rho_{\alpha\beta}(n, n_0, h | n_1) = k^4 (-1)^{\mu + \nu + \nu} C_{\alpha 1}^{\mu} C_{\beta 1}^{\nu} C_{\gamma 1}^{\nu} \beta_{pq}^{\nu} (a) \beta_{pq}^{\nu} (a) \rho_{\alpha\beta}^{(0)}(n_0).$$

(A.3)
\[ K(I, J, i, j, L) = (-1)^{i+j} \frac{2i+1}{2j+1} \frac{(2j+1)(2I+1)}{(2J+1)} \begin{bmatrix} 1 & 1 & I \\ 1 & 1 & J \\ i & j & L \end{bmatrix}, \] (A.4)

where \[ \begin{bmatrix} 1 & 1 & I \\ 1 & 1 & J \\ i & j & L \end{bmatrix} \] is the symbol for \(\ell\).

\[ \beta_{pq}(\ell) \] depends on the vector \(q = (k \cdot k_0) = (n \cdot n_0)\). To perform the transformation of the tensor \(\beta_{pq}\beta^*_{pq}\) from the reference frame \(X_0Y_0Z_0\) to the \(X_YZ_0\) one can use the identity

\[ \beta_{pq}\beta^*_{pq} = \sum_{\ell m} Y_{\ell m}(\hat{q}, q_0 | n_0) D_{\ell m}(n_0, a) (A_{pq\ell}^{\infty}(q))_a \] (A.5)

where \(\hat{q}\) and \(q_0 = 0\) are the angles of \(q\) in \(X_0Y_0Z_0\). \(D_{\ell m}(n, a)\) determines the rotation from the frame \(X_0Y_0Z_0\) to \(X_YZ_0\). \(Y_{\ell m}(\theta, \phi)\) are the spherical harmonics. The tensor \(A_{pq\ell}^{\infty}(q)_a\) in the frame \(X_YZ_0\) has the form

\[ (A_{pq\ell}^{\infty}(q)_a) = \int Y_{\ell m}(\hat{q}, q_0 | a) (\beta_{pq}\beta^*_{pq})_a dQ_q. \] (A.6)

In the particular case if \(\beta_{pq} = \alpha_{pq} f(q)\) one has

\[ (A_{pq\ell}^{\infty}(q)_a) = \alpha_{pq}^* a^*_{pq} V^{-2} \int Y_{\ell m}(\hat{q}, q_0 | a) f(q)^2 dQ_q. \] (A.7)

Then

\[ \rho_{\ell m}(n, n_0, h | n_1) = k^4 C_{\ell m}^{LM} \rho_{\ell m}(n_0) H(J, M, I, N), \] (A.8)
\[ H(J, M, I, N) = C_{J M I N}^{\ell m} D_{\ell m}(n_0, n) S_{\ell m}^2 P_{\ell m}, \] (A.9)
\[ S_{\ell m}^2 = K(I, J, i, j, L) g_{ij}^{\ell m}, \] (A.10)

\[ g_{ij}^{\ell m} = (-1)^{\ell+i} C_{\ell m i j}^{\ell m} a_{ij} n^*_a, \] (A.11)

\[ P_{\ell m} = (-1)^{i+j} C_{k_0}^{\ell m} C_{-k_0}^{\ell m} (D_{k_0}(n_0, a)) Y_{\ell m}(\hat{q}, q_0 | n_0) V^2 \int Y_{\ell m}(q' | a) f(q')^2 dQ_q, \] (A.12)

\[ a_{ij} = \Sigma (-1)^i C_{\ell}^{\ell m} a_{ij} n^*_a, \] (A.13)
\[ a_{ij} = \delta_{pq} - (4 \pi i / \omega) d_{pq} - \delta_{pq} / 4 \pi, \] (A.14)
\[ f(q) = V^{-1} \int_{(V)} \exp(-i q r) dr, \] (A.15)

where \(f(q)\) is the form-factor of the grain. The integral is taken over the volume \(V\) of the grain.

The averaging over orientation implies the integration of the \(D_{\ell m}(n_0, a)\) with the weight-function \(W(a | h)\) over the solid angle \(\sin \theta \ d\theta \ d\phi \ d\gamma\) and the angle of rotation \(\gamma\) around the axis of the grain, i.e.

\[ \langle D_{\ell m}(n_0, a) \rangle = D_{\ell m}(k_0, \ h) \int D_{\ell m}(h, a) \sin \theta \ d\theta \ d\phi \ d\gamma, \] (A.16)