Angle variables for numerically fitted orbital tori

James Binney and Sanjiv Kumar
Theoretical Physics, University of Oxford, 1 Keble Roqd, Oxford OX1 3NP

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ABSTRACT
Angle variables are constructed for orbital tori least-squares fitted to general potentials by the method of McGill & Binney. These angle variables enable one to determine the densities \( \rho_j(x) \) associated with the orbit that has given actions \( J \). They also make it possible to treat any non-integrable potential as a perturbation on a nearby integrable one. As an illustration of this approach, Hamiltonian perturbation theory is used to derive the width of the 1:1 resonant-orbit family in a realistic model of the potential of a disc galaxy.

Key words: celestial mechanics, stellar dynamics – galaxies: kinematics and dynamics – galaxies: spiral.

1 INTRODUCTION
The development of a technique for the construction of sufficiently general steady-state stellar systems remains a fundamental problem of dynamical astronomy. In the classical approach to galaxy modelling, one assumes that the distribution function depends on the phase-space coordinates only through classical isolating integrals of motion such as the particle energy \( E \) and the components of the angular momentum vector \( \mathbf{L} \). Unfortunately, we now know that this approach is insufficiently general to be applicable to more than a small minority of galaxies. The other widely used technique is direct \( N \)-body simulation. This approach is wonderfully general, but it does suffer from several serious drawbacks: (i) even when very large numbers of particles are employed, it is plagued by particle noise; (ii) there is no general procedure for tailoring a model to fit a given body of observational data, and (iii) understanding exactly what is going on in an \( N \)-body model can be exceedingly difficult and indeed requires one to draw on analytic or semi-analytic model-building techniques (e.g. Wilkinson & James 1982; Sparke & Sellwood 1987).

These deficiencies of classical and \( N \)-body modelling techniques have stimulated a good deal of work over the last 15 years on alternative techniques. In these alternatives, one generally takes the view that steady-state galaxies are made up of orbits rather than of stars. Numerical experiments show that in many realistic galactic potentials most orbits are either quasi-periodic or nearly so (e.g. Binney & Spergel 1982, 1984). In that case, each orbit of a \( d \)-dimensional galaxy model has the topology of a \( d \)-torus in \( 2d \)-dimensional phase space, and the orbits as a whole form a \( d \)-dimensional continuum. The natural approach to galaxy modelling then starts by surveying this continuum for a given potential and proceeds to the construction of a specific galaxy model by populating these orbits in an appropriate way.

Schwarzschild (1979) pioneered this line of attack. His survey of orbit space was carried out by directly integrating the equations of motion. While this procedure is in principle very straightforward, in practice it requires a high degree of intuition and application because it does not generate any integrals with which to organize orbit space. An important development was de Zeeuw’s (1985) demonstration that precisely the orbit families Schwarzschild had identified numerically for non-rotating barred potentials are supported by the potentials introduced by Stäckel a century ago, for which three classical integrals are known. Thus, in the special case of Stäckel potentials, the necessary survey of orbit space can be carried out analytically and only the population of orbits required to build up a galaxy need be found numerically. Moreover, the integrals of Stäckel potentials provide a natural way of organizing orbit space. Numerous workers have since built models of systems with Stäckel potentials, or ‘Eddington systems’ as they are sometimes called (Bishop 1986; Dejonghe & de Zeeuw 1988; Statler 1988; Evans & Lynden-Bell 1989; de Zeeuw & Hunter 1990; Hunter et al. 1991; Dejonghe & Laurent 1991). Unfortunately, the density distributions of real galaxies are likely to differ significantly from those of Eddington systems because the non-spherical components of the latter must fall with radius as \( r^{-4} \), whereas studies of the rotation curves of galaxies and of the shapes of objects formed in cosmological \( N \)-body simulation suggest that real galaxies have non-spherical components of density that fall roughly as \( r^{-2} \).

What one would like to do is to extend to more general potentials the property enjoyed by Stäckel potentials of having explicitly available orbital tori. McGill & Binney
(1990, hereafter Paper I) introduced such a method and demonstrated its effectiveness in the case of axisymmetric systems. This technique involves numerically determining the generating function of a canonical transformation which maps the orbital tori of some ‘toy’ potential, in practice the isochrone potential, into the phase space of the given galactic potential to form a ‘target’ torus. Proceeding in this way, torus by torus, the galactic phase space can be filled as densely as one pleases with approximate orbital tori. Unfortunately, possession of such tori does not by itself suffice for a galaxy model à la Schwarzschild; in addition to knowing where the tori run in phase space, we need to know how stars are distributed upon them.

The natural labels for the tori are the action integrals $J'_i (i = 1, \ldots, d)$, which are the magnitudes over $2\pi$ of each torus’s $d$ independent cross-sections. The variables canonically conjugate to the $J'_i$, namely the angle variables $\theta'_i$, provide the key to the population of individual tori, since stars are uniformly distributed in the $\theta'_i$ (e.g. Binney & Tremaine 1987, section 3.5.2). Unfortunately, the $\theta'_i$ do not follow immediately from the generating functions $S(J', \theta)$ yielded by the technique of Paper I. The problem is that the direct route from the generating function to the $\theta'_i$ involves differentiating $S$ with respect to $J'$ and, for each value of $J'$, Paper I obtains $S$ independently. Consequently, differences in these independent values of $S$ will be dominated by numerical noise.

In Section 2, we formalize the problem in hand and show that it may be circumvented by solving an infinite set of equations for an infinite number of unknowns. In Section 3, we describe the numerical procedure we have found effective in the approximate solution of these equations. Section 4 illustrates the application of numerically fitted action-angle coordinates to Hamiltonian perturbation theory. Section 5 sums up.

2 ANALYTIC PRELIMINARIES

The generating function which maps the toy action-angle coordinates $(J, \theta)$ into the target variables $(J', \theta')$ is of the form

$$S + \theta J' + s(J', \theta)$$

(1), where $s$ is periodic in each component of $\theta$ with period $2\pi$. The target and toy variables are related to one another by

$$J = J' + \frac{\partial S}{\partial \theta'}$$

(2)

$$\theta = \theta + \frac{\partial S}{\partial J'}.$$ 

Thus the relation between the angle variables involves differentiating the numerically evaluated function $s$ with respect to $J'$, which is in practice inadvisable. Now differentiating the first of equations (2) we have

$$\left(\frac{\partial J}{\partial J'}\right)_\theta = \delta_{ij} + \frac{\partial^2 s}{\partial \theta_i \partial \theta_j}.$$ 

(3)

Hence the vector of frequencies of motion on the target torus, $\omega' = \partial H/\partial J'$, is given by

$$\omega'_i = \left(\frac{\partial H}{\partial J'}\right)_{\theta_j} = \left(\frac{\partial H}{\partial \theta'_i}\right)_{J'_j}.$$ 

(4)

Equation (4) is a linear partial differential equation for $\theta'(\theta)$. Since we seek a solution such that $\theta' - \theta$ is a periodic function of $\theta$, it is natural to Fourier expand everything. We write

$$\frac{\partial H}{\partial J} = \sum_n h_n e^{in\theta},$$

(5)

$$s = -i \sum_n S_n e^{in\theta},$$

where $n$ is a vector with integer components. Paper I specifically determines the $S_n$ and in doing so repeatedly evaluates $\partial H/\partial J$ all round the target torus. Consequently, the $h_n$ are available for the trivial extra cost of taking a discrete Fourier transform (DFT). Fourier transforming equation (4), whose left side is independent of $\theta$, we obtain with this notation the infinite set of equations

$$\omega' = h_0 + \sum_n (h^*_n \cdot n) \frac{\partial S_n}{\partial J'_n},$$

$$0 = h_1 + \sum_n (h^*_n \cdot n) \frac{\partial S_n}{\partial J'_1},$$

(6)

$$0 = \ldots,$$

$$0 = h_n + \sum_n (h^*_n \cdot n) \frac{\partial S_n}{\partial J'_n},$$

$$0 = \ldots.$$ 

Since the $h_n$ are known quantities, the set formed by these equations, less the first, forms an infinite set of equations for the $\partial S_n/\partial J'$ in terms of the $h_n$. We postpone until Section 4 discussion of whether they can in fact be solved, and simply assume that they can. Once their solution is known, the first equation can be used to determine $\omega'$. The vector notation used in (6) has the disadvantage of obscuring the fact that the equations, less the first, actually form $d$ independent sets of equations, one set for each set of numbers $\partial S'_n/\partial J'_i$ with $i = 1, \ldots, d$ fixed.

3 NUMERICAL IMPLEMENTATION

Since the equation sets defined by (6) in principle contain an infinite number of equations in an infinite number of unknowns ($\partial S'_n/\partial J'_i$ for all $n$ on an infinite $d$-dimensional lattice), one must in practice select suitable subsets of unknowns and equations for solution. Valuable insight into how this should be done can be obtained by studying equations (6) for the case of a Stäckel potential. In this case, it is not necessary to resort to least-squares minimization to obtain $J$ and $\partial H/\partial J$ on a grid of values of $\theta$ for the torus $J'$ (Paper I, section 4). Hence the $h_n$ are readily calculated and

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the $\partial S_n/\partial J'$ may be evaluated with modest extra effort as described in the Appendix. It is not hard to obtain values of $\partial H/\partial J$ and $\partial J/\partial J'$ (from which the $\partial S_n/\partial J'$ are calculated) accurate to five or six significant figures. Unfortunately, the values of $h_n$ and $\partial S_n/\partial J'$ obtained by taking the DFTs of $\partial H/\partial J$ and $\partial J/\partial J'$ contain errors of the order of a per cent; these errors are dominated by the difference between a true Fourier integral and a DFT. They can be reduced only by increasing the number of sampling points on the torus. The results reported here were obtained with $32 \times 32$ points per torus.

In what follows, we concentrate on meridional-plane orbits. The two material actions are then the radial action $J$, and the latitudinal action $J'$. Moreover, we have

$$S_n = -s_n, \quad h_n = h_n.$$  \hspace{1cm} (7)

Equations (6) can therefore be written

$$\sum_{n>0} M_{n,k} \frac{\partial S_n}{\partial J'} + h_k = 0 \quad (k \neq 0).$$  \hspace{1cm} (8a)

Here $n > 0$ implies summation over just half of $n$-space, the components of the vectors $J'$ and $h$ should match, and

$$M_{n,k} = (h_{n-k} + h_{n+k}) \cdot n.$$  \hspace{1cm} (8b)

Fig. 1 shows what happens when one compares the $\partial S_n/\partial J'$ obtained from the $h_n$ by solving equation (8a) with the values calculated in the Appendix. (The orbit is that shown in fig. 8 of Paper I.) The Appendix values are shown by the vertices of the curves ($\partial S_n/\partial J'$, lower curve, $\partial S_n/\partial J'$, upper curve) ordered by decreasing absolute magnitude. The estimates from equation (8a) are shown by squares and triangles; full symbols show the results of using 80 equations and open symbols those obtained with 50 equations. It will be seen that, though the first few coefficients from (8a) lie reasonably near the corresponding vertex, the last third of the coefficients are frequently orders of magnitude larger than they should be. These wildly inaccurate coefficients occur, for smaller $n$, sooner with only 50 equations than they do with 80 equations, but the improvement between 50 and 80 equations is small in relation to the $\sim (8/5)^3$ increase in the cost of solving the equations.

A change in the number of points sampled on the torus prior to Fourier transforming demonstrates that the mischief in Fig. 1 is generated by errors associated with the use of DFTs in place of true Fourier transforms. Since the $h_n$ obtained by least-squares fitting will inevitably contain errors in addition to those derived from the use of DFTs, there seems little point in seeking another method of taking transforms; it is essential to be able to solve equation (8a) in the presence of significant noise.

One solution to this problem is to feed into the system one’s prior knowledge that the $\partial S_n/\partial J'$ trail off for large $|n|$. This one can do by choosing the $\partial S_n/\partial J'$ which minimize the sum of the squares of errors in the satisfaction of equation (8a) plus the sum of the squares of appropriately renormalized $\partial S_n/\partial J'$. In symbols, we seek the $\partial S_n/\partial J'$ which minimize

$$\chi^2 = \sum (M' x + y)^2 + x \cdot \Xi \cdot x,$$  \hspace{1cm} (9a)

where

$$x_n = \partial S_n/\partial J', \quad y_n = h_n, \quad (i = r, l),$$

$$\Xi_{nm} = \tilde{\xi}_{nm} \delta_{nm}.$$  \hspace{1cm} (9b)

**Figure 1.** Estimates for the Stäckel orbit shown in fig. 8 of Paper I of $\partial S_n/\partial J'$ obtained both by the technique given in the Appendix and from (8a). The estimates from the Appendix are shown by the vertices of the lower and upper curves for $i = r$ and $i = l$, respectively. (These fall monotonically because by definition the scalar index $n$ orders $\partial S_n/\partial J'$ from the Appendix by decreasing absolute value.) Estimates from (8a) are shown by squares ($i = r$) and triangles ($i = l$); filled symbols correspond to the use of 80 equations and open symbols to 50 equations. The equations were ordered by the sum of the absolute values of their terms, with $\partial S_n/\partial J'$ taken from the Appendix.
Here the $\xi_a$ are some suitable normalizing constants. It is easy to show that the desired vector $x$ satisfies
\begin{equation}
[[M^T \cdot M] + \Xi] \cdot x = -y \cdot M.
\end{equation}

Fig. 2 shows the results of solving (10) with the same data as were used to obtain Fig. 1. Again the lower curve shows the values of $\partial S_\theta / \partial J_r$ from the Appendix and the upper curve shows the corresponding values of $\partial S_\theta / \partial J_r$. The filled points show the values obtained by solving (10) truncated to 80 equations, with
\begin{equation}
\xi_a = (10^{-n})/\sqrt{\Xi + (\partial S_a / \partial J_r)^2}.
\end{equation}
Here the radical is evaluated using the Appendix, and $n$ gives the rank ordering of coefficients derived from the Appendix and used as the ordinate in Figs 1-3. In the left-hand half of Fig. 2, each filled point now lies close to its vertex, and in the right-hand half of the figure, where filled points sometimes stray some distance from their vertices, they tend to lie below their vertices. This is good because it is much better to underestimate a poorly determined quantity than to overestimate it, as frequently happens in Fig. 1. A useful measure of the error in the solution to equations (10) is provided by the quantity
\begin{equation}
\bar{e} = 100 \frac{\sum_a |x_a - x_a^{(0)}|}{\sum_a |x_a^{(0)}|},
\end{equation}
where $x_a^{(0)}$ denotes values of $\partial S_a / \partial J_r$ from the Appendix. The filled points in Fig. 2 yield $\bar{e} = (2.9, 3.1)$ for the radial and latitudinal equations, respectively. The first of equations (6) returns frequencies $(\omega_r, \omega) = (0.6607, 0.5372)$, compared with the exact values $(0.6607, 0.5365)$.

Clearly, determination of $\xi$ with the help of the Appendix is in general unacceptable because one does not usually know the values of the $\partial S_a / \partial J_r$ in advance of solving equation (10) for them. However, one only really needs to know the values of the $\partial S_a / \partial J_r$ accurately enough to be able to order them by magnitude and to estimate which equations are most important. A guess that produces satisfactory results when used in (12) is
\begin{equation}
\frac{\partial S_\theta}{\partial J_r} = h_a/[(1 + \omega) [1 + (k_r + 1 - k_l/2)^2]]
\end{equation}
\begin{equation}
+ h_b/[(1 + (k_r + 1 + k_l/2)^2)].
\end{equation}
This fact is illustrated by the open symbols in Fig. 2, which show the values of $\partial S_\theta / \partial J_r$ obtained when $\xi$ is determined by replacing the Appendix values for $\partial S_a / \partial J_r$ in (11) with the rough estimate (13). This replacement increases $\bar{e}$ to $\bar{e} = (5.0, 8.9)$. The first of equations (6) now returns frequencies $(\omega_r, \omega) = (0.6607, 0.5389)$. Thus, while the scatter of the points in Fig. 2 does increase appreciably, the overall accuracy of the results remains quite good.

Fig. 3 shows the same quantities as Fig. 2 except that now the $h_a$ are obtained by least-squares fitting of a generating function as in Paper I. The two figures are remarkably similar, demonstrating that the largest source of error in solving the equations is in the choice of preliminary estimates of the $\partial S / \partial J_r$.

How does this machinery cope with an orbit of real astronomical interest, such as the orbit in the meridional plane of a flattened singular logarithmic potential shown in fig. 1o(b) of Paper I? The upper panel of Fig. 4 shows the time evolution along this orbit of the true angle variables $\theta_r$. These do not increase linearly in time, reflecting the fact that they are not the true angles. The points in the lower panel of the figure show the corresponding plot for the ‘true’ angle variables $\theta_r$ obtained by solving 80 of equations (10) with $\xi$.

**Figure 2.** The curves are the same as those of Fig. 1. The symbols are obtained by solving 80 equations of the set (10) with $\xi$ given by equation (11). The full symbols are obtained when the values of $\partial S_a / \partial J_r$ used in this equation derive from the Appendix. The open symbols are obtained when $\partial S_a / \partial J_r$ is evaluated from equation (13).
determined with the aid of (13). Now all points lie close to one of the straight-line segments of the figure, which show the trajectory \( \theta'(i) = \theta'(0) + \omega' i \), where \( \omega' \) has been obtained from the first of equations (6).

4 CONNECTION WITH PERTURBATION THEORY

This paper completes the job of tailoring a set of action-angle variables from an arbitrary Hamiltonian. Once a dense set of tori is to hand, one has in effect constructed an integrable Hamiltonian \( \tilde{H} \); the value of \( \tilde{H} \) at any point \((p, q)\) in phase space is the mean value of the originally prescribed Hamiltonian \( H \) on the torus through that given point.

\( \tilde{H} \) behaves like a separable Hamiltonian in that (i) all its orbits are quasi-periodic, and (ii) its tori form a single nested sequence. Consequently, surfaces of section for \( \tilde{H} \) have invariant curves which all wrap around a single closed orbit; the surfaces of section contain no resonant islands. It is interesting to compare this simple structure with the richer structure that may be generated by the original Hamiltonian \( H \).

The dots in Fig. 5 show some invariant curves for the Miyamoto & Nagai (1975) potential,

\[
\Phi_\mu = -\frac{GM}{\sqrt{R^2 + (a + \sqrt{2^2 + b^2})^2}},
\]

with \( b/a = 0.45 \); \( \Phi_\mu \) is generated by a density distribution that resembles the luminous component of an early-type disc galaxy remarkably closely (e.g. Binney & Tremaine 1987, section 2.2.1). All invariant curves in Fig. 5 are at a common energy \(-0.39GM/a\) and squared angular momentum \( L_\mu^2 = 0.075GMa \). The outermost curve is that of the orbit which lies in the equatorial plane, while the innermost is the orbit reaching furthest above and below this plane. A tight cluster of dots near \((r, \nu_r) = (1.2, 0.45)\) encircles the closed orbit associated with the 1:1 resonance between the vertical and the radial oscillations – variously called a ‘tube orbit’ (Ollongren 1965), a ‘reflected banana’ (Lees & Schwarzschild 1992) or a ‘saucer orbit’, since in three-dimensional space it looks like a saucer with a hole bored through its centre. This cluster of dots is surrounded by two further invariant curves of orbits trapped by the resonance, the outer of which lies close to the boundary of the long, thin, curved island that these orbits form.

The five full curves in Fig. 5 mark the intersection of five numerically constructed tori with the surface of section. The central curve of this set is seen to slice straight through the resonant island. Does Hamiltonian perturbation theory allow one to consider the island as arising from resonant trapping of orbits on the tori by the small difference \( \delta H = H - \tilde{H} \) between the Miyamoto & Nagai Hamiltonian and that arising from the tori?

In standard secular perturbation theory (e.g. Arnold 1978), one handles a 1:1 resonance by transforming to new actions and angles:

\[
\begin{align*}
\psi &= \theta_r - \theta_p, \\
J_r &= J_r - J_p, \\
\theta_r &= \theta_r - \theta_p.
\end{align*}
\]

\( \theta_r \) circulates rapidly, with the result that its conjugate action \( J_r \) is approximately constant, while \( \psi \) varies slowly and its conjugate action \( J_\psi \) changes significantly. The motion of the star in the \((J_\psi, \psi)\) plane is governed by the approximate Hamiltonian

\[
H(J', \theta') = \tilde{H}(J') + \sum_m h_{m, -m} e^{im\psi}.
\]

A resonantly trapped star moves in the \((J_\psi, \psi)\) plane around an island. Since this motion has to be at constant \( H \), by expanding \( \tilde{H} \) in powers of \( J_p \) and recalling that on the reson-
next-largest resonant term is $h_{(2,-2)} = 1.9 \times 10^{-5}$. From the outermost curve inwards, the frequencies $\omega_\nu$, recovered for the tori plotted as full curves in Fig. 5 are $\omega_\nu = (-0.0853, -0.0306, 0.0004, 0.0255, 0.0520)$. Thus $\omega_\nu$ passes through zero almost exactly on the torus that cuts through the closed resonant orbit in $H$. Numerically differencing these values of $\omega_\nu$, we find $\delta_\nu = 1.1 \pm 0.15$. With these numbers, equation (17) yields a half-width $\delta J = 0.015$. For comparison, the tori that hug the untrapped orbits on either side of the resonance in Fig. 5 have $J_\nu = 0.161$ and $J_\nu = 0.202$, respectively, while the torus that passes through the resonant island has $J_\nu = 0.182$. Since the resonant island’s half-width in Fig. 5 is significantly less than the half-width $\delta J = 0.02$ of the gap between these non-resonant orbits, we conclude that Fig. 5 is in excellent agreement with secular perturbation theory. This is encouraging, as our application of the latter has employed both the frequencies and the angle variables of the numerically recovered tori.

It is perhaps worth noting that, as one proceeds in Fig. 5 from the smallest to the largest full curve, the rms variation in $H$ around the corresponding torus decreases monotonically irrespective of the presence of the resonant island. Thus, on the island, orbits in $H$ deviate significantly from the numerically constructed tori not because the fluctuations in $H$ are unusually large on the local tori, but because for these tori $\omega_\nu$ is small; when $\omega_\nu$ is small, fluctuations act in one direction for extended periods and can trap the star.

Very few integrable Hamiltonians are known, and it is important to examine closely the claim made here that the present technique enables one to find an integrable Hamiltonian $\tilde{H}$ which closely approximates any Hamiltonian $H$ of a specified general form. First note that nothing can prevent the torus-fitting procedure of Paper I producing a function $s(J_\nu, \theta)$ and a corresponding two-parameter family of tori with actions $J_\nu$. As long as equations (2) can be solved for the primed variables in terms of the unprimed ones, these tori will be images of null tori under a bona fide canonical map. Consequently, they will themselves be null tori and the average value $\overline{H(J)}$ of the given Hamiltonian $H$ on each torus will uniquely define an integrable Hamiltonian for which the target tori are invariant tori.

Our ability to tailor integrable Hamiltonians therefore stands or falls by the solubility of equations (2). In particular, $s$ must be differentiable. Differentiability with respect to $\theta$ is guaranteed by the construction. Only differentiability with respect to $J_\nu$ can be in doubt. We can take as many or as few terms in the Fourier expansion of $s$ as we like, so we need not worry about the convergence of series. Moreover, since all the functions $H(x, v)$, $J(x, v)$, etc., in the problem are arbitrarily differentiable, one feels that infinitesimal changes in the target actions $J_\nu$ must induce infinitesimal changes in $s(J_\nu, \theta)$. Consequently, it seems likely that $s$ is differentiable. If this is so, equations (2) will be solvable unless two image tori $J_\nu$ for different values of $J_\nu$ somewhere intersect; in this case, at a point of intersection two values of $(J_\nu, \theta)$ will correspond to a single value of $(J, \theta)$ and the transformation (2) will be undefined.

It is hard to be certain that image tori generated by the method of Paper I do not intersect, but at this stage one can say that (i) in numerous trials with a number of flattened, axisymmetric potentials, we have not found intersecting tori to be a problem, and (ii) one is most likely to be plagued by this

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Figure 4. The trajectory in two angle-angle planes of a star on the orbit shown in Fig. 10(b) of Paper I in a flattened logarithmic potential. The upper panel shows the trajectory in the plane of the toy angles, while the lower panel shows the motion in the plane of the ‘true’ angles derived from equation (4).
Figure 5. The dots show the invariant curves of seven orbits of a common energy and angular momentum in the Miyamoto & Nagai potential (14) of a disc–bulge galaxy. The full curves show the intersections of numerically generated tori with this surface of section. \( r \) is in units of the parameter \( a \) of the potential and \( \nu \), is in units of \((GM/a)^{1/2}\).

4.1 Does \( \mathcal{H} \) derive from a potential?

It would be very convenient if the integrable Hamiltonian \( \mathcal{H} \) derived from a potential, that is, if \( \mathcal{H} \) were of the form \( p^2/2m + \Phi(x) \). We can explore this possibility by evaluating the difference Hamiltonian \( \delta \mathcal{H} = \mathcal{H} - \bar{\mathcal{H}} \) for several different values of \( p \) at a given \( x \); if \( \Phi \) exists, \( \delta \mathcal{H} \) will be independent of \( p \). In Fig. 6, the values of \( \delta \mathcal{H} \) at eight different values of \( p \) are shown for each of 15 points \((R, z)\) in the meridional plane of the Miyamoto & Nagai potential examined in Fig. 5. While the spread in the values of \( \delta \mathcal{H} \) at a given point \( x \) is always smaller than the full range of variation of \( \delta \mathcal{H} \), it is clear that \( \delta \mathcal{H} \) does have non-negligible dependence on \( p \). We conclude that \( \Phi \) does not exist.

5 CONCLUSIONS

We have presented a general technique for recovering the frequencies and angular variables associated with tori least-squares fitted to any given Hamiltonian by the method of Paper I. This technique has been applied to orbits in three flattened, axisymmetric potentials: a Stäckel potential in which all orbits are precisely integrable, a scale-free logarithmic potential like that generated by a flattened heavy halo, and the disc–halo potential of Miyamoto & Nagai (1975).

The frequencies and angles are recovered by solving what is in principle an infinite set of linear equations. In practice, one can solve a truncated subset of these equations only if
one has a rough idea of what the solution vector looks like. For the Stäckel potential, the required information was available from other sources and it was subsequently found possible to solve the equations for the other two potentials under the assumption that their equations are structured in the same way as those of the Stäckel potential.

The quantities recovered here can be used in two rather different ways. A straightforward application is to recover the real-space density $\rho_J(x)$ associated with the torus $J$ by exploiting the fact that the probability of finding a star on a torus is uniform in the angle variables. In future publications, we plan to use this principle to obtain the star densities and line-of-sight velocity dispersions of realistic models of the halo of the Milky Way. However, the most cost-effective route to orbital densities, if these are all one wishes to know, is likely to be integration of the equations of motion from randomly chosen starting points on the tori recovered in Paper I. In this hybrid technique, one would be using the torus-fitting machinery only to label orbits with action integrals with a view to weighting each orbit according to a distribution function $f(J)$. In other respects, model-building would proceed along the lines pioneered by Schwarzschild (1979).

A more subtle application of the technique developed in this paper is to use the tori as a basis for Hamiltonian perturbation theory. The latter has long provided the standard framework for celestial mechanics, and via the epicycle approximation has played a key role in studies of cold stellar discs. It has to date, however, been comparatively little used in studies of hot stellar systems, although Dehnen & Gerhard (1993) have recently constructed some axisymmetric models whose distribution functions depend on approximate integrals obtained by the application of first-order perturbation theory to the total angular momentum integral of spherical systems. Undoubtedly, a significant obstacle to the application of perturbation theory to a wider range of systems has been the want of suitable integrable Hamiltonians around which to perturb. Under certain plausible assumptions, the approach to modelling orbits developed in these papers amounts to a technique for recovering an integrable Hamiltonian $\tilde{H}$ which closely approximates a given Hamiltonian $H$, thus providing a secure base from which to apply perturbation theory than the spherical Hamiltonians employed by Dehnen & Gerhard (1993). $\tilde{H}$ will in general be non-integrable and thus support both resonant orbits and a degree of stochasticity. In the case of the Miyamoto & Nagai potential, we found that an important island of resonant orbits could be understood by treating the actual Hamiltonian as a perturbation on $\tilde{H}$. In particular, $\tilde{H}$ has a resonance precisely on the torus which cuts through the middle of the island of resonant orbits supported by $H$. The appropriate Fourier component of the residual Hamiltonian $\delta H = H - \tilde{H}$ is of just the magnitude required to explain the size of the island. $\delta H$ appears not to be derivable from a potential $\delta \Phi(x)$.

In these papers, we have concentrated on flattened, axisymmetric potentials. In the integrable case, such potentials support only one orbit family, and the integrable Hamiltonians $\tilde{H}$ introduced here centre on this family. It will be interesting to see whether the torus-mapping technique is equally applicable to potentials which support more than one major orbit family in the integrable limit. Preliminary work suggests that this is, in fact, the case. If these hopes are borne out, the present small study may open the way to the application of Hamiltonian perturbation theory to a wide range of phenomena associated with fragile resonances in realistic galactic potentials. There seems reason to hope that this development will significantly improve our ability to model such phenomena as moving star clusters in the Milky Way and rings, shells and other fine structure in early-type stellar systems.

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APPENDIX: EVALUATING $\partial S_p/\partial J'_f$ FOR A STÅCKEL POTENTIAL

From equations (1) to (5), it follows that the $n_i(\partial S_p/\partial J'_k)$ are the Fourier expansion coefficients of $(\partial J_i/\partial J'_k)_p$. For a Stäckel potential, one now has analytic expressions for $J(J', \Theta)$ at each point around the torus $J'$ (Paper I, section 4). Thus, in principle, the required derivative can be taken numerically. However, when one constructs Stäckel tori without resort to least-squares-fitting, the independent parameters are not the $J_i$ but the integrals $I_1 = H_1, \ldots, I_d$ that arise when separating the Hamilton–Jacobi equation. It is necessary, therefore, to extract the required derivatives from

$$\left(\frac{\partial J_i}{\partial J'_j}\right)_\Theta = \sum_{k=1}^d \left(\frac{\partial I_k}{\partial J'_j}\right)_\Theta \left(\frac{\partial I_k}{\partial J_i}\right),$$

(A1)

where the subscript $J$ indicates that the $I_i$ other than $I_j$ are to be held constant, and similarly for the $J'$ subscript. The first partial derivative on the right of this equation is readily evaluated by numerical differencing. The second requires some thought because it is not numerically practicable to hold constant the $J'_i$, while varying the $I_j$.

For simplicity confining ourselves to the case $d = 2$ of two-dimensional orbits, we write

$$dJ'_i = \left(\frac{\partial J'_i}{\partial I'_1}\right)_{I'_2} dI'_1 + \left(\frac{\partial J'_i}{\partial I'_2}\right)_{I'_1} dI'_2.$$  

(A2)

Dividing this through in turn by $dJ'_1$ and $dJ'_2$ and solving the resulting equations for $(\partial I'_1/\partial J'_2)_{I'_1}$ and $(\partial I'_2/\partial J'_1)_{I'_2}$, we find

$$\left(\frac{\partial I'_1}{\partial J'_2}\right)_{I'_1} = \frac{1}{\Delta} \left(\frac{\partial J'_2}{\partial I'_1}\right)_{I'_1},$$

(A3a)

$$\left(\frac{\partial I'_2}{\partial J'_1}\right)_{I'_2} = -\frac{1}{\Delta} \left(\frac{\partial J'_1}{\partial I'_2}\right)_{I'_2},$$

(A3b)

where

$$\Delta = \frac{\partial J'_1 J'_2}{\partial (I'_1 I'_2)}.$$  

(A3c)

Equation (A3a) gives us the remaining partial derivatives needed to calculate $(\partial J/\partial J')_p$ from (A1).