On the Law of Force to any Point in the Plane of Motion, in order that the Orbit may be always a Conic.

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§ 1. In a paper "Sur la possibilité de déduire d'une seule des lois de Kepler le principe de l'attraction," printed in the Comptes Rendus, t. 84, pp. 671–674 (April 9, 1877), M. Bertrand remarks that it would be interesting to solve the following problem: "En sachant que les planètes décrivent des sections coniques, et sans rien supposer de plus, trouver l'expression des composantes de la force qui les sollicite, exprimées en fonction des coordonnées de son point d'application," and adds, "Nous connaissons deux solutions: La force peut-être dirigée vers un centre fixe et agir proportionnellement à la distance, ou en raison inverse de son carré. En existe-t-il d'autres?"

The problem was completely solved by M. Darboux (Comptes Rendus, t. 84, pp. 760–762 and 936–938, April 16 and 30), and M. Halphen (Ibid., pp. 939–941), who each proved that a body moving in the plane of xy under the action of a central force, equal to either

$$\frac{\mu r}{(ax + by + c)^2} \cdot \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (1)$$

$$H$$
or
\[ \frac{\mu r}{(ax^2 + bxy + cy^2)^{\frac{3}{2}}} = \ldots \ldots \ldots (2) \]

must describe a conic whatever be the initial conditions of the motion, and that there are no other laws of force for which this is true.

It is also shown that all the conics described under the action of the law (1) have \( ax + by + c = 0 \) as polar of the origin, and those described under the action of (2) have \( ax^2 + bxy + cy^2 = 0 \) as tangents (real or imaginary).

M. Darboux obtained his results by substituting the value of \( u \) given by the general equation of a conic,
\[ u = a \cos \theta + b \sin \theta + \sqrt{(A \cos 2\theta + B \sin 2\theta + H)}, \]
in the differential equation
\[ \frac{d^2 u}{d\theta^2} + u = \frac{P}{\frac{1}{2}m^2}; \]
while M. Halphen's process, which is very elegant, depends upon the differential equation of the fifth order satisfied by conics.

If the motion is not restricted to take place in the plane of \( xy \), the laws of force are
\[ \frac{\mu r}{(ax + by + cz + d)^{\frac{3}{2}}} \]
and
\[ \frac{\mu r}{(ax^2 + a'y^2 + a'z^2 + 2byz + 2b'zx + 2b''xy)^{\frac{3}{2}}} \]

and, as remarked by M. Halphen, in the former case all the polars of the centre of force with regard to the conics lie in the plane \( ax + by + cz + d = 0 \), and in the latter case all the conics have double contact with the cone
\[ ax^2 + a'y^2 + a'z^2 + 2byz + 2b'zx + 2b''xy = 0. \]

In what follows I shall, for the sake of simplicity, only consider motion in the plane of \( xy \); it is evident that the extension to the case of motion in three dimensions presents no difficulty.

§ 2. Newton, in the scholium to Prop. xvii. of the Principia, finds the law of force tending to any point in order that any given conic may be described about it; and I now proceed to consider the expression for the force given by Newton, and to connect it with the results obtained by MM. Darboux and Halphen.

The law of force in the scholium to Prop. xvii. is deduced by Newton from Corollary 3 to Prop. vii., in which it is shown that the force under the action of which a body \( P \) revolves in any
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The orbit about any centre of force C is to the force under the action of which it can revolve in the same orbit in the same periodic time about any other centre of force O as OP: CP to CG, CG being drawn parallel to OP and cutting the tangent at P in G.

Now let C be the centre of the conic, so that the law of force is \(\mu r\); then (fig. 1)

\[
\text{force to } O = \frac{CG}{CP}, \quad \text{force to } C = \frac{OP}{OP}
\]

therefore

\[
\text{force to } O = \mu \left(\frac{CG}{OP}\right)^3 OP,
\]

which is the law of force given in the scholium to Prop. xvii.

If CY and OZ are the perpendiculars let fall upon the tangent at P, we see that

\[
\text{force to } O = \mu \left(\frac{CY}{OZ}\right)^3 OP.
\]

Now if UV is the polar of O, and CY', PZ' are perpendiculars let fall upon it,

\[
\frac{CY}{OZ} = \frac{CY'}{PZ'}
\]

so that

\[
\text{force to } O = \mu \left(\frac{CY'}{PZ'}\right)^3 OP
\]

\[
= \mu \left(\frac{p_0}{P}\right)^3 r,
\]

\[\text{H 2}\]
and $p_o$ denoting the perpendiculars from $P$ and the centre upon the polar of $O$.

Thus any conic may be described under the action of a force $\frac{\mu r}{p^3}$ tending to $O$; and since the periodic time is the same for the force $\mu r$ to $C$ and $\mu \left(\frac{p_o}{p}\right)^3$ to $O$, it is equal to $\frac{2\pi}{\sqrt{\mu}}$; and therefore when the force is $\frac{\mu r}{p^3}$, the periodic time is $\frac{2\pi}{\sqrt{\mu}}p_o^3$.

In the case of the parabola the centre passes to infinity, so that $p_o$ is infinite; but if we diminish $\mu$ so that $\mu p_o^3 = \mu'$, the force to $O$ is still $\frac{\mu' r}{p^3}$, the extension to the case of the parabola being similar to that in the scholium to Prop. x. of the Principia. But the law of force to any point $O$ of a parabola can be deduced directly from that to the focus without difficulty. This is done in § 5.

§ 3. The equation (3), viz. that

$$\frac{CY}{OZ} = \frac{CY'}{FZ'}$$

is easily proved geometrically, for if $PL$ (fig. 2) be drawn parallel to the polar $UV$, then

![Fig. 2](https://via.placeholder.com/150)

perpendicular from $O$ on tangent at $P$ = $OT$
perpendicular from $C$ on tangent at $P$ = $CT'$
perpendicular from $P$ on polar of $O$ = $LN$
perpendicular from $C$ on polar of $O$ = $CN'$

and

$$\frac{OT}{CT} = \frac{LN}{CN'}$$
for
\[ CO \cdot CN = CR^2 = CL \cdot CT, \]
giving
\[ \frac{CO}{CT} = \frac{CL}{CN}; \]
and therefore
\[ \frac{CT - CO}{CT} = \frac{CN - CL}{CN}, \]
that is
\[ \frac{OT}{CT} = \frac{LN}{CN}; \]
but the proposition is evident at once analytically, for if
\[ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \]
be the equation of the ellipse, and if \( h, k \) be the point \( O \), \( x, y \) the point \( P \), then
\[ \frac{OZ}{CY} = \frac{hx + ky}{a^2 + b^2 - 1} = \frac{PZ'}{CY'}. \]

It may be observed that (3) is only a particular case of a more general proposition in which \( P \) and the tangent at \( P \) may be replaced by any point \( P \) and its polar; for let the coordinates of \( O \) and \( P \) be \( h, k \) and \( h', k' \), then

\[ \frac{\text{perpendicular from } P \text{ on polar of } O}{\text{perpendicular from centre on polar of } O} = \frac{hh' + kk'}{a^2 + b^2 - 1} \]

\[ = \frac{\text{perpendicular from } O \text{ on polar of } P}{\text{perpendicular from centre on polar of } P}. \]

This proposition is proved for circles in Salmon's *Conics*, 5th edition, Art. 101, p. 93.

§ 4. The law of force in the Newtonian form \( \mu \left( \frac{CY}{OZ} \right)^3 OP \) leads directly to the analytical expressions for the force in terms of the coordinates of any point; for let the point \( O \) be the origin, and suppose that the conic
\[ ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \]
is described about \( O \). The tangent at \( x, y \) is
\[ ax + h(x + \eta) + by + g(x + \xi) + f(y + \eta) + c = 0; \]
and, \( x_0, y_0 \), being the coordinates of the centre,
\[ \frac{CY}{OZ} = \frac{ax_0 + h(xy + yx_0) + by_0 + g(x + x_0) + f(y + y_0) + c}{gs + fy + c}, \]

where \(x, y\) are given by the equations

\[
\begin{align*}
ax + by + g &= 0, \\
ax + by + f &= 0.
\end{align*}
\]

Multiplying these equations respectively by \(x\) and \(y\), and subtracting them from the numerator of the expression for \(\frac{CY}{OZ}\), we have

\[
\frac{CY}{OZ} = \frac{gx + fy + c}{gx + fy + c},
\]

and therefore

\[
\text{force to } O = \mu \left(\frac{CY}{OZ}\right)^2 \text{ OP} = \mu \left(\frac{gx + fy + c}{gx + fy + c}\right)^2 r = \frac{\mu r}{(gx + fy + c)^2}.
\]

§ 5. Consider the case of the parabola separately. The force to the focus \(S\) is \(\frac{\mu}{SP^2}\), and therefore

\[
\text{force to any point } O = \left(\frac{Sg^2}{OP^2 \cdot SP}\right) \frac{\mu}{SP^2} = \mu \left(\frac{Sg}{OP \cdot SP}\right)^3 \text{ OP}
\]

where \(Sg\) is drawn parallel to \(OP\) cutting the tangent at \(P\) in \(g\), and \(Sy\) is the perpendicular upon the tangent at \(P\).

Now \(\frac{Sy}{SP} = \sin \theta\), where \(\theta\) is the inclination of the tangent at \(P\) to the axis of the parabola, and therefore

\[
\text{force to } O = \mu \left(\frac{\sin \theta}{OZ}\right)^3 \text{ OP} \ldots \ldots \ldots \quad (4)
\]

where \(PZ'\) is the perpendicular upon the polar of \(O\), and \(\alpha\) is the inclination of the polar of \(O\) to the axis of the parabola.

Thus the force to \(O = \frac{\mu r}{P^2}\) as before.

The theorem

\[
\frac{OZ}{\sin \theta} = \frac{PZ'}{\sin \alpha}
\]

is true when \(P\) and the tangent at \(P\) are replaced by any point \(P\) and the polar of \(P\); viz. we have

\[
\begin{align*}
\text{perpendicular from } P \text{ on polar of } O &= \sin \alpha \\
\text{perpendicular from } O \text{ on polar of } P &= \sin \beta'
\end{align*}
\]
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where $\alpha$, $\beta$ are the inclinations of the polars of $O$ and $P$ to the axis of the parabola.

Enunciated in a geometrical form, this theorem asserts that if $O$ and $P$ be any two points, and if $OO'$ be drawn parallel to the axis, cutting the polar of $P$ in $O'$, and $PP'$ be drawn parallel to the axis, cutting the polar of $O$ in $P'$, then $OO' = PP'$. The proof is very simple; for, $y^2 = 4ax$ being the equation of the parabola and $h$, $k$ the point $O$, $h'$, $k'$ the point $P$, then the equation of the polar of $P$ is $yk' = 2a (x + h')$, and therefore

$$OO' = \frac{hk'}{2a} - h' - k',$$

which involves the coordinates of $O$ and $P$ symmetrically.

The equation (4) gives a simple expression for the law of force to any point in a parabola; for if a body be describing a parabola under the action of a force tending to any point $O$, then we see from (4) that

$$\text{force to } O = \mu \frac{OP}{OM^3},$$

where $OM$ is drawn parallel to the axis cutting the tangent at $P$ in $M$ (fig. 3).

Fig. 3.

$\S$ 6. It follows therefore that under the action of the force

$$\frac{\mu r}{(ax + by + c)^2} \ldots \ldots \ldots \ldots \ldots (1)$$

any conic having $ax + by + c = 0$ as the polar of $O$ may be described. Taking $O$ as the origin, the general equation of all such conics is

$$(ax + by + c)^2 = Ax^2 + 2Hxy + By^2; \ldots \ldots \ldots (5)$$

and since, the line $ax + by + c = 0$ being given, this equation contains two arbitrary constants in addition to an arbitrary quantity $\lambda$ by which $a$, $b$, $c$ must admit of being multiplied in virtue of
the factor $\frac{\mu}{h^2}$ in the differential equation, it follows that this is the most general form of orbit that can be described under the action of the force (1). From (5) we see that the conic can be also described under the action of the force

$$\frac{\mu r}{(Ax^2 + 2Hxy + By^2)^{\frac{3}{2}}}$$

and, since $Ax^2 + 2Hxy + By^2 = 0$ being two given lines, the equation (5) contains two arbitrary constants besides the quantity $\lambda$ by which $A, B, C$ may be multiplied, it follows that, under the action of (2), the body will always describe a conic having $Ax^2 + 2Hxy + By^2 = 0$ as the (real or imaginary) tangents from the origin.

§ 7. There are no other laws of force for which the orbit is always a conic; for suppose that

$$\frac{\mu r}{\{\phi(x, y)\}^3}$$

expresses such a law, then

$$ax + by + c = \phi(x, y)$$

must represent a conic of which $ax + by + c = 0$ is the polar of the origin, and containing two arbitrary constants in addition to the constants in $\lambda \phi(x, y)$.

The most general rational equations of the form (6) for which $ax + by + c = 0$ represents the polar of the origin are

$$ax + by + c = Ax^2 + 2Hxy + By^2 + v(2ax + 2by + c) + \frac{1}{2} c.$$  \hspace{1cm} (7)

$$(ax + by + c)^2 = Ax^2 + 2Hxy + By^2 + v(2ax + 2by + c).$$  \hspace{1cm} (8)

and it is not difficult to assure oneself, by examining the different cases and transformations, that the only transformation satisfying the requisite conditions is obtained by putting $v = 0$ in (8), which then becomes identical with (5). M. Darboux gives a proof that (1) and (2) are the only laws, on pp. 936, 937 of his second paper in the *Comptes Rendus*.

From (7) we see that if the force be

$$\frac{\mu r}{(Ax^2 + 2Hxy + By^2 + f)^{\frac{3}{2}}}$$

the body will describe a conic having its asymptotes parallel to the lines $Ax^2 + 2Hxy + By^2 = 0$ if properly projected.

It will be observed that the problem solved by Newton is, "Given any conic (or curve, the law of force for the description of which about any one point is known), find the law of force to
any point”; while M. Bertrand’s problem is, “Find the general law of force, expressed in terms of $x$ and $y$, such that the orbit is always a conic, whatever the initial conditions may be,” and it is remarkable that there should be two distinct laws of force for which this is true.

§ 8. As far as I know, the pair of expressions (1) and (2) were first obtained by MM. Darboux and Halphen; but the law $\frac{\mu r}{p^3}$, which is identical with $\frac{\mu r}{(ax+by+c)^3}$, was discovered by Sir W. R. Hamilton, and is thus enunciated in the Proceedings of the Royal Irish Academy, No. 57, vol. iii., p. 308 (November 30, 1846).

“Sir William R. Hamilton stated the following theorems of central forces, which he had proved by his calculus of quaternions, but which, as he remarked, might be also deduced from principles more elementary. ‘If a body be attracted to a fixed point, with a force which varies directly as the distance from that point and inversely as the cube of the distance from a fixed plane, the body will describe a conic section, of which the plane intersects the fixed plane in a straight line, which is the polar of the fixed point with respect to the conic section.’

The second theorem, which is analogous to the first, relates to motion on a sphere in a spherical conic. The account then proceeds: “The first theorem had been suggested to Sir W. Hamilton by a recently resumed study of a part of Sir Isaac Newton’s Principia.”

A short analytical proof of the converse of this theorem, viz. that “if $X$ be any given conic, $O$ any point in its plane, $F$ a central force at $O$ which causes a material particle to describe $X$, then $F$ varies as $\frac{r}{p^3}$,” was given by Professor Casey in a paper, “On M’Cullagh’s property of a self-conjugate triangle and Sir W. Hamilton’s law of force for a body describing a conic section” (Quarterly Journal of Mathematics, vol. v. (1862), pp. 233–235).

It thus appears that a reply to M. Bertrand’s question with regard to the existence of other laws besides $\mu r$ and $\frac{\mu}{p^2}$ for which the orbit was always a conic had been given by Sir W. R. Hamilton’s theorem that the orbit is always a conic when the law of force is $\frac{\mu r}{p^3}$, $p$ being the perpendicular on any fixed plane; but M. Bertrand’s general problem seems never to have been considered. It is not unlikely that Sir W. R. Hamilton deduced his law from Newton’s Prop. xvii., and it is curious that the general formula which are deducible from the translation of Newton’s results into analysis should not have been examined.

§ 9. The elegant expression for the periodic time to which Newton’s corollary leads, viz. that if a conic be described under
the action of the force $\frac{\mu r}{p^3}$ tending to a point O, then the periodic time is $\frac{2\pi}{\sqrt{\mu}} p_o^{\frac{3}{2}}$, where $p_o$ is the perpendicular from the centre of the conic upon the polar of O, seems not to have been previously noticed. It is interesting to apply it to the cases in which O coincides with the centre and focus. If O coincides with the centre, then the polar is at infinity, so that $\frac{\mu}{p^3} = \frac{\mu}{p_o^3} = \text{constant} = \mu'$ suppose; thus the force is $\mu' r$ and the periodic time is $\frac{2\pi}{\sqrt{\mu'}}$.

If O coincides with the focus, the polar is the directrix, so that $p = r/e$; thus the force is $\frac{\mu r}{r^3}$ and the periodic time is $\frac{2\pi}{\sqrt{\mu'}} \left(\frac{\mu}{e^2}\right)^{\frac{3}{2}}$, that is, the force is $\frac{\mu'}{r^3}$ and the periodic time is $\frac{2\pi}{\sqrt{\mu'}} a^3$.

If $p$ denotes, as in Sir W. R. Hamilton's theorem, the perpendicular upon a given fixed plane, the periodic time will be $\frac{2\pi}{\sqrt{\mu}} p_o^{\frac{3}{2}}$, $p_o$ being the perpendicular from the centre upon the fixed plane; so that the periodic time in all orbits having their centres in a plane parallel to the fixed plane is the same. If the plane of motion be parallel to the fixed plane $p$ is constant and we have the case of motion under the action of a force to the centre.

§ 10. By means of the formula $\frac{2\pi}{\sqrt{\mu}} p_o^{\frac{3}{2}}$, it will be found that if the conic

$$(ax + by + c)^2 = Ax^2 + 2Hxy + By^2$$

be described about the origin under the action of the force

$$\frac{\mu r}{(ax + by + c)^2}$$
or

$$\frac{\mu r}{(Ax^2 + 2Hxy + By^2)^{\frac{3}{2}}}$$

the periodic time will be

$$\frac{2\pi}{\sqrt{\mu}} (ax_o + by_o + c)^{\frac{3}{2}}$$,

$x_o$, $y_o$ being the coordinates of the centre,

$$= \frac{2\pi}{\sqrt{\mu}} \left(\frac{c (AB - H^2)}{AB - H^2 - (Ab^2 + Ba^2 - 2Hab)}\right)^{\frac{3}{2}}$$,

and that if the conic

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$
be described about the origin under the action of the force

\[ \frac{\mu r}{(gx + fy + c)^3} \]

or

\[ \frac{\mu r}{(y^2 - ac)x^2 + 2(fg - hc)xy + (f^2 - hc)y^2 + g^2} \]

the periodic time will be

\[ \frac{2\pi}{\sqrt{\mu}} \left( \frac{gx + fy + c}{h^2} \right)^{\frac{3}{2}} \]

\[ = 2\pi \left( \frac{c(ab - h^2) - (a(f^2 + bg^2 - 2fhg))}{ab - h^2} \right)^{\frac{3}{2}} \]

and the same expression will denote the periodic time if this
orbit be described about the point \( p, q \) under the action of the
force

\[ \frac{\mu r}{(apx + h(xy + yg) + b(y + y) + f(y + q) + c)^3} \]

\( r \) being measured from \( p, q \).

§ 11. The expression \( \mu \left( \frac{CY}{OZ} \right)^3 \) OP for the force to any point
O may be investigated directly in the same manner and with the
same facility as the force to the focus is obtained by Newton in
Prop. xi.; for if \( F \) denote the force to \( O \), then (fig. 4)

\[ F = \frac{2h^2}{OP} \]

\[ QR = \frac{QT}{Q} \]

By similar triangles \( QT_x, PMO \),

\[ \frac{QT}{QT} = \frac{PM}{OP}, \]

and

\[ \frac{Qr^2}{Fv, Gv} = \frac{CD^2}{CP^2}, \ldots \ldots \ldots (9) \]
Now

\[
\frac{P_\gamma}{QR} = \frac{P_\gamma}{P_x} = \frac{P_\gamma}{OP} = \frac{CP}{PF} \cdot \frac{PM}{OP};
\]

therefore

\[
\frac{Q_\gamma^2}{QR \cdot GV} = \frac{CD^2}{OP} \cdot \frac{PM}{PF \cdot OP};
\]

whence, since ultimately \(Q_x = Q_\gamma\), and by (9),

\[
\frac{Q_T^2}{QR} = 2 \frac{CD^2}{PF} \left( \frac{PM}{OP} \right)^3.
\]

Thus

\[
F = 2 \frac{\mu^2}{OP} \cdot 2 \frac{PF}{CD^2} \left( \frac{OP}{PM} \right)^3
\]

\[
= \frac{\mu^2}{a^2 \nu^2} \left( \frac{PF}{PM} \right)^3 \cdot OP
\]

\[
= \frac{\mu^2}{a^2 \nu^2} \left( \frac{CY}{OZ} \right)^3 \cdot OP.
\]

Newton himself uses Corollary 3 of Prop. vii. in his second demonstration of Prop. xi. to deduce the law of force to the focus from that to the centre.

§ 12. The law of force

\[
\frac{\mu r}{(Ax^2 + 2Hxy + By^2)^\frac{3}{2}}
\]

is connected with the known equations for motion about the focus in an elliptic orbit in a plane inclined to the plane of reference; for if a sphere described with the focus as centre cut the plane of the orbit and the plane of reference in NP and NM, and if PM be a great circle perpendicular to NM, then PM = s, and if \(u\) denote the reciprocal of the projection of the radius vector upon the plane of reference, it is well known that

\[
l u = \sqrt{(1 + s^2) + \epsilon \cos (\theta - \omega)}. \ldots \ldots \ldots (10)
\]

where \(l\) is the semi-latus rectum of the orbit. But if \(h\) denote twice the area described in unit of time by the projection of the radius vector,

\[
\frac{d^2u}{d\theta^2} + u = \frac{P \cos PM}{h^2 \nu^2}
\]

\[
= \frac{\mu}{h^2} \cos^3 PM
\]

\[
= \frac{\mu}{h^2 (1 + s^2) \nu^2}.
\]
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Now \( s = k \sin (\theta - \gamma) \), so that the differential equation is

\[
\frac{d^2u}{d\theta^2} + u = \frac{\mu}{k^2 \left(1 + k^2 \sin^2 (\theta - \gamma)\right)^{\frac{3}{2}}}
\]

while (10) becomes

\[lu = \sqrt{1 + k^2 \sin^2 (\theta - \gamma)} + e \cos (\theta - \theta_0)\]

and therefore, transforming from polar to rectangular coordinates, if the law of force is

\[
\frac{\mu'}{(1 + k^2 \sin^2 \gamma) x^2 - 2k^2 \sin \gamma \cos \gamma xy + (1 + k^2 \cos^2 \gamma) y^2}^{\frac{1}{2}}
\]

the equation of the orbit is

\[(1 + k^2 \sin^2 \gamma) x^2 - 2k^2 \sin \gamma \cos \gamma xy + (1 + k^2 \cos^2 \gamma) y^2 = (Ax + By + l)^2,
\]

agreeing with (5). The ordinary equations of elliptic motion thus indicate the laws (1) and (2); and these laws may be found directly by considering the orthogonal projections of a conic on a plane through the focus.

For, the equation of a conic having the origin for focus is

\[x^2 + y^2 = (Ax + By + C)^2,
\]

and if this be projected orthogonally upon a plane through the focus, the axis of \( z \) being the line of intersection of the two planes, the equation of the projection is of the form

\[x^2 + m^2 y^2 = (Ax + Bmy + C)^2,
\]

that is, any conic having the tangents from the origin equally inclined to the axes, and as we may turn the axes of coordinates through an arbitrary angle, we thus obtain any conic.

§ 13. Newton's law, \( \mu \frac{CG^3}{OP^2} \), for the force to O may be put in the form

\[
\mu \frac{DD'}{OP^2, PP'} \ldots \ldots \ldots \ldots (11)
\]

where \( PP' \) is the chord through O and \( DD' \) is the diameter parallel to the chord \( PP' \); for if \( PM \) be drawn parallel to the line joining \( C \) to the middle point of \( PP' \), cutting \( CD \) in \( M \), then \( CG \cdot CM = CD^2 \), and therefore \( 2CG \cdot PP = DD'^2 \). The form (11) was given in an examination paper set in St. John's College, Cambridge, in December 1869, and I am indebted to Mr. E. J. Routh for the reference to it.
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By a known theorem, if $a$, $b$ be the semi-axis major and semi-axis minor of an ellipse

$$PP' = \frac{1}{2a} DD' \sqrt{\left(1 - \frac{b^2}{a^2}\right)},$$

$p_1$, $p_2$ being the perpendiculars upon the chord $PP'$ from the foci $S$, $H$; and therefore, if the body describe an elliptic orbit about $O$,

$$\text{force to } O = \frac{\mu}{r^2 (b^2 - p_1 p_2)^{\frac{3}{2}}},$$

where $OP = r$ and $p_1$, $p_2$ have the same or opposite signs according as $S$, $H$ are on the same or on opposite sides of $PP'$.

If the orbit be hyperbolic

$$\text{force to } O = \frac{\mu}{r^2 (b^2 + p_1 p_2)^{\frac{3}{2}}},$$

$b$ being the semi-conjugate axis.

Consider the case when the orbit is an ellipse. Since

$$\frac{CG^3}{OP^3} = \left(\frac{p_0}{p}\right)^3 r, \text{ by § 2,}$$

and also,

$$= \frac{1}{r^2} \left(DD'\right)^3 = \frac{1}{r^2} \left(\frac{(ab)^3}{(b^2 - p_1 p_2)^{\frac{3}{2}}}\right),$$

it follows that

$$\frac{p_0}{p} = \frac{1}{r} \sqrt{\frac{ab}{(b^2 - p_1 p_2)}}.$$

Now, when the force $= \frac{\mu v}{p^3}$, the periodic time $= \frac{2\pi}{\sqrt[3]{p_0}}$, and therefore when the force

$$= \frac{\mu \cdot ab^3}{p_0^3} \cdot \frac{1}{r^2} \frac{1}{(b^2 - p_1 p_2)^{\frac{3}{2}}},$$

the periodic time $= \frac{2\pi}{\sqrt[3]{\mu}} p_0^{\frac{2}{3}}$. Putting $\mu' = \frac{ab^3}{p_0^3}$, the periodic time

$$= \frac{2\pi}{\sqrt[3]{\mu}} p_0^{\frac{2}{3}} = \frac{2\pi}{\sqrt[3]{\mu}} (ab)^{\frac{2}{3}},$$

and therefore when the force to $O$

the periodic time

$$= \frac{\mu}{r^2 (b^2 - p_1 p_2)^{\frac{1}{2}}},$$

$$= \frac{2\pi}{\sqrt[3]{\mu}} (ab)^{\frac{2}{3}}.$$
When $O$ coincides with a focus, $p_1 p_2 = 0$; the force to
$S = \frac{\mu}{b^3} \frac{1}{r^2} = \mu' \frac{1}{r^2}$ suppose, and the periodic time
$$= \frac{2\pi}{\sqrt{\mu}} a^3 b^3 = \frac{2\pi}{\sqrt{\mu'}} a^3 b^3.$$

When $O$ coincides with the centre,
$$p_1 p_2 = -a^2 e^2 \sin^2 \theta,$$
where $\theta$ is the inclination of CP to the major axis, therefore

force to $O = \frac{\mu}{r^2 (a^2 \sin^2 \theta + b^2 \cos^2 \theta)}$

$$= \frac{\mu}{r^2 (a^2 \sin^2 \theta + b^2 \cos^2 \theta)}$$

$$= \frac{\mu}{r^2} \frac{r^2}{a^2 b^2} = \frac{\mu}{a^2 b^2} r = \mu' r,$$

and the periodic time
$$= \frac{2\pi}{\sqrt{\mu}} (ab)^3 = \frac{2\pi}{\sqrt{\mu'}} (ab)^3.$$

§ 14. The substance of the above paper was communicated
to the Royal Astronomical Society on December 14, 1877, and
an abstract appeared in the account of that meeting in the
Observatory, vol. i., p. 268 (January 1878); but I have been
prevented by other occupations from writing out the paper till
now. The theorem that if a body moves under the action of a
central force $\frac{\mu r}{p^3}$ the periodic time is $\frac{2\pi}{\sqrt{\mu}} p^3$ was set by me in the
Mathematical Tripos, Wednesday morning, January 2, 1878 (see
Solutions of the Senate-House Problems and Riders for 1878,
pp. 41–43).

Trinity College, Cambridge,
1878, August 29.

On a Portable Star Finder for Altitude and Azimuth Telescopes.

By Edward H. Liveing, A.R.S.M.

The following is a description of an instrument that I
invented in the early part of last year, and that I have had in
actual use about twelve months.

My object in describing it is that I think it may be very
useful with the altitude and azimuthly mounted reflectors so
much in use among amateurs and others at the present day.