
In astronomical investigations it often happens that we construct a table to exhibit the number of stars having successive values of a certain measured property, e.g. the number of stars of successive magnitudes, or the number between successive limits of proper motion. In general the observations are subject to a probable error, which is at least approximately known. The error is in general small, otherwise our table would not be of much value, but it must have some effect on the numbers of the table. It is not, however, usual to take account of the probable error, probably because it is not generally realised that it can be eliminated and an improved table formed in a very simple way.

To avoid the awkwardness of general terms, suppose we are concerned with counts of stars between given limits of magnitude, and, knowing the average accidental error of our determinations of magnitude, we wish to apply corrections to our counts to eliminate these errors.

Let $u(m)dm = \text{observed number of stars between magnitudes } m \text{ and } m + dm$

$v(m)dm = \text{true number.}$

Let the probable error of the observed magnitudes be $\sigma 477/h$, so that the frequency of an error $x$ is proportional to $e^{-hx^2}$.

We have

$$u(m) = \frac{h}{\sqrt{\pi}} \int_{-\infty}^{\infty} v(m + x)e^{-hx^2}dx;$$

for, of the stars having a true magnitude $m + x$, the proportion $\frac{h}{\sqrt{\pi}} e^{-hx^2}$ will have an error of measurement $-x$, and will therefore be observed as of magnitude $m$.

By the symbolic form of Taylor’s theorem

$$v(m + x) = e^{\frac{\partial}{dm}} v(m),$$

therefore

$$u(m) = \frac{h}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{\frac{\partial}{dm} - x^2} v(m)dx.$$

The integral is of the well-known form

$$\int_{-\infty}^{\infty} e^{-ax^2} dx = \sqrt{\frac{\pi}{a}} \exp \frac{a x^2}{4a^2},$$

thus integrating

$$u(m) = \exp \left( \frac{1}{4h^2} \frac{d^2}{dm^2} \right) v(m);$$
therefore \[ v(m) = \exp\left(-\frac{1}{4h^2} \frac{d^2}{dm^2}\right) u(m) = u(m) - \frac{1}{4h^2} u''(m) + \frac{1}{2!} \left(\frac{1}{4h^2}\right)^2 u^{(3)}(m) - \ldots \]

The result so far is accurate.

For a small accidental error, confining ourselves to the first two terms, the result can be given in a form convenient for computation as follows:

The tabular second difference for intervals \( \alpha \)
\[ = u(m + \alpha) + u(m - \alpha) - 2u(m) = \alpha^2 u''(m) \] approximately.

Also
\[ \frac{1}{2h} = 1.046 \times \text{probable error}. \]

Thus the approximate correction is
\[ -\left(\frac{1.046 \times \text{probable error}}{\text{tabular interval}}\right)^2 \times \text{tabular second difference}. \]

**Note on the Convergency of the Series.**—The Astronomer Royal has pointed out to me that the series in some typical cases is divergent, e.g. \( u(m) = (1 + m^2)^{-1} \). The operator \( \frac{d^{2n}}{dm^{2n}} \) introduces a factor of the order \( 2^n n! \), whilst the divisor is only \( n! \). Apparently, however, in these cases the expansion is asymptotic; and it seems certain that the first few terms give the approximate correction quite correctly. The divergence arises from using the Taylor expansion beyond its range of convergence.

The difficulty does not really arise in practice. In a table with a finite number of entries, we are actually dealing with a polynomial, in which case all the series terminate, and no question of divergence arises. Thus, if \( \alpha \) is the tabular interval, the tabular values of \( v(m) \) may be represented from \( m - na \) to \( m + na \) by a polynomial of the \( 2n \)th degree, say \( v_{2n}(m) \). By taking \( n \) sufficiently great the whole range of \( v \), which contributes appreciably to \( u(m) \), can evidently be included. Beyond the limits \( m \pm na \), the divergence between \( v_{2n}(m) \) and \( v(m) \) will generally increase rapidly; it can be shown, however, that
\[ \int_{-na}^{na} x^{2n} e^{-bx^2} dx \]
tends to zero as \( n \) is increased, and hence the part of \( v_{2n}(m) \) beyond the limits \( m \pm na \) will (when \( n \) is great) make no contribution to \( u(m) \).

Thus the polynomial and the true function agree for the \( 2n + 1 \) tabular entries \( m - na \) to \( m + na \), and beyond these limits they both make contributions to \( u(m) \), which tend to zero. Hence the polynomial can be used for our purpose, and a terminating series results.

Substituting the polynomial only requires that we should deduce the differential coefficients from the tabular differences—a procedure which we should naturally adopt in any case.