above is $+600 \text{ km./sec.}$ If for the average distance we take $10^5 \text{ parsecs} = 2 \times 10^{10}$, then we find

$$R = 3 \times 10^{11}.$$  

Of course this result, derived from only three nebulae, has practically no value. If, however, continued observation should confirm the fact that the spiral nebulae have systematically positive radial velocities, this would certainly be an indication to adopt the hypothesis B in preference to A. If it should turn out that no such systematic displacement of spectral lines towards the red exists, this could be interpreted either as showing A to be preferable to B, or as indicating a still larger value of $R$ in the system B.

Doorn: 1917 July.

The Equations of Radiative Transfer of Energy.

By J. H. Jeans.

1. In a gaseous star it is probable that much more energy is transferred by radiation than by ordinary gaseous conduction, so that an accurate determination of the laws of radiative transfer is a necessary preliminary to many problems of stellar physics. In two recent papers* Professor Eddington has given an equation of transfer which I believe to be erroneous (at least when considered with reference to its application), and which has, I believe, led him to an erroneous result. In the present paper I have examined independently the laws of radiative transfer.

These laws, like those of ordinary conduction of heat, can of course be found in a general form independently of the special problem to which they are ultimately to be applied; we need not complicate the laws by introducing (as Professor Eddington does) the curvature of the star's figure.

Consider a medium arranged so that layers of equal temperature are normal to the axis of $x$, and consider the stream of radiation making an angle $\theta$ with this axis. At the plane $x = \xi$ let the stream of radiation crossing a plane of cross-section $d\sigma$ per unit time in directions contained within a small cone $d\omega$ be $I d\sigma d\omega$, where $I$ is a function of $x$ and $\theta$.

After traversing a length of path $ds$, the main part of the radiation will reach the plane $x = \xi + ds \cos \theta$. On this path it will have been diminished in a ratio $1 - c \rho ds$, where $c$ is a coefficient of opacity for this particular radiation, measured per unit mass, and $\rho$ is the density of the medium. It will also have been reinforced by radiation emitted by the matter traversed. The volume of this matter is $d\sigma ds$, so that if $E$ is the emission

per unit mass per unit time, the emission will be $E \rho ds d\sigma$, of
which a fraction $d\omega/4\pi$ will lie within the cone $d\omega$.

Thus the beam, when it arrives at the plane $x = \xi + ds \cos \theta$, will be of
intensity

$$I d\sigma d\omega (1 - c \rho ds) + E \rho ds d\sigma d\omega/4\pi,$$

whence we obtain, as the increment of $I$ in passing over unit
length of path,

$$\cos \theta \frac{dI}{dx} = \frac{E \rho}{4\pi} - c \rho I \quad . \quad . \quad . \quad (1)$$

This is the general equation of radiative transfer.

2. In a gas which is in equilibrium at uniform temperature
$\partial I/\partial x$ of course vanishes, so that

$$E = 4\pi c I \quad . \quad . \quad . \quad (2)$$

I now being independent of $\theta$.

In the general problem in which the temperature is not
uniform, $I$ is not independent of $\theta$, the whole transfer of energy
arising from the variations in $I$. Following Eddington, let us assume

$$I = A + BP_1 + CP_2 + \ldots \quad . \quad . \quad . \quad (3)$$

where $P_1, P_2 \ldots$ are Legendre's coefficients and $A, B, C \ldots$
are functions of $x$.

Equation (1) now assumes the form

$$\cos \theta \left( \frac{dA}{dx} + P_1 \frac{dB}{dx} + P_2 \frac{dC}{dx} + \ldots \right) = \frac{E \rho}{4\pi} - c \rho(A + BP_1 + CP_2 + \ldots) \quad (4)$$

Using the harmonic relation

$$(2n + 1)P_n \cos \theta = (n + 1)P_{n+1} + nP_{n-1},$$

this equation becomes

$$P_1 \frac{dA}{dx} + \frac{1}{2}(2P_2 + 1) \frac{dB}{dx} + \frac{1}{3}(3P_3 + 2P_1) \frac{dC}{dx} + \ldots = \frac{E \rho}{4\pi} - c \rho(A + BP_1 + CP_2 + \ldots).$$

Since this must be true for all values of $\theta$, we may equate
coefficients of the different harmonics, and obtain

$$\frac{1}{3} \frac{dB}{dx} = \frac{E \rho}{4\pi} - c \rho A \quad . \quad . \quad . \quad (5)$$

$$\frac{1}{5} \frac{dA}{dx} + \frac{2}{5} \frac{dC}{dx} = -c \rho B \quad . \quad . \quad . \quad (6)$$

$$\frac{2}{5} \frac{dB}{dx} + \frac{3}{5} \frac{dD}{dx} = -c \rho C, \text{ etc.} \quad . \quad . \quad . \quad (7)$$
The total absorption of energy per unit volume per unit time is
\[ \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \cos \theta \, d\theta = 4\pi \epsilon \rho A. \]

The total emission of the same unit volume has been assumed to be \( E \rho \). Let us assume that the energy of this emitted radiation comes partly from the re-emission of the absorbed radiation, but partly also from radioactive or similar sources which generate energy at a rate \( 4\pi \epsilon \) per unit mass. Then the equation of energy of the unit volume under consideration, divided throughout by \( \rho \), is
\[ 4\pi cA + 4\pi \epsilon = E. \]  
(8)

Combining this with equation (5), we find
\[ \frac{1}{3} \frac{dB}{dx} = \epsilon \rho. \]  
(9)

The total net transfer of radiation per unit area per unit time across the plane \( x = \xi \) is \( \iint \cos \theta \, d\omega \) or \( \frac{1}{3} \pi B \). Thus equation (9), which can be expressed in the form
\[ \frac{d}{dx} \left( \frac{1}{3} \pi B \right) = 4\pi \epsilon \rho, \]
is obviously the "equation of continuity" of energy.

Combining (7) with (9), we obtain
\[ \frac{3}{4} \frac{dD}{dx} + \epsilon \rho C = -2 \epsilon \rho. \]  
(10)

In Professor Eddington's two recent papers \( \epsilon \) is supposed to be finite, while \( C, D \) are neglected as being small. Our exact equation (10) shows this procedure to be illegitimate. The error involved is of the order of magnitude of \( \epsilon \), which in turn is the order of magnitude of Professor Eddington's main result.

3. In the simplest and most natural case in which \( \epsilon = 0 \), equation (9) shows that \( B \) must be constant, and equations (5), (6), (7) now reduce to
\[ E = 4\pi cA. \]  
(11)

\[ \frac{dA}{dx} + \frac{3}{3} \frac{dC}{dx} = -\epsilon \rho B. \]  
(12)

\[ \frac{3}{4} \frac{dD}{dx} = -\epsilon \rho C, \text{etc.} \]  
(13)

For a uniform temperature gradient, or a gradient which may be treated as uniform over a range which is large compared with \( 1/\epsilon \rho \), \( dA/dx \), \( dC/dx \), etc., will all be constants. Hence the right-hand members of equations (12), (13), etc., are constants,
so that B, C, D . . . are constants. Thus equations (13) and (14) reduce to \( C = D = E = \ldots = 0 \), while equations (11) and (12) become

\[
\frac{dA}{dx} = -cB \quad (15)
\]

If \( c \) denote the velocity of light, the energy of radiation per unit volume is

\[
\frac{1}{c} \int \int \int d^3\omega = \frac{4\pi}{c} A \quad (16)
\]

and this may be put equal to \( aT^4 \) where \( a \) is the universal constant \( 7.1 \times 10^{-15} \) C.G.S. units. Since \( ac = 4\sigma \) where \( \sigma \) is Stefan’s constant \( (5.32 \times 10^{-8}) \), we have

\[
\pi A = \sigma T^4 \quad (17)
\]

so that, from equation (15), the net flow of radiant energy per unit area is

\[
\frac{4\pi}{3}B = -\frac{16\sigma T^3}{3c} \frac{dT}{dx}.
\]

Thus the flow may be put in the usual form \( -\kappa \frac{dT}{dx} \), where \( \kappa \) is a coefficient of radiative conduction. Its value in energy-units is

\[
\kappa = \frac{16\sigma T^3}{3c} \quad (18)
\]

while its value in heat-units is

\[
\kappa = \frac{16\sigma T^3}{3cJ} = 6.8 \times 10^{-8} \frac{T^3}{J} \quad (19)
\]

4. In the interior of a star \( T \) will be of the order of \( 10^6 \), and for a giant star we may assign to \( \rho \) a value of the order of \( 10^{-8} \). Thus \( \kappa \) is of the order of \( 10^8/c \).

At \( \theta \) C. the coefficient of ordinary gaseous conduction of heat is of the order of \( 5 \times 10^{-8} \) for \( \text{CO}_2 \) and gases of similar molecular weight. Its increase is at most proportional to the absolute temperature, so that at \( 10^6 \) degrees its value will be at most of the order of \( 0.2 \). Clearly, then, whatever reasonable value we assign to \( c \), gaseous conduction will be quite insignificant in comparison with radiative conduction.

In a gas at the high temperature of the interior of a star there must be a great amount of ionisation, so that electronic conductivity must in any case be expected to outweigh gaseous conductivity. It is very difficult to form an estimate of the degree of conductivity under such conditions, but various lines of reasoning (e.g. comparison with metallic conduction, etc.) agree in suggesting that ordinary conductivities of all kinds are insignificant in comparison with radiative conductivity in a star’s interior.
5. The simple solution found in § 3, namely,

\[ I = A + B \cos \theta, \]

will be accurate in the interior of a star, but will fail as the surface is approached. For at the surface there must be no entrant flow of radiation, so that the true value of \( I \) must vanish for all values of \( \theta \) between 90° and 180°; there will, moreover, be a discontinuity in the value of \( I \) when \( \theta = 90° \). It will be found that an expansion of \( I \) in harmonics is unsuitable for the investigation of the flow of energy near the boundary, such an expansion becoming divergent as the boundary is approached.

We accordingly return to the fundamental equation (1). Write \( \mu \) for \( \cos \theta \), and put

\[ \int \sigma d\mu = \nu, \]

making \( \nu = 0 \) at the boundary. Thus \( \nu \) is negative inside the star, \( -\nu \) measuring the aggregate absorbing power of the matter between an interior point and the star's surface. The fundamental equation (1) now assumes the form

\[ \frac{\partial I}{\partial \nu} = \frac{E}{4\pi c} - I \quad \ldots \ldots \ldots (20) \]

This equation is true for all values of \( \mu \) from \( -1 \) to +1. Multiply by \( d\mu \) and integrate through this range and we obtain

\[ \frac{\partial}{\partial \nu} \int_{-1}^{+1} \mu I d\mu = \frac{E}{2\pi c} - \int_{-1}^{+1} I d\mu \quad \ldots \ldots \ldots (21) \]

Here \( 2\pi \int_{-1}^{+1} \mu I d\mu \) measures the net transfer of energy across a plane parallel to the axis of \( x \), so that the left-hand member is equal to \( 2\varepsilon/c \). If, as before, we take \( \int_{-1}^{+1} I d\mu \) to be equal to \( 2A \), equation (21) is readily seen to be identical with our former equation of energy (8), while equation (20) becomes

\[ \frac{\partial I}{\partial \nu} = A + \frac{\varepsilon}{c} - I \quad \ldots \ldots \ldots (22) \]

Consider for the present only the case of \( \varepsilon = 0 \). The simplest solution is that already discussed, namely,

\[ I = A + B\mu, \quad A = A_0 - \nu B \quad \ldots \ldots \ldots (23) \]

the general solution is found to be

\[ I = e^{-\nu/\mu} \left[ \int e^{\nu/\mu} A d\nu + \phi(\mu) \right] \quad \ldots \ldots \ldots (24) \]

in which \( \phi(\mu) \) is an arbitrary function of \( \mu \). To investigate the flow of radiation near the boundary we require to find a solution
of the type (24), which shall reduce to (23) in the far interior of the star \((\nu = -\infty)\). Let us suppose the required solution to be

\[
A = A_0 - \nu B + C,
\]

where \(C\) is a function of \(\nu\) only, which vanishes when \(\nu = -\infty\).

Inserting this value for \(A\) into equation (24) and simplifying, we find

\[
I = A_0 - \nu B + \mu B + e^{-\nu/\mu} \left[ \int_{-\infty}^{\nu} e^{\nu/\mu} C \frac{d\nu}{\mu} + \phi(\mu) \right].
\]

(Since \(\phi(\mu)\) is already at our disposal, the lower limit of integration is also at our disposal. To keep the integrals convergent, we must take the lower limit to be \(\nu = \infty\) when \(\mu\) is negative and \(\nu = -\infty\) when \(\mu\) is positive. We may accordingly suppose the value of \(I\) at \(\nu = 0\) to be

\[
I = A_0 - \nu B + \mu B + e^{-\nu/\mu} \left[ \int_{-\infty}^{\nu} e^{\nu/\mu} C \frac{d\nu}{\mu} + \phi(\mu) \right] \quad (26)
\]

when \(\mu\) is positive (emergent wave) and

\[
I = A_0 - \nu B + \mu B + e^{-\nu/\mu} \left[ \int_{-\infty}^{\nu} e^{\nu/\mu} C \frac{d\nu}{\mu} + \psi(\mu) \right] \quad (27)
\]

when \(\mu\) is negative (entrant wave).

The values of \(\phi(\mu)\) and \(\psi(\mu)\) must now be determined from the boundary conditions. In the far interior of the star \((\nu = -\infty)\) the emergent wave must be unaffected by the presence of the boundary, so that \(\phi(\mu) = 0\). At the surface of the star the emergent wave must be of zero intensity for all negative values of \(\mu\), so that

\[
\psi(\mu) = -(A_0 + \mu B).
\]

This completes the evaluation of \(I\) in terms of \(C\); it remains to determine \(C\), which can be evaluated from the relation

\[
A = \frac{1}{2} \int_{-1}^{+1} I d\mu \quad . \quad . \quad . \quad (28)
\]

Inserting the values for \(A\) and \(I\), this relation becomes

\[
A_0 - \nu B + C = A_0 - \nu B + \frac{1}{2} \left[ \int_{-\infty}^{1} e^{-\nu/\mu} \left[ \int_{-\infty}^{\nu} e^{\nu/\mu} C \frac{d\nu}{\mu} \right] d\mu \right.
\]

\[
+ \frac{1}{2} \left. \int_{-1}^{0} e^{-\nu/\mu} \left[ \int_{-\infty}^{\nu} e^{\nu/\mu} C \frac{d\nu}{\mu} - A_0 - \mu B \right] \right] d\mu.
\]

On rearranging the integrals, this becomes

\[
C = \frac{1}{2} \left[ \int_{-\infty}^{\nu} C \int_{0}^{\nu} e^{-\nu/\mu} \frac{d\mu}{\mu} d\nu + \frac{1}{2} \int_{-1}^{0} C \int_{0}^{\nu} e^{-\nu/\mu} \frac{d\mu}{\mu} d\nu - \frac{1}{2} \int_{-1}^{0} e^{-\nu/\mu} \left( A_0 + \mu B \right) d\mu \right].
\]

\[
\quad . \quad . \quad . \quad (29)
\]
This integral equation will determine $C$ exactly, but I have not succeeded in solving it except by approximation.

6. A first approximation is clearly

$$C = -\frac{1}{2} \int_{-1}^{0} e^{-\nu_0 \mu} (A_0 + \mu B) d\mu \quad \quad (30)$$

We shall examine this first, and consider the error of the approximation later. The value of $C$ at the boundary ($\nu_0 = 0$) is

$$C = -\frac{1}{2} (A_0 - \frac{1}{2} B) \quad \quad (31)$$

and in the general value of $C$ each term falls off to zero in the interior of the star.

At the boundary there is no entrant wave ($I = 0$), while from equation (26) the emergent wave is given by

$$I = A_0 + \mu B + C'$$

where

$$C' = \int_{-\infty}^{0} e^{-\nu_0 \mu} C \frac{d\nu}{\mu}$$

If $C$ were constant for all values of $\nu$, the value of $C'$ would be identical with that of $C$, but as $C$ varies, $C'$ must be regarded as an average value of values of $C$ taken some distance into the star. When $\mu = 0$, $C'$ becomes identical with the value of $C$ at the boundary; for other values of $\mu$, $C'$ is less than this.

At the boundary the general relation (28) becomes

$$A_0 + C = \frac{1}{2} \int_{0}^{1} I d\mu$$

$$= \frac{1}{2} A_0 + \frac{1}{4} B + \frac{1}{2} \int_{0}^{1} C' d\mu$$

in which the last term is rather less than the value of $\frac{1}{2} C$ at the boundary, say $\frac{1}{2} \eta C$, where $\eta < 1$. Thus the equation becomes

$$C(1 - \frac{1}{2} \eta) = -\frac{1}{2} (A_0 - \frac{1}{2} B) \quad \quad (32)$$

This is consistent with (31) only if

$$A_0 = \frac{1}{2} B \quad \quad (33)$$

in which case $C = 0$ at the boundary. It now follows from equation (30) that $C$ is small for all values of $\nu$, and hence that equation (30) gives a good approximation to the true solution of equation (29).

One equation remains, namely,

$$\int_{-1}^{+1} I \mu d\mu = \frac{3}{2} B.$$
At the surface this assumes the form
\[ \int_0^1 (A_0 + \mu B + C')\mu d\mu = \frac{2}{3}B \]
or
\[ \frac{1}{2}A_0 + \frac{1}{3}B + \frac{1}{2}C'' = \frac{2}{3}B, \]
where \( C'' \) is another average value of \( C \). This becomes
\[ C'' = -(A_0 - \frac{2}{3}B), \]
and is again satisfied approximately by equation (33).

7. Thus a solution which approximately satisfies all the necessary conditions is \( C = 0, A_0 = \frac{4}{3}B \). The value of the temperature is given by (cf. equation (17)).
\[ \sigma T^4 = \pi A = \pi (A_0 - \nu B) = \pi B(\frac{1}{2} - \nu) \quad . \quad . \quad (34) \]

In this approximate solution it appears that \( \partial A/\partial \nu = -B \) right up to the boundary, and this equation is identical with equation (15), which was found to be exactly true in the star's interior. Thus the results given in § 3, which are accurately true inside the star, are also approximately true up to the surface, and even in the outermost layers there may be supposed to be a radiative flow of heat \( -\kappa \sigma T^4/\kappa x \) where \( \kappa \) is given by equations (18) and (19).

From equations (26) and (27) the stream of radiation at any internal point is given by
\[ I = (\frac{1}{2} - \nu + \mu)B \quad . \quad . \quad . \quad (35) \]
when \( \mu \) is positive (emergent wave), and
\[ I = [(\frac{1}{2} + \mu)(1 - e^{-\nu/\mu}) - \nu]B \quad . \quad . \quad . \quad (36) \]
when \( \mu \) is negative (entrant wave).

At the surface (\( \nu = 0 \)) the entrant wave is zero for all values of \( \mu \), while the emergent wave is given by
\[ I = (\frac{1}{2} + \cos \theta)B \quad . \quad . \quad . \quad (37) \]

This last formula ought of course to give the darkening at a star's limb, \( \theta \) being the angle between the normal to the surface and the line of sight. It is at once seen to be of the same form as the ordinarily assumed formula *
\[ I = I_0(1 - x + x \cos \theta) \]
and agrees with this if \( x \), the "coefficient of darkening," is taken equal to \( \frac{2}{3} \). This is about the actual value found for the sun by Schwarzschild and Abbot.

8. The total outward flow of energy per unit area is \( \frac{4}{3} \pi B \). If this is regarded as coming from a photosphere at temperature \( T_1 \), we must have
\[ \sigma T_1^4 = \frac{4}{3} \pi B \quad . \quad . \quad . \quad . \quad (38) \]

so that, by comparison with equation (34), the position of the photosphere will be given by \(-v = \frac{8}{3}\), or, replacing \(v\) by its value,

\[
\int cpdx = \frac{8}{3}
\]

where the integral is taken from the boundary to the photosphere.

From equation (34) the temperature \(T_0\) at the outer surface is given by

\[
\sigma T_0^4 = \frac{1}{2} \pi B
\]

so that, from equations (38) and (40), \(T_1^4/T_0^4 = \frac{8}{9}\), or \(T_1 = 1.28 T_0\).

If, however, we consider only the radiation which leaves the star normally, a different result is obtained. The normal stream of radiation per unit area is, from equation (37), \(I = \frac{3}{2}B\), and if this is regarded as coming from a photosphere at temperature \(T_2\), we have

\[
\sigma T_2^4 = \frac{3}{2} \pi B,
\]

giving, by comparison with (34), \(v = -1\) and \(T_0 = 1.32 T_2\).

The light coming from the limb, naturally enough, may be regarded as coming from a photosphere at zero depth and at temperature \(T_0\).

9. To avoid any possibility of misunderstanding, it may be well to repeat in conclusion that equations (34)–(39) are only approximations. There would not, I think, be any serious trouble in extending these approximations to higher orders of accuracy, but it seems probable that the gain in accuracy would only be small compared with the error involved in the neglect of scattering.

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*The Evolution and Radiation of Gaseous Stars.*

By J. H. Jeans.

1. In a well-known lecture, "Relations between the Spectra and other Characteristics of the Stars," * Professor H. N. Russell gave a diagram in which the absolute brightness of giant stars was seen to be approximately independent of their spectral type. One possible interpretation—although, of course, not the only one—is that the total emission of light of these stars does not change as their evolution progresses.

If this is the true interpretation, it is of interest to inquire whether the result is universally true, or is true only of certain stars. If the law expresses some general property of matter or of stellar structure, we should expect it to extend throughout the universe. On the other hand, it may be only a peculiarity of stars of our own system or near us—Russell's stars all being selected for well-determined parallax,—in which case other laws may be expected to obtain in other parts of space. Shapley has recently found that, of the giant stars in the Hercules cluster, the

*Popular Astronomy, xxii. 5 and 6.*