up. To take a concrete instance, the gravitational attraction between the two constituents of β Lyræ is about $4 \times 10^{32}$ dynes; the utmost possible value for the repulsion between them produced by the pressure of radiation is $4 \times 10^{34}$ dynes. Thus pressure of radiation can at most balance one part in $10^{13}$ of the gravitational attraction; the remaining $10^{13} - 1$ parts are balanced by the presence of angular momentum which must have been in the system from the outset, and would have broken the star up even had the pressure of radiation been non-existent.

6. The conclusion of the whole matter appears to be that pressure of radiation introduces no new factors, and is of comparatively little importance, in the problems we have had under discussion. The old-fashioned sphere of gas, in which radiation was left entirely out of account, still provides a remarkably good model of a star. It reproduces all phenomena with fidelity qualitatively, but requires modification quantitatively in two respects. In the first place, the capacity of the old model for transfer of energy from centre to surface was far too small, the transfer by radiation being millions of times as great as that by gaseous conduction; and in the second place, the capacity of the old model for heat-energy was too small, a fully ionised gas having about twenty times the heat-capacity of unionised gas of molecular weight 40. To this increase of heat-capacity a further contribution is added by the presence of radiation, ranging from about 10 per cent. in our sun to perhaps 50 per cent. in a B-type star, and about 120 per cent. in a star of mass 20. To these may be added the somewhat different circumstance, so convincingly demonstrated by Eddington, that the ionised star can attain to far higher densities than the mass of plain gas. But apart from the need for these quantitative corrections, a sphere of ordinary gas without radiation appears to provide a good model, and the various paradoxical theorems recently enunciated do not seem to me to correspond to anything in the facts of nature.


1. It is generally accepted that the great majority of spectroscopic and short-period binaries have been formed by fission. Newly formed binaries are distinguished by low eccentricity of orbit and by orbital dimensions so small that the two constituents are almost in contact. As was first clearly shown by Campbell, the majority of these newly formed binaries exhibit spectra of the "early" types, O, B, and A.

So long as an "early" spectral type was supposed to indicate an early stage of stellar evolution, this last circumstance admitted of a simple, although perhaps not entirely convincing, explanation; it could be taken to mean that a star broke up either very early in its career or not at all. With the abandonment of the old view of stellar evolution, this interpretation of the spectra of binaries had also to be
abandoned, and there is need for a new interpretation which shall be in conformity with new views of stellar evolution. The first part of the present paper suggests such an interpretation; the second part deals with the actual process of fission.

2. In a recent paper* I gave a diagram showing how all the theoretically possible configurations of equilibrium for a star could be represented in a plane in which a star's mass was taken for abscissa and its luminosity for ordinate. The star's effective temperature was shown to be uniquely determined by its mass and luminosity, so that all

stars having the same specified effective temperature lay on a definite curve in this diagram. The curve $T=0$ coincided with the axis of mass, the curve $T=\infty$ with the axis of luminosity, and the various curves representing intermediate effective temperatures lay as shown in fig. 1. Since spectral type depends almost solely on temperature, these curves may also be regarded as the loci of configurations of given spectral type.

Imagine a star of mass $M$, generating energy at a rate $E$, and so of luminosity proportional to $E$, to divide into two precisely similar constituents, each of mass $\frac{1}{2}M$ and each generating energy at a rate $\frac{1}{2}E$. Let P in fig. 1 represent the configuration of the parent star, then the straight line OP will represent all stars for which $E/M$ has the

same value as it has for this star. Thus each of the two constituents \((\frac{1}{4} M, \frac{1}{4} E)\) will lie on this line, and will of course be represented by \(B\), the middle point of \(OP\). A glance at the arrangement of curves in fig. 1 will show that the effective temperature of \(B\) is necessarily higher than that of \(P\). Or, to put the matter in terms of spectral type, the process of fission advances a star to an "earlier" spectral type. It is on these lines that it is suggested that the early spectra of newly formed binaries are to be explained, and we turn to quantitative calculations.

3. In another recent paper,* I have obtained the equation

\[
m + \log T_e + 11.75 \log M + 32.5 \log \Psi_0 = 8.332 \quad (1)
\]

This determines \(T_e\) the effective temperature of a star, and so also its spectral type, in terms of its absolute magnitude \(m\) and its mass \(M\), for \(\Psi_0\) is a function of \(M\) alone.† To understand the meaning of equation (1) more clearly, let us replace \(m\) by its value \(-2.5 \log E\) + a constant, where \(E\) is the rate of emission of radiation from the star. Further, since \(\Psi_0\) is a function of \(M\) which only varies slowly with \(M\) let us put

\[
32.5 \log \Psi_0 = 11.75 \log \theta \quad (2)
\]

so that \(\theta\) is a function of \(M\) which only varies slowly with \(M\). In actual fact \(\theta\) is a positive quantity which is equal to unity for stars of very small mass, is equal to 0.93 for stars equal to the sun in mass, and is equal to 0.57 for a star of five times the sun's mass. After this it decreases only very slowly, being equal to 0.36 for a star of twenty times the sun's mass, and ultimately vanishes for a star of infinite mass.

Equation (1) may now be written in the form

\[
\log T_e = 2.5 \log E - 11.75 \log (\theta M) + a \text{ constant} \quad (3)
\]

showing that \(T_e\) is determined by the balance between the rate \(E\) at which radiation is generated in the star's interior and its mass \(M\), or, if we prefer, its "corrected mass" \(\theta M\). An increase in \(E\) increases \(T_e\), while a decrease in \(M\), \(E\) remaining constant, also increases \(T_e\). If \(E\) decreases more rapidly than \((\theta M)^{4.7}\), a star's effective temperature is decreasing, so that it is moving to a "later" spectral type, while if \(E\) is decreasing less rapidly than \((\theta M)^{4.7}\), \(T_e\) is increasing and the star is moving to an earlier spectral type, and of course conversely as regards increases in \(E\). On the view of stellar evolution I have recently put forward, the value of \(E\) is uninfluenced (to a first approximation at least) by the temperature and density of the star's interior, depending solely on the spontaneous decay of the stellar matter into radiation. The mass also changes as required by the spontaneous decay, so that equation (3) enables us to trace the value of the effective temperature, and consequently also the spectral type, of a star throughout the whole course of its evolution.

4. Let us now return to the simple idealised case already con-

---

* M.N., 1925 March, p. 394.
† This is tabulated in M.N., 1925 March, p. 395.
sidered, in which a star of mass $M$, generating energy at a rate $E$, divides into two precisely similar constituents each having mass $\frac{1}{2}M$ and generating energy at a rate $\frac{1}{2}E$. Equation (3) gives the value of $T'_{\theta}$ for the parent star, while the value for each of the constituents after fission is $T'_\theta$, given by

$$\log T'_\theta = 2.5 \log (\frac{1}{2}E) - 11.75 \log (\frac{1}{2}\theta M) + a \text{ constant} \quad (4)$$

where $\theta'$ is the value of $\theta$ corresponding to mass $\frac{1}{2}M$, and the constant, on the right is the same as in equation (3).

On subtracting corresponding sides of equations (3) and (4), we obtain

$$\log (T'_\theta / T_\theta) = (11.75 - 2.5) \log 2 - 11.75 \log (\theta'/\theta) = 2.785 - 32.5 [\log \Psi_0' - \log \Psi_0] \quad (5)$$

A numerical investigation shows that the right-hand member of this equation is positive for all masses, so that $T'_\theta$ is always greater than $T_\theta$, confirming our inference from the diagram shown in fig. I. Thus when a star divides into two equal and similar parts, the effective temperature of each constituent is higher than that of the parent star, so that the star appears to move towards an "earlier" spectral type.

It will, of course, be understood that this change of type is something quite distinct from the slow evolutionary change imagined by current theories of stellar evolution. The rate at which it occurs, when once the fissional process has set in, is determined solely by the rate at which fission proceeds; it has nothing whatever to do with the star's evolution in the ordinary sense. Speaking with reference to the astronomical scale of time, we can describe the change of type as being almost instantaneous.

5. The Lick "Third Catalogue of Spectroscopic Binary Stars"* contains thirty stars, apparently of recent formation, for which either $m + m'$ or $(m + m')\sin^3 i$, or both, can be determined with fair certainty. These are shown in the following table:

<table>
<thead>
<tr>
<th>Boss.</th>
<th>Star.</th>
<th>Type.</th>
<th>$e.$</th>
<th>$(m + m')\sin^3 i.$</th>
<th>$m + m'$.</th>
<th>$m'/m$.</th>
</tr>
</thead>
<tbody>
<tr>
<td>...</td>
<td>TV Cas.</td>
<td>B9</td>
<td>0.0</td>
<td>...</td>
<td>2.84</td>
<td>0.55</td>
</tr>
<tr>
<td>...</td>
<td>π Cas.</td>
<td>A5</td>
<td>0.010</td>
<td>2.69</td>
<td>...</td>
<td>0.99</td>
</tr>
<tr>
<td>...</td>
<td>ν And.</td>
<td>B3</td>
<td>0.0</td>
<td>2.60</td>
<td>...</td>
<td>0.73</td>
</tr>
<tr>
<td>497</td>
<td>ι Tri f</td>
<td>A5</td>
<td>0.010</td>
<td>1.761</td>
<td>...</td>
<td>1.00</td>
</tr>
<tr>
<td>844</td>
<td>α Per.</td>
<td>B1</td>
<td>0.00</td>
<td>9.21</td>
<td>...</td>
<td>0.70</td>
</tr>
<tr>
<td>1076</td>
<td>d Tau.</td>
<td>A3</td>
<td>0.0</td>
<td>4.0</td>
<td>...</td>
<td>0.47</td>
</tr>
<tr>
<td>1213</td>
<td>66 Eri.</td>
<td>B9</td>
<td>0.09</td>
<td>5.0</td>
<td>...</td>
<td>0.79</td>
</tr>
<tr>
<td>1301</td>
<td>7 Ori.</td>
<td>B1</td>
<td>0.016</td>
<td>21.8</td>
<td>...</td>
<td>0.95</td>
</tr>
<tr>
<td>1314</td>
<td>ψ Ori.</td>
<td>B2</td>
<td>0.0651</td>
<td>9.72</td>
<td>...</td>
<td>0.76</td>
</tr>
<tr>
<td>1478</td>
<td>θ Aur.</td>
<td>A0p</td>
<td>0.00</td>
<td>4.38</td>
<td>...</td>
<td>0.98</td>
</tr>
<tr>
<td>...</td>
<td>B.D. 6° 1309</td>
<td>B0p</td>
<td>0.035</td>
<td>13.89</td>
<td>...</td>
<td>0.84</td>
</tr>
<tr>
<td>2119</td>
<td>V Pup.</td>
<td>B1p</td>
<td>0.08</td>
<td>33</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>3476</td>
<td>α Vir.</td>
<td>B2</td>
<td>0.10</td>
<td>15.4</td>
<td>...</td>
<td>0.61</td>
</tr>
</tbody>
</table>

* L.O.B., No. 385 (1924).

<table>
<thead>
<tr>
<th>Bos.</th>
<th>Star</th>
<th>Type</th>
<th>e</th>
<th>(m+m') sin^2 i</th>
<th>m+m'</th>
<th>m'/m.</th>
</tr>
</thead>
<tbody>
<tr>
<td>..</td>
<td>U Cor. Bor.</td>
<td>B8</td>
<td>0</td>
<td>5.7</td>
<td>5.99</td>
<td>0.38</td>
</tr>
<tr>
<td>4277</td>
<td>μ² Sco</td>
<td>B3p</td>
<td>0.05</td>
<td>16.5</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4328</td>
<td>ζ Her.</td>
<td>A0</td>
<td>0.023</td>
<td>2.7</td>
<td></td>
<td>0.62</td>
</tr>
<tr>
<td>..</td>
<td>U Oph.</td>
<td>B8</td>
<td>0.00</td>
<td>9.90</td>
<td></td>
<td>0.88</td>
</tr>
<tr>
<td>4388</td>
<td>ι Her.</td>
<td>B3</td>
<td>0.053</td>
<td>9.4</td>
<td></td>
<td>0.38</td>
</tr>
<tr>
<td>..</td>
<td>TX Her.</td>
<td>A5</td>
<td>0.00</td>
<td>3.78</td>
<td>3.81</td>
<td>0.86</td>
</tr>
<tr>
<td>4643</td>
<td>108 Her.</td>
<td>A2</td>
<td>0.00</td>
<td>2.90</td>
<td></td>
<td>0.69</td>
</tr>
<tr>
<td>4776</td>
<td>β Lyr.</td>
<td>B8p</td>
<td>small</td>
<td>23.41</td>
<td>30.57</td>
<td>0.45</td>
</tr>
<tr>
<td>4788</td>
<td>50 Dra.</td>
<td>A0</td>
<td>0.012</td>
<td>1.85</td>
<td></td>
<td>0.95</td>
</tr>
<tr>
<td>..</td>
<td>RS Vul.</td>
<td>B8</td>
<td>0.053</td>
<td>5.70</td>
<td>7.09</td>
<td>0.31</td>
</tr>
<tr>
<td>..</td>
<td>Z Vul.</td>
<td>B3</td>
<td>0.00</td>
<td>7.60</td>
<td></td>
<td>0.45</td>
</tr>
<tr>
<td>4947</td>
<td>2 Sge.</td>
<td>A0</td>
<td>0.05</td>
<td>1.56</td>
<td></td>
<td>0.71</td>
</tr>
<tr>
<td>5173</td>
<td>18 Vul.</td>
<td>A2</td>
<td>0.012</td>
<td>4.33</td>
<td></td>
<td>0.91</td>
</tr>
<tr>
<td>..</td>
<td>Y Cyg.</td>
<td>B2</td>
<td>0.00</td>
<td>31.57</td>
<td>31.9</td>
<td>0.92</td>
</tr>
<tr>
<td>5375</td>
<td>57 Cyg.</td>
<td>B3</td>
<td>0.137</td>
<td>3.46</td>
<td></td>
<td>0.93</td>
</tr>
<tr>
<td>5764</td>
<td>2 Lac.</td>
<td>B5</td>
<td>0.015</td>
<td>1.58</td>
<td></td>
<td>0.82</td>
</tr>
<tr>
<td>6142</td>
<td>..</td>
<td>Bo</td>
<td>0.105</td>
<td>31.2</td>
<td></td>
<td>0.69</td>
</tr>
</tbody>
</table>

The average value of m+m' for the seven stars for which this can be determined is 12.80. For the remaining twenty-three stars (m+m') sin^2 i has the average value 13.20. If we follow Schlesinger in supposing that the chance of detection of a binary is proportional to sin i, then the average value of sin^2 i for all detected binaries must be 0.68, whence, neglecting any small correlation there may be between m+m' and sin i (for discovered binaries), we find that the average value of (m+m') for these twenty-three stars is 19.40. The average value of m+m' for the whole list of thirty stars is now 17.90.

The values of T_e'/T_e calculated from equation (5) for different values of M (or m+m') are as follows:

- \[ M=50 \] log (T_e'/T_e)=0.575 or T_e'=3.76T_e
- \[ M=25 \] log (T_e'/T_e)=0.673 or T_e'=4.71T_e
- \[ M=17.9 \] log (T_e'/T_e)=0.816 or T_e'=6.55T_e
- \[ M=10.0 \] log (T_e'/T_e)=1.017 or T_e'=10.40T_e
- \[ M=5.0 \] log (T_e'/T_e)=1.417 or T_e'=26.12T_e

6. The ideal case of a star dividing into two equal and similar masses cannot ever be realised in nature, for the circumstance that instability first sets in through the unsymmetrical third zonal harmonic introduces a dissimilarity from the outset. The average mass-ratio of the twenty-eight stars in the above table, for which m'/m is known, is found to be 0.7326, representing the fission of a mass M into constituents of masses 0.577M and 0.423M. Aitken* gives the average mass-ratio of binaries of types B to B8 as 0.70, representing fission into constituents of masses 0.588M and 0.412M. The luminosity also is divided unevenly. The larger constituent is almost invariably

* The Binary Stars, p. 207.
the most luminous (β Lyrae possibly providing a solitary exception), but the ratio of \( E/M \) can vary widely for the two components.

Consider now the general case of a parent star dividing so that its mass \( M \) forms constituents \( \mu M, (1-\mu)M \) between which its rate of generation of energy \( E \) is divided in amounts \( \eta E, (1-\eta)E \). In place of equation (5) we readily find the equation

\[
\log \left( \frac{T_e'}{T_e} \right) = 2.5 \log \eta - 11.75 \log \mu - 32.5 \log \eta \log \Psi_0' - \log \Psi_0 \, .
\]  

(6)

The following table gives values of \( T_e'/T_e \) for different values of \( \eta \) and \( \mu \), the combined mass of the star being assumed to be 20\( \odot \). The values \( \mu = 0.42 \) and 0.58 represent average values for the two components of a binary, the values \( \mu = 0.33 \) and 0.67 represent quite extreme values, and the value \( \mu = 0.50 \) of course represents the ideal case already considered of division into two equal parts.

<table>
<thead>
<tr>
<th>( \mu )</th>
<th>0.33</th>
<th>0.42</th>
<th>0.50</th>
<th>0.58</th>
<th>0.67</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \eta = 0.33 )</td>
<td>22.9</td>
<td>5.89</td>
<td>2.34</td>
<td>1.00</td>
<td>0.48</td>
</tr>
<tr>
<td>( \eta = 0.42 )</td>
<td>40.7</td>
<td>10.5</td>
<td>4.17</td>
<td>1.78</td>
<td>0.85</td>
</tr>
<tr>
<td>( \eta = 0.50 )</td>
<td>63.1</td>
<td>16.2</td>
<td>6.31</td>
<td>2.75</td>
<td>1.32</td>
</tr>
<tr>
<td>( \eta = 0.58 )</td>
<td>91.2</td>
<td>23.4</td>
<td>9.33</td>
<td>3.98</td>
<td>1.91</td>
</tr>
<tr>
<td>( \eta = 0.67 )</td>
<td>128.8</td>
<td>33.1</td>
<td>13.2</td>
<td>5.62</td>
<td>2.69</td>
</tr>
</tbody>
</table>

Some of the entries obviously correspond to cases which never occur in nature: in particular, all those printed in italics violate the condition that the more massive constituent is to be the brighter.

Shapley * has found that for the average binary the ratio between \( \mu \) and \( \eta \) is expressed approximately by the equation

\[
2\mu = 0.4 + 1.2\eta \, .
\]  

(7)

The values of \( T_e'/T_e \) which occur when \( \mu \) and \( \eta \) are connected by this relation are found to be as follows:

\[
\mu = \begin{array}{c}
0.33 \\
0.42 \\
0.50 \\
0.58 \\
0.67 \\
\end{array}
\quad \eta = \begin{array}{c}
0.21 \\
0.37 \\
0.50 \\
0.63 \\
0.77 \\
\end{array}
\]

\[
(M=20\odot) \quad T_e'/T_e = \begin{array}{c}
11.5 \\
7.59 \\
6.31 \\
4.90 \\
3.89 \\
\end{array}
\]

\[
(M=10\odot) \quad T_e'/T_e = \begin{array}{c}
27.5 \\
16.6 \\
10.5 \\
7.08 \\
5.25 \\
\end{array}
\]

We see that the less massive constituent has the "earlier" spectral type. This is a consequence solely of Shapley's suggested law (7) and of our equation (1): it is not in any sense dependent on any special hypothesis as to the origin of binary stars.

7. It may be thought that all our calculated values for \( T_e'/T_e \) are impossibly high. It has to be admitted that they are very sensitive to changes in the value of the coefficient of \( \log T \) in equation (1). We have taken this coefficient to be unity, but a value 2 is not altogether inadmissible. The effect of changing the coefficient from 1 to 2 would be to halve all our calculated values for \( \log T_e'/T_e \), and so to replace all values of \( T_e'/T_e \) by their square roots. The values of \( T_e'/T_e \) is also

calculated subject to the supposition that the rate of generation of energy in the two constituents after fission is equal to that in the parent star before fission. Our values for $T'_s/T_s$ would require correction if the process of fission affected the rate of generation of energy. We ought, perhaps, to interpret our figures merely as showing that there are ample possibilities for fission to raise the temperature of a star up to that of types B and A.

8. It is of course possible to work the tables backwards and compute the state before fission of the binary stars shown in the table of § 5. A reconstruction of the states of known binaries before fission took place shows that their states during fission must have occupied something like the range occupied by the long-period and Cepheid variables. This inevitably suggests that the long-period and Cepheid variables may be stars in process of fission into binaries. The hypothesis is by no means free from difficulties, but it would seem to be well worthy of discussion, and the remainder of the present paper is devoted to a preliminary discussion of the hypothesis.

9. The hypothesis as to the physical nature of Cepheid variables which at present holds the field is the well-known "pulsation theory," which was first brought into favour by the arguments of Plummer,* Shapley,† and others, and to which precision has been given by the mathematical investigations of Eddington.‡ In so far as any hypothesis of the nature of long-period variables holds a corresponding position, it is perhaps that of Merrill, which supposes the light variation to be the consequence of a periodic condensation of clouds in the star's atmosphere, and their subsequent dispersal by the radiation imprisoned by them.

For the purpose of mathematical treatment, Eddington treats a Cepheid as being spherical and devoid of rotation, the period of light-variation being supposed to represent the period of a natural free pulsation of the star. Merrill's hypothetical condensations and dispersals must be accompanied by a rhythmical heating and cooling of the outer layers at least of the star, and so by a pulsation of the star. It is perhaps more natural to regard this pulsation as the fundamental phenomenon, so that again the period must be that of a natural free pulsation.

Any such interpretation of the periodicity appears to be open to fatal objections. Taking the widest possible view of its structure and mechanism, a spherical star at rest will possess a certain number of possible configurations of equilibrium, and these can be divided perfectly sharply and definitely into stable and unstable configurations. The characteristic of a vibration about an unstable configuration is that its amplitude increases without limit until finally the star settles down to some entirely different, and stable, configuration of equilibrium. The characteristic of a vibration about a stable configuration is that its amplitude is gradually damped by viscosity and other dissipative forces until it falls to zero. The dynamics of oscillations of a non-rotating system has no knowledge of finite oscillations maintained at a constant amplitude.

* M.N., Vols. 73–75.
† Astrophys. Journ., 40 (1914), 448.
‡ M.N., 79 (1918), 1.
It is somewhat difficult to estimate what would be the rate of decay of the pulsations of a gaseous star. If \( \gamma \) is an average coefficient of viscosity for its interior, the time in which viscosity would reduce the amplitude of a vibration by one-half is of the order of \( R^2 \rho / \gamma \) or again of \( M/4 \gamma p \). The radii and masses of a number of Cepheids have been tabulated by Eddington.* If we take \( M=5 \odot \) and \( R=2 \times 10^{13} \) (or \( M=10 \odot \) and \( R=4 \times 10^{13} \)), the value \( \gamma=0.002 \) would give a time of about \( 2 \times 10^8 \) years for the reduction of an oscillation to half its amplitude. But the value \( \gamma=0.002 \), which is appropriate to an ordinary gas under atmospheric conditions, is far too low for the ionised gas which we believe to constitute a star's interior. The transport of momentum in such a gas results mainly from the motion of the atomic nuclei, and at \( 10^7 \) degrees C. these have a velocity of about two hundred times the velocities at ordinary atmospheric temperature. Thus our value ought to be increased some two hundredfold, thereby reducing the time in question to about \( 10^7 \) years. Judging from the number of Cepheids in the sky, this probably gives too short a life to the pulsations of a Cepheid. A more serious objection, however, is that a mass of gas has a large number of free vibrations whose periods are, in general, incommensurable. The free pulsations of a sphere of gas could only exhibit regularly recurring maxima in the very improbable event of all the vibrations except one having been damped to zero amplitude, while this particular one persisted with vigour. Finally, if the Cepheids were masses of gas whose pulsations were dying down, we ought to find pulsations of all amplitudes from zero upwards. In actual fact the light-variations cluster, with comparatively little dispersion, around a mean of about 0.7 m.; on the pulsation theory this would only be intelligible if all the pulsations had been set going at about the same time and with about equal amplitudes.

10. When we pass to systems in motion, and in particular to rotating systems, we no longer find that configurations of equilibrium or of steady motion can be sharply divided into stable and unstable configurations. For a rotating system there is an intermediate class of configurations which are "ordinarily" stable, but are secularly unstable, and a rotating system can execute oscillations of permanently finite amplitude about such a configuration. As the suggestion of the present paper is that these are precisely the oscillations observed in variable stars, it may be worth giving a short account of the main conclusions of the theory of secular instability.

A system oscillating about a state of rest keeps its amplitude of oscillation constant except in so far as this amplitude is altered by dissipative forces. In this case the effect of dissipative forces is invariably to lessen the amplitude of vibration. The reason is simple: dissipative forces use up energy and the only source from which energy can be fed to them is the energy of the oscillation itself.

The case is different with a system oscillating about a state of dynamical steady motion or about a position of relative equilibrium while in rotation. Here dissipative forces may either decrease or increase the amplitude of oscillation. A new source of energy, from

* M.N., 79 (1918), 4.
which the dissipative forces can be fed, is present in the motions of the constituent bodies of the system, or in the case of a single rotating body, in the rotation itself. There is no automatic regulation of the rate at which energy is fed to the dissipative forces; it may be either faster or slower than the rate at which they consume energy, so that the amplitude of a vibration may either increase or decrease.

11. A good instance of secular instability was provided, although unconsciously, by Sir G. Darwin in his investigations on tidal friction. Let us consider two bodies such as the earth and moon rotating about one another in orbits which are initially circular. The position of the moon is fixed by two co-ordinates \( r, \theta \); and in the case we are considering \( r \) and \( \theta - \omega t \) (which is the value of \( \theta \) referred to rotating axes) have constant values, say \( a \) and \( e \). If there are no dissipative forces, the moon, if knocked slightly out of its orbit, will describe a new somewhat elliptic orbit and will repeat this orbit indefinitely. Its motion can now be regarded as one of oscillation about a new circular orbit \((a', \omega')\), so that if \( a' \) and \( \omega' \) are suitably chosen the two co-ordinates \( r \) and \( \theta - \omega t \) will both be performing small oscillations about values \( a', \omega' \). In the absence of dissipative forces these oscillations remain permanently of the same period; we may notice that they both have the same frequency as the orbit about which they oscillate or, if we prefer, the same frequency as the rotation of the axes to which they are referred.

If, however, the dissipative forces of tidal friction are in play, the oscillations do not gradually decrease in amplitude, as might perhaps à priori be expected. As Darwin showed, they both increase, the moon describing an orbit of ever-increasing radius and of ever-increasing eccentricity. The dissipative forces are in fact being fed from the earth's energy of rotation, and as soon as this source is used up, the increase in the oscillations ceases.

12. An instance of a simpler kind has been worked out by Lamb.* Imagine a small spherical particle placed inside a spherical bowl of radius \( a \) which is rotating with angular velocity \( \omega \) about a vertical axis. If there is no friction between the bowl and the particle, there is a stable position of equilibrium at the lowest point of the bowl, and if the particle is displaced from this position, it will execute oscillations of constant amplitude about it. But if the interior of the bowl is rough, the particle refuses to stay in this equilibrium position; it picks up the motion of the bowl and describes spirals of ever-increasing radius about the lowest point of the bowl until it ends by describing permanently a circle at height \( a - g/\omega^2 \) above this point, at the same speed as the bowl. This is the same circle as would be described by a conical pendulum rotating with angular velocity \( \omega \). Lamb shows that in its earlier stages the motion of the particle may be resolved into two circular motions in opposite directions, that whose motion is opposite to the bowl being gradually extinguished, while that whose motion is with the bowl gradually increases in radius until it attains to the value already stated.

13. Returning to astronomical problems, it is known that a mass of rotating matter, which has not got too great a central condensation of

mass, becomes secularly unstable as soon as it passes a certain clearly defined point on the Jacobian series of ellipsoids. The oscillation through which instability first sets in is one in which the surface of the ellipsoid undergoes a third harmonic deformation. The star begins by being pear-shaped, the extent of divergence from the ellipsoidal shape being oscillatory in amount, but showing a general tendency to increase. Although it is not possible to trace out the course of events in detail, it is fairly certain that the process ends by fission into two stars which at first describe orbits of small eccentricity about one another. After this point the story is carried on by tidal friction. As soon as the body divides, the dissipative forces which have so far produced secular instability become identical with those which now produce tidal friction, while the oscillations of the star become resolved into the oscillations of the co-ordinates of the two constituents which, in the way already described, constitute the description of an elliptic orbit.

Thus the beginning and end of the process of fission are known in some detail. At the beginning there is a rotation and an oscillation, and the periods of the two are different and in general incommensurable. At the end there is a rotation (or rather by now a revolution) and an oscillation, and the periods of the two are the same, except for a slight difference arising from the fact that the two constituents, not being perfectly spherical, do not attract according to the exact inverse square law. Except for this insignificant correction, we may conclude that during the process of fission the periods of oscillation and of rotation gradually approximate to one another, and that they have become identical by the time the fission is complete.

Thus at the beginning of fission the light curve would exhibit two (or possibly more) superposed and generally incommensurable periods. There would be no regularly recurring maxima, and in its general characteristics the light curve would resemble that of the "irregular" variables. Before the end of the fissional process there would be only one period, that of the star's rotation, and the light curve would show a definite periodicity with regularly recurring maxima. In this respect the light curve would be similar to that of the Cepheids and of the majority of the long-period variables. After the fission is complete this period persists as that of the star's rotation and is gradually modified in accordance with the theory of tidal friction. We are thus led to conjecture that the following scheme may represent the process of fission:

<table>
<thead>
<tr>
<th>Shape of Star</th>
<th>Class of Star</th>
<th>Light Curve</th>
<th>Spectrum</th>
<th>Range</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pear-shaped</td>
<td>Irreg. variable.</td>
<td>Periods superposed.</td>
<td>M, N.</td>
<td>Up to 6 m.</td>
</tr>
<tr>
<td>,</td>
<td>Long-period variable.</td>
<td>Traces of irregularity.</td>
<td>M.</td>
<td></td>
</tr>
<tr>
<td>,</td>
<td>Cepheid.</td>
<td>Regular periodicity.</td>
<td>M to A.</td>
<td>0.4 to 1.0 m.</td>
</tr>
<tr>
<td>Two ellipsoids.</td>
<td>Early spectroscopic and eclipsing binaries.</td>
<td>Orbital periodicity.</td>
<td>A, B.</td>
<td>Up to 0.7 m.</td>
</tr>
</tbody>
</table>
The last column but one shows the progress of spectral type which, as we have seen, must necessarily accompany the progress of fission, no matter how rapidly the process of fission takes place.

14. It will be seen that this scheme requires that the mass of the long-period variable or of the Cepheid variable shall be equal to the sum of the masses of the two constituents of the binary which it will ultimately form by fission. That is to say, the mass of certain stars of types M, K, G, F must be supposed to be about double that of certain other stars of types B and A. Some astronomers may feel that this is a grave objection; there is a general belief, supported to some extent by the recent researches of Seares, that even in the giant stars the early type stars are the most massive. As Shapley has pointed out, however, general conclusions of this kind are often better formed and better tested from a study of distant star-clusters than from a study of the galactic system, since in studying the star-clusters the errors arising from uncertainties of parallax and the disturbances resulting from selection by nearness are avoided. Shapley’s studies of the star-clusters seem to indicate quite definitely that in them at least the M stars are brighter than the B stars, whence, a fortiori, the M stars must be more massive than the B stars.

Newly formed binaries exhibit a large range in mass and density, whence we must suppose that stars in which the process of fission is just commencing must exhibit a similar range. Thus no single star must be expected to cover the whole range of density from the most diffuse of irregular variables, such as α Orionis, up to the density of the shortest period eclipsing binaries. A certain amount of control is provided by the principle of conservation of angular momentum, so that it can be shown that α Orionis can never break up into a binary of very short period, while RR Lyre cannot, when fission commenced, have been a star of very low density. The full discussion of this question must, however, be reserved for a later paper.

15. From what has been said, it is clear that when a star which is in process of breaking up shows a definite periodicity, the periodicity must be that of the star’s rotation. Thus according to our scheme a Cepheid of twenty days’ period must be rotating in twenty days.

This provides a check on our theory, since in any star in which fission is in progress there is a simple relation between a star’s density and its period of rotation. Let us consider first the ideal case of an incompressible mass. When instability first sets in, the density and the velocity of rotation are connected by the relation

$$\frac{\omega^2}{2\pi\gamma\rho} = 0.14200$$  \hspace{1cm} (8)

When the star has broken into two equal masses which are assumed for convenience to be spherical and rotating in contact, the relation is

$$\frac{\omega^2}{2\pi\gamma\rho} = 0.16667$$  \hspace{1cm} (9)

Inequality in the two constituent masses does not greatly alter this
figure, and neither does homogeneity of the matter provided that $\rho$ is taken to denote the mean density.

Thus when a star shows a quite regular period $w^2$ and $\rho$ ought to be connected by an equation of the general type of (8) and (9). Probably equation (9) provides a good enough approximation. Denoting the period in days by $P$, this can be written in the form

$$\rho = \frac{0.075}{P^2} \quad . \quad . \quad . \quad (10)$$

In the following table the average period for Cepheids of different type are those estimated by Shapley*; the fourth column gives the density of the Cepheid as calculated from relation (10). The last two columns give the average densities of stars of the same spectral type as estimated respectively by Seares and Russell.

<table>
<thead>
<tr>
<th>Star. Type</th>
<th>Average Period</th>
<th>$\rho$ (calc.)</th>
<th>$\rho$ (Seares)</th>
<th>$\rho$ (Russell)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Long period</td>
<td>M 350</td>
<td>.0000006</td>
<td>.0000006</td>
<td>.</td>
</tr>
<tr>
<td>Cepheid</td>
<td>M 33</td>
<td>.00007</td>
<td>.00001</td>
<td>.005</td>
</tr>
<tr>
<td></td>
<td>K 18</td>
<td>.0002</td>
<td>.0002</td>
<td>.040</td>
</tr>
<tr>
<td></td>
<td>G 11</td>
<td>.0006</td>
<td>.0002</td>
<td>.040</td>
</tr>
<tr>
<td></td>
<td>F 6</td>
<td>.0021</td>
<td>.0004</td>
<td>.</td>
</tr>
<tr>
<td></td>
<td>A 0.4</td>
<td>.047</td>
<td>.008</td>
<td>.14</td>
</tr>
</tbody>
</table>

As Shapley has pointed out, a relation of the type of (10), but with a different numerical factor, would hold on the pulsation theory also. The calculated densities agree with observation about equally well on either hypothesis; both hypotheses survive this test, but the pulsation theory appears to the present writer to be ruled out by the objections previously stated.

16. A further test is provided by calculating $\rho$ from our theoretical equation (1). If $a$ is the radius of the star, assumed for simplicity to be spherical, we have $M = \frac{4}{3} \pi a^3$ and $E \propto T_e^4 a^2$. In terms of logarithms these equations and equation (10) become

$$\log M = \log \rho + 3 \log a + \text{cons.} \quad . \quad . \quad (11)$$
$$-0.4m = 4 \log T_e + 2 \log a + \text{cons.} \quad . \quad . \quad (12)$$
$$\log P + 2 \log \rho = \text{cons.} \quad . \quad . \quad (13)$$

Let us write the theoretical equation (1) in its approximate form (cf. equation (3)).

$$c_1 \log T_e + c_2 \log M + c_3 m = \text{cons.} \quad . \quad . \quad (14)$$

We have already noticed that over the range of mass $5\odot$ to $20\odot$, which is approximately that of the Cepheid variables, the quantity $\theta$ varies only slowly with $M$, so that this equation ought to give a good approximation for the Cepheid variables.

Eliminating $a$, $T_e$, and $M$ from equations (11) to (14), we obtain

$$(12c_2 - 2c_3)m + 4c_2 \log P + (12c_2 - 2c_1) \log T_e = \text{cons.} \quad . \quad (15)$$

It is difficult to determine the constants $c_1$, $c_2$, $c_3$ in equation (14) with any accuracy, but all investigations agree in showing that $c_3$ is quite large in comparison with $c_1$ and $c_2$. If, as a first rough approximation, we neglect the ratios $c_1/c_2$ and $c_2/c_3$ altogether, equation (15) becomes

$$0.3m + \log P + 3 \log T = \text{cons.} \quad (16)$$

If we treat $T_\epsilon$ as a constant, thereby limiting ourselves to a small range of spectral type, equation (16) becomes

$$\log P + 0.3m = \text{cons.} \quad (17)$$

This expresses the period-luminosity law. The curve given by Shapley to express this law has for its equation in the region for stars brighter than abs. mag. $-2$, approximately

$$\log P + 0.29m = \text{cons.} \quad (18)$$

which is in satisfactory agreement with (17).

Further approximations to the period-luminosity law present a more difficult problem which is reserved for another paper.

17. At any instant let $E$ denote the total rate of emission of energy of a variable star, let $A$ be the total area of its surface, and let $\theta A$ be the area visible to a terrestrial observer. Disregarding the "darkening" effect at the limb, the star's brightness will appear to a terrestrial observer to be proportional to $\theta E$, so that its estimated absolute magnitude will be of the form

$$-0.4m = \log \theta + \log E + \text{cons.} \quad (19)$$

In this equation both $\theta$ and $E$ will vary periodically, the period of $\theta$ being either that of the star's rotation or else half of this period, while the period of $E$ will be that of the star's oscillations. We have seen that in the earlier stages of fission these two periods will in general be unequal and incommensurable, so that the light curve represented by equation (19) will not exhibit regularly spaced maxima and the star will be classified as an "irregular" variable. At a later stage the periods of $\theta$ and $E$ will become equal and the light curve will show regular periodicity.

When this latter stage is reached, let the frequency of the star's oscillation be $n/2\pi$, so that $\log E$ will be of the form $f(nt)$, where $f(nt)$ is periodic in $nt$ with period $2\pi$. Then $\log \theta$ will be of the form $\phi(2nt)$ if the rotating body is of such a shape that $\theta$ repeats itself twice in a revolution; otherwise it will be of the form $\Phi(nt)$. The rotational terms will, of course, differ in phase to any extent from the term $f(nt)$ which arises from oscillations of the star's mass. If for simplicity we suppose $\phi(2nt)$ and $\Phi(nt)$ to be simple cosine terms with constants added, equation (19) will assume the alternative forms

$$m = m_0 + A \cos (2nt - e) + Bf(nt) \quad . \quad . \quad . \quad (20)$$

$$m = m_0 + A' \cos (nt - e') + B'f(nt) \quad . \quad . \quad . \quad (21)$$

18. An analysis of the light curves of Cepheid variables suggests that these can be expressed by equations of these forms (or possibly by equation (20) alone) where the graphs of $f(nt)$, as a function of $nt$,
are very similar for all Cepheids, the observed differences in the light curves arising from variations in the phase $e$ and in the ratio of the coefficients $B/A$ or $B'/A'$. Further, the spectral variations as recorded by Shapley* appear to follow the value of $f(nt)$ very closely, the maximum of $f(nt)$ coinciding with the earliest type of spectrum attained by the star, and vice versa. This is, of course, precisely what ought to be expected if the term $f(nt)$ represents the effect of the physical oscillation of the star, while the cosine terms depend on the star's orientation relative to our line of sight. Fig. 2 shows the light curve of the Cepheid variable S Sagittæ† analysed in the way suggested.

* *Astrophys. Journ., 44 (1916), 273.*

† *Cf. Shapley, loc. cit., p. 288.*
The continuous curve above the axis is the observed light curve as given by Gore and Luizet, the dotted curve is the suggested graph of $f(mt)$, and the entries along the time-axis denote the spectral type as recorded by Shapley. In passing, we may notice how entirely impossible it is to interpret a light curve of this shape in terms of a pure pulsation hypothesis.

According to this view of Cepheid variation, the Cepheids ought to change continuously into long-period variables at one end and into eclipsing or spectroscopic binaries at the other end. There ought to be no clearly defined line of demarcation between Cepheids and binaries, and when the attempt to interpret the light and velocity curves in terms of the orbital motion of a binary leads to satisfactory and consistent results, there would seem to be no reason why this interpretation should not be adopted. When this interpretation fails, it seems natural to suppose that the process of fission is not yet complete and to attempt an explanation in terms of the oscillations of a rotating body, after the manner suggested in equations (20) and (21).

The constituents of a fully formed binary cannot legitimately be treated as particles, so that their mutual attraction, as has already been remarked, will not be precisely $mm'/r^2$, and the orbit will not be precisely elliptical. To a first approximation the correction to be applied to the ordinary Keplerian motion will consist of a rotation of the apsidal line of the orbit and a consequent slow variation in the light curve. The case of a binary in which the constituents are not yet fully separated is in general similar, but the irregularity is now more pronounced. There is a rotation of the line passing through the longest diameter of the mass when at its greatest elongation, the line which ultimately will develop into the apsidal line of the orbit, and this may be represented analytically by slow changes of the phases $e$, $e'$ entering into equations (20) and (21). To a terrestrial observer it will appear that the light curve is gradually changing its character, and he will further observe fluctuations in the periods between successive maxima. Prolonged observation ought to suggest a cyclical period of the form

$$P = P_0 + Q \cos at$$

where $2\pi/\alpha$ is the period of rotation of the embryo apsidal-line.

19. Thanks to the valuable work of Phillips, the discussion of the light curves of long-period variables is a much simpler matter. Phillips has analysed the light curve, as far as the third harmonic term, in the form

$$m = P_0 + P_1 \cos (\theta - \pi) + P_2 \cos (2\theta - \phi_2) + P_3 \cos (3\theta - \phi_3) \quad (22)$$

He finds that all the stars fall into two groups, the first being distinguished by the fact that $\phi_2$ has the approximately constant value $202^\circ\cdot1$, while the values of $\phi_2$ and $\phi_3$ in the second group are connected approximately by a linear relation which Phillips expresses as

$$\phi_3 = 1.67\phi_2 - 126^\circ\cdot8$$

(23)

These values for $\phi_2$ are, of course, only approximate, and I find that the value of $\phi_3$ for the second group can almost as well be expressed by

$$\phi_3 = 1.5\phi_2 - 107.5$$

(24)

This change involves the transfer of a few stars from Group II. to Group I.; it has the advantage, as we shall see, that it permits of a physical interpretation being given to the two groups.

Phillips finds for the coefficients $P_1, P_2, P_3$ values which, to a rough approximation, are in the same ratio for all the stars of the same group. Using the mean values for these coefficients, the light curves of the two groups are found to be

(Group I.)

$$m = 2.03 \cos (nt - \pi) + 0.23 \cos (2nt - \phi_2) + 0.11 \cos (3nt - 202.1)$$

(25)

(Group II.)

$$m = 2.41 \cos (nt - \pi) + 0.44 \cos (2nt - \psi_2) + 0.18 \cos (3nt - 1.5\psi_2 + 107.5)$$

(26)

in which the phase $\psi_2$ in Group II. has been replaced by its value as given by equation (24).

If we refer the light curves of Group II. to new origins by the change of phase involved in writing $nt + 0.5\psi_2 - 103.2$ for $nt$, equation (26) becomes

(Group II.)

$$m = 2.41 \cos (nt + 0.5\psi_2 - 283.2) + 0.44 \cos (2nt - 206.4) + 0.18 \cos (3nt - 202.1)$$

(27)

Let us now put

$$f(nt) = 3.32 \cos (nt - \pi) + 0.44 \cos (2nt - 206.4) + 0.18 \cos (3nt - 202.1),$$

and we find that equations (25) and (27) assume the forms

(Group I.)

$$m = \Lambda \cos (2nt - \epsilon) + \frac{1}{3}f(nt)$$

(Group II.)

$$m = \Lambda' \cos (nt - \epsilon) + f(nt)$$

which are precisely identical with our theoretical equations (20) and (21).

This last result suggests strongly that the long-period variables may admit of explanation in terms of the theory which led to equations (20) and (21)—in other words, that they may be interpreted as stars in the process of fission. The details of course remain to be worked out; at present the suggestion is merely put forward on broad lines for consideration and, it is hoped, for discussion.

20. One interesting possibility may be noticed in conclusion. The majority of M-type long-period variables show a single definite period, which, on our theory, suggests that they are well advanced in the process of fission. This, together with the numerical values we calculated in § 5, combine to suggest that normally fission must start while a star is still non-luminous.

This indicates a possible way of clearing up certain difficulties which beset the general question of the birth of stars in spiral nebulae.

The total luminosity of a nebula has always seemed to be far too small compared with that of the stars which calculation shows ought to be.
born out of it. So long as we are concerned only with visible radiation this is certainly the case, but it now appears possible that the great bulk of the radiation both from the nebulae and the stars may be invisible. Just as fission raises the temperature of a star's surface, so the break-up of a nebula into stars may raise the temperature of the surface of the stellar matter involved. To put the matter in an extreme way, it is conceivable that the generation of energy in the matter of a nebula may fail to raise it to a temperature at which it emits any radiation capable of detection while it is in nebular form. When the matter has divided into stars the same total generation of energy may raise these stars to somewhere near to the limit of visibility, and further fission into binaries may raise them to about B-type. This would give an immediate explanation of Scarpa's observation that the outer parts of spiral arms are bluer than the inner parts.

Calculation * indicates that fission ought to commence in stars born out of spirals when they reach densities of the order of $10^{-8}$ or lower, and at this density a star of ordinary mass would be in a state of complete darkness. If stars of this low density were of such high temperature that their substance was fully ionised there would be some difficulty in understanding the process of fission, for fission requires more uniformity of density throughout the mass than we should expect in a mass of uniform substance at a density of the order of $10^{-8}$. It is, however, possible that we have to deal with a mass whose outer parts consist of unbroken molecules and in which the ionisation steadily increases as we move towards the centre. If so, the difficulty would, to a large extent, disappear. A great deal more of mathematical analysis appears to be required before a final opinion can be expressed on this and on many similar questions.


1. In any interpretation of the connection between planetary and diffuse nebulae and the stars which they surround,† it becomes desirable to know to what extent atoms of any element can be driven out from the central star. For instance, the emission of corpuscular rays from a very hot star has more than once been suggested as a means of excitation of the spectrum of a surrounding nebulosity. The problem of the escape of atoms from an atmosphere in gravitational equilibrium has been worked out rigorously by Milne.‡ The result of his work is as follows:—

(a) Atoms of hydrogen, and a fortiori all heavier atoms, cannot

* Cosmogony and Stellar Dynamics, § 255.