THE SPECTROSCOPIC DETERMINATION OF STELLAR ROTATION AND ITS EFFECTS ON LINE PROFILES.

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In a previous communication to the Society* I discussed the effects of axial rotation of a star on the observed form of its absorption lines, and formulated the integral equation connecting the observed form of the line with the "true" form. By the true form is meant the form which would be observed by an observer at rest with respect to axes fixed in the star, or, in other words, the form of the line yielded by an element of the disc moving only across the observer's line of sight.

The problem existing is twofold. Firstly, to determine the rotational speed of the star, or, more correctly, its equatorial velocity in the line of sight, and secondly, given this quantity, to deduce from the observed line the form of the line emitted by an element of the stellar surface.

At the time of my earlier discussion few measurements of the shapes of stellar lines existed. As it was not thought that stellar rotational speeds exceeded some 20–30 km./sec. at the star's equator, I obtained a solution of the equations valid only when the distortion of the line due to rotation is slight, and dealt for the most part with the change in the central intensity of a stellar absorption line. Since then many observations have been made of the profiles† of stellar absorption lines, and in particular the observations of Struve and Elvey have shown‡ that velocities of the order of ten times the limit formerly supposed to exist are quite likely.

Shajn and Struve‡ showed that in rapidly rotating stars the lines are often of a broad shallow form to which the name "dish-shaped" was very appropriately attached. Elvey showed, by working out numerically from a line of ordinary shape the observed form due to various rotational speeds, that this is the shape to be expected when the speed is high. Strictly speaking, as will be shown below, this shape is characteristic not specifically of high rotational speeds but of great distortion of the original line, though since stellar lines are not usually very narrow a high speed will as a rule be required to produce the effect.

Struve and Elvey have made estimates of rotational speeds for a number of stars, by a process of matching the observed form of the line with shapes calculated from an assumed original or true shape at various speeds. It

† At the 1932 Meeting of the I.A.U. it was agreed informally, after discussion by the Solar Physics Commission, that the use of the word "contour" to denote the shape of a spectrum line is undesirable, and the word "profile" was recommended as more suitable.
will appear later that this method gives, under certain restrictions, a fair estimate of rotational speed, assuming this to be the cause of the deformation; the main defect of the method is that it assumes a knowledge of the original line we do not possess.

In view of these discoveries I re-examined the problem and have obtained a complete solution of the equations, leading to several interesting results, and some valuable formulae for use in various cases occurring in practice.

Since two unknown quantities are involved in the production of the observed line, namely, the original form of the line and the rotational speed of the star, it is to be expected that they cannot be determined individually without either some further experimental evidence or some assumption in addition to a knowledge of the observed line. Surprisingly, this expectation turns out, as will be shown, to be wrong; and theoretically, if only an observed line is distorted by stellar rotation, it is possible to discover this fact and to find both the rotational speed of the star and the original form of the line.

In my earlier discussion I pointed out that a priori knowledge of relative central intensities of lines would suffice, but, under conditions existing then and now, this is in practice of no service. To make any progress some additional assumption appeared to be necessary. Elvey and Struve assume a knowledge of the original form of the line. This is of course adequate, but is obviously to be avoided if possible. In the present discussion it is shown that far less sweeping assumptions suffice to determine the rotational speed (with considerable precision, if it be large) and hence the original form of the line. This has two advantages: it eases the practical calculation of the quantities required, and, what is more important, makes the belief that the effects observed are due to rotation of the star much more probable.

In the discussion of the problem which follows I shall show how it is possible to find out whether an observed line has been distorted by rotation of the emitting star and how the rotational speed may be found, then certain other devices for finding the rotational speed will be described. These latter involve assumptions not strictly necessary theoretically, but useful in certain cases occurring in practice, and facilitating calculation and examination of particular observed lines. The problem of computing the "original" form of the line will be treated and practical methods described. It will be convenient, however, since some of the results there obtained will be required later, to discuss the solution of the integral equation first, i.e. how to find the original line given the observed line and the rotational speed. This will be followed by the analysis of line shape to test for rotational distortion, and the next part of the paper will consist of numerical examples in illustration and test of the theory developed.

2. The Integral Equation for a Spectrum Line in a Rotating Star.—It is unnecessary to repeat the algebra of the first paper * leading to the equation

\[ O(\zeta) = \frac{2}{\pi} \int_{-1}^{+1} I(\zeta + \beta t) \sqrt{1 - t^2} dt, \]

\[ (2.1) \]

* M.N., 88, 553, 1928.
where $I(\zeta)$ is the "true" shape of a line and $O(\zeta)$ the observed shape, and $V \sin \theta / c$, where $V$ is the equatorial velocity of the star, $\theta$ the inclination of its axis of rotation to the normal to the line of sight and $c$ the velocity of light. $\zeta$ is $\lambda - \lambda_m / \lambda_m$, where $\lambda_m$ is the wave-length of the centre of the line and $\lambda$ the wave-length at any other point. It should be noted that the introduction of this quantity has the effect of removing the irrelevant $\lambda_m$ from the equations and dealing only with the actual shape of the line, ignoring its position, which only enters as a sort of scale factor. This simplification is important in the treatment, as without it the solution would be much more complicated algebraically. This variable, which might be termed profile distance or intrinsic wave-length, is also the one naturally occurring in theoretical formulations of line profiles, in Unsold's formula for example.

The next step is the generalisation of this equation, obtained originally for a spherical body with no darkening towards the limb, rotating as a rigid body, at least on the surface.

It is hardly practicable to allow for variation of rotational speeds at different latitudes in the present state of knowledge. It seems unlikely that the effect would be great, but we may in any event regard $\beta$ as an average value. If a law giving the variation of rotation with latitude be known it can readily be inserted without much complication, but, unless it amounts to a difference of angular velocity of over 10 per cent. from equator to pole, will produce inappreciable disturbing effects so far as we are concerned. It is easily seen that the equation for a star which is an ellipsoid of revolution rotating about its axis of figure is identical with (2.1), since the velocity in the line of sight is purely a function of the $x$ co-ordinate on the disc where axes Ox, Oy are taken through the centre of the disc, perpendicular to and in, respectively, the plane through the axis of rotation and the line of sight.

The equation is in fact

$$O(\zeta) = \frac{2}{\pi ab} \int_{-a}^{+a} I\left(\zeta + \frac{\omega \sin \theta x}{c}\right) b \sqrt{a^2 - x^2} dx,$$

where the elliptical central section of the star is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

and on putting $x = at$ this becomes

$$O(\zeta) = \frac{2}{\pi} \int_{-1}^{+1} I(\zeta + \beta t) \sqrt{1 - t^2} dt,$$

where $\beta$ has the same meaning as before, and $V$ is the equatorial velocity of the ellipsoidal star, of semi-major axis $a$.

The effect of darkening is definite but slight, merely altering the coefficients in the solution somewhat. In the first paper a law of darkening
\( \frac{1}{2} (2 + 3 \cos \theta) \) is assumed, and since this is known to be a good approximation it will be retained in this work and the appropriate coefficients calculated. It will be further assumed that the law of darkening on an elliptical disc leads to the same equation, namely,

\[
O(\zeta) = \frac{5}{4\pi} \int_{-1}^{+1} \frac{8 \sqrt{1 - t^2} + 3\pi(1 - t^2)}{10} \beta t \left[ I(\zeta + \beta t) - I(\zeta) \right] dt. \tag{2.2}
\]

It should be noted that it is of no importance whether the profile be that of an emission line or an absorption line; in either case \( I(\zeta) \) is simply the intensity relative to \( I(o) \) on some suitable scale; if the line be an absorption line \( I(o) \) can be taken as the height of \( I(\zeta) \) relative to the continuous background, \textit{i.e.} \( I(o) - \) the residual central intensity.

\textit{Note on the "Law of Darkening."}—The law used in (2.2) is the approximate solar law for white light, \textit{i.e.} integrated radiation. It is used here as a "combined law of darkening" merely for purposes of comparison with no darkening, and does not imply constancy of line profile over the disc or that monochromatic radiation is not being considered. The equation (2.2) is arrived at in the manner indicated in my first paper, p. 549, as follows: Allowing for change of background intensity, line intensity and line profile we need to consider an expression \( s(\theta)I(\zeta, \theta) \), where \( s(\theta) \) represents the darkening of the continuous spectrum, and \( I(\zeta, \theta) \) the change in the relative intensity of the line and its change of form over the disc, \( \theta \) being \( \circ \) at the centre and \( \pi/2 \) at the limb. The only accurate information about the behaviour of \( I(\zeta, \theta) \) is that obtained by Professor H. H. Plaskett (\textit{M.N.}, 91, 870, 1931), and his formula is complicated. It seemed to me idle to complicate this discussion by an attempt to include an accurate expression for \( I(\zeta, \theta) \), at least until more information about its importance in this connection is available from the reduction of observations. Some allowance for this factor can be made, however, by supposing \( I(\zeta, \theta) \) can be split up into \( a(\theta) \cdot I(\zeta) \). This implies a change in the "depth" of the line from centre to limb, its form remaining unchanged. This is certainly better than taking \( a(\theta) \) constant, and an expression of the form \( a(\theta) = \frac{a_1 + b_1 \cos \theta}{a_1 + b_1} \) allows of a fair approximation to the truth. Thus \( I(\zeta, \theta) \) becomes

\[
\frac{a_1 + b_1 \cos \theta}{a_1 + b_1} \cdot I(\zeta)
\]

as a first approximation, or we may assume that \( s(\theta)a(\theta) \) has an expansion of the form \( \frac{a + b \cos \theta + c \cos 2\theta + \ldots}{a + b + c + \ldots} \), and this is the procedure followed here, where the first two terms only are retained and the values \( a = 2, b = 3 \) adopted provisionally.

If it turns out that the observational material permits the law of darkening to be tested we can then try more elaborate expressions, and since the test can be performed on individual lines the "continuous" and "monochromatic" parts may even be separable. As will be seen, at present the
indications are that the test of the law is possible, but it is premature to attempt to fix better values for \( a \) and \( b \).

As will be clear from § 3 and § 6.1, it is to some extent a matter of indifference which, in the integral equation (3.1), is regarded as the unknown function; we can assume a law of darkening and find \( \beta \) and \( I(\zeta) \), or we can assume \( I(\zeta) \) and find \( \beta \) and the law of darkening or change of form. At present the former is the proper procedure, later we may be in a position to assume \( I(\zeta) \) with sufficient certainty to calculate the other factor.

3. The Solution of the Integral Equation.—The general form of equation (2.1) is

\[
O(z) = a \int_{-1}^{+1} I(z + \beta t)g(t)dt, \tag{3.1}
\]

where \( g \) is any given function of \( t \), and

\[
\frac{1}{a} = \int_{-1}^{+1} g(t)dt. \tag{3.2}
\]

Let \( K(u) \) be defined by the relation

\[
K(u) = uI(u). \tag{3.3}
\]

Then on multiplying both sides of (3.1) by \( e^{-zx} \) we obtain

\[
e^{-zx}O(z) = a \int_{-1}^{+1} e^{-zx} \frac{K(z + \beta t)}{z + \beta t} g(t)dt. \tag{3.4}
\]

Let \( C \) be a contour in the complex \( (z) \) plane such that all singularities of \( O \) and \( K \) lie inside it, or a line \( c-i\infty \) to \( c+i\infty \) such that they all lie to the left of it, then

\[
\frac{1}{2\pi i} \int_{C} e^{-zx}O(z)dz = a \int_{-1}^{+1} e^{-zx} \frac{K(z + \beta t)}{z + \beta t} g(t)dt. \tag{3.5}
\]

Let

\[
\frac{1}{2\pi i} \int_{C} e^{-zx}O(z)dz = \phi(x), \tag{3.6}
\]

\[
\frac{1}{2\pi i} \int_{C} e^{-zx}K(z)dz = \psi(x). \tag{3.7}
\]

Then

\[
\frac{1}{2\pi i} \int_{C} \frac{K(z + \beta t)}{z + \beta t} dz = e^{-\beta iz}\psi(x). \tag{3.7}
\]

Thus

\[
\phi(x) = a\psi(x) \int_{-1}^{+1} e^{-\beta iz}g(t)dt = \psi(x)G(\beta x), \tag{3.8}
\]

where

\[
G(u) = a \int_{-1}^{+1} e^{-ut}g(t)dt. \tag{3.9}
\]

Thus

\[
\psi(x) = \frac{1}{G(x)}\phi(x). \tag{3.91}
\]
Now in virtue of (3.6) by the Bromwich-Wagner theorem

\[ O(u) = \int_{0}^{\infty} e^{-uz} \phi(x) dx \]

and

\[ I(u) = \frac{K(u)}{u} = \int_{0}^{\infty} e^{-uz} \psi(x) dx, \]

so that finally on multiplying (3.91) by \( e^{-z\tau} \) and integrating from 0 to \( \infty \)

\[ I(u) = \int_{0}^{\infty} e^{-uz} \frac{1}{G(\beta x)} \phi(x) dx \]

\[ = \frac{1}{2\pi i} \int_{0}^{\infty} e^{-uz} \frac{1}{G(\beta x)} \int_{c} e^{-z\tau} O(z) d\tau dx. \]

This is the solution of equation (3.1), as may be verified by direct substitution. In effect this procedure amounts to a symbolic solution where \( z \) in equation (3.1) is regarded as the symbolic operator.\(^*\) We have retained the full detail of the complex variable theory here, since (a) it appears inevitably in the result (3.94), and (b) it is of service in discussing any restrictions necessary on the functions employed, \( e.g. \) to secure convergence or in transforming (3.94) by deforming the contour \( C \).

The solution as it stands in equation (3.94) is useless, since the observed quantity \( O(z) \) is to be integrated over a contour in the complex plane, while it is known only numerically for real values of the argument.

We may note here a result that will be of use later and is of general value as a test of approximate formulæ, namely, the areas of the original and the distorted profiles are the same, \( i.e. \)

\[ \int_{-\infty}^{\infty} O(\zeta) d\zeta = \int_{-\infty}^{\infty} I(\zeta) d\zeta = A; \]

this can be seen directly by integration of equation (2.2), or is otherwise obvious considering the fact that the total absorption cannot be altered by rotation.

Two modes of procedure are open to us at this stage. We may be able to reduce (3.94) to a form involving knowledge of \( O(z) \) for real values of \( z \) only. This will depend on the form of \( G(x) \); we shall see presently that this reduction can be effected in two ways that are of service in practice. Otherwise we may seek a function \( O(z) \), which represents the observed quantity over the real axis, and is of a form amenable to treatment by equation (3.94), so that the complex integration can be carried out and in effect \( \phi(x) \) found in terms of the observed quantities, thus reducing the determination of \( I(u) \) to the evaluation of (3.93), which can always be done since \( G(x) \) is known.

There exists a method due to Schwarzschild \(^*\) for representing over the

\(^*\) In Carson’s method, as developed by Van der Pol and Niessen, \( zO(z) \) is the image of \( \phi(x) \), and \( K(z) \) the image of \( \psi(x) \).

\(^*\) Astr. Nach., 4422 and 4557.
whole complex plane a function known only for real values of the argument, but it is troublesome to apply and involves rather precise knowledge of the function to be represented. In view of the special form that \( O(\xi) \) has in the problem of stellar-line profiles, the following is suggested as a convenient method. \( O(\xi) \) is zero or negligible everywhere except over a small range of the argument near \( \xi = 0 \), and in this range it is finite, single valued and with at most one or two maxima and minima. Crudely speaking, it is some sort of flattened-out error curve. It can therefore be represented to any accuracy desired by an expression of the form

\[
O(\xi) = \frac{O_0}{1 + a_1\xi + a_2\xi^2 + \ldots + a_n\xi^n}
\]  

(3.96)

where \( a_1, a_2 \ldots \) may be calculated directly from the observations (by solving a set of \( n \) linear simultaneous equations of the form

\[
\frac{O_0}{O(\xi)} = 1 + a_1\xi + a_2\xi^2 + \ldots + a_n\xi^n
\]

for \( n \) selected values of \( \xi \). \( O(x) \) can have no poles on the real axis and must have a set of poles at conjugate complex points which are found from the equation \( 1 + a_1x + a_2x^2 + \ldots + a_nx^n = 0 \). Thus \( O(x) \) can be broken up into partial fractions of the form

\[
O(x) = \sum \frac{f(x)}{(x - \gamma)^2 + \delta^2}
\]

(3.97)

allowing immediate complex integration and yielding terms of the form \( \sinh mx \cos nx \) to be inserted in (3.93). This method (which we will call Method I) is applied in § 6.2. Its accuracy depends on the number of terms used in equation (3.96), which in turn depends on the accuracy of our knowledge of \( O(\xi) \). In any event, paucity of knowledge of the form \( O(\xi) \) will impair the determination of \( I(\xi) \), but this method is not very exacting as will be seen from the example. Against it must be set the considerable labour involved if great accuracy is wanted, and also the fact that for small values of \( \beta \) the integrals of the oscillating functions converge so slowly as to be impractical for values of \( u \) near the centre of \( I(u) \).

For the convenience of anyone wishing to try this method I give the formula for the polynomial of highest even degree usable in practice, namely, the sixth.

If

\[
O = \frac{O_0}{1 + Ap^8 + Bp^4 + Cp^2}
\]

\[ A, B, C \] are found from the observations for three selected values of \( p \), and the expression put in the form

\[
O = O_0 \cdot \left( \frac{N}{(p^2 + a^2)(p^4 - b^2p^2 + c^2)} \right)
\]

(3.97)
where $a^2$ is found as the only positive root of $1 - Ap^2 + Bp^4 - Cp^6 = 0$.

This is factorised as $(p^2 + a^2)(p + a + \beta^2)(p - a - \beta^2)^2$, where $a^2 + \beta^2 = c$, $a^2 - \beta^2 = b^2/2$, and after splitting into partial fractions and integrating we have

$$I \left\{ \frac{\sin ax}{a} + \frac{\beta^2 - a^2 - 3a^2}{2a(a^2 + \beta^2)} \sinh ax \cos \beta x + \frac{a^2 + a^2 - 3\beta^2}{2\beta(a^2 + \beta^2)} \cosh ax \sin \beta x \right\},$$

which, to give $I(u)$, is to be multiplied by $e^{-ux} \frac{\beta x}{2I_1(\beta x)}$ (here $\beta$ means $V \sin \theta / c$, the rotational speed) and integrated between the limits 0 and $\infty$.

**Method II.**—Another method is to try to find some amenable functional form for $O(\zeta)$ which, with but few disposable constants, will fit a good many observed curves.

Such an empirical formula is

$$O = O_0 e^{b \sqrt{b^2 + a^2z^2}}, \quad (3.98)$$

containing three constants only, which has been found to fit several observed curves well. We require then

$$\frac{1}{2\pi i} \int_c e^{zx} \cdot O_0 e^{b \sqrt{b^2 + a^2z^2}} dz,$$

which has branch points of the integrand at $t = \pm i$,

$$t = \frac{ax}{b} \quad \text{and} \quad O_0 e^{b} \int_a^b \frac{t^x - b\sqrt{1 + \beta^2}}{2\pi a} dx,$$

then reduces to

$$\frac{O_0 e^{b} \beta}{2\pi a} \sum_{r=0}^{\infty} \frac{1}{a} \binom{r+1}{2} \frac{\Gamma\left(\frac{r+1}{2}\right)}{\Gamma\left(\frac{r}{2}\right)} \left(\frac{bx}{a}\right)^{r+1} \left(\frac{b}{a}\right)^{r+1} \quad (3.981)$$

which can be inserted in (3.93).

The integrals should be fairly rapidly convergent.*

The most satisfactory methods are these depending on a reduction of (3.94) to expressions involving $O(\zeta)$ as known numerically only. These fall into the following classes: (a) where $G(u)$ is of such a form that $\int_0^\infty e^{-au} G(u) du$ can be evaluated in functional form and where the integral possesses singularities that allow the complex integral to be reduced (in this class fall the use of approximations to $G(u)$ which admit of simple reduction), (b) expansions of $\frac{1}{G(u)}$ admitting of term-by-term integration and consequent reduction of the complex integral yielding a solution in series.

* It is probably simpler not to use the series of Bessel functions, but to evaluate the real form of the integral numerically as it stands with the given values of $a$ and $b$. 

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In our particular example of the equation (3.94) the second alternative is the most useful one and may be combined, as will be seen, with the first. We shall deal with two cases.

**Method III**, where \( \frac{1}{G(u)} \) can be expanded in a Maclaurin series of powers of \( u \). Let

\[
\frac{1}{G(u)} = a_0 + \sum a_n u^n. \tag{3.991}
\]

Then (3.94) reduces to terms of the form

\[
a_n \beta^n \int_0^\infty e^{-uzx} \int_0^\infty e^{-xz} O(z) dz, \tag{3.992}
\]

which is immediately integrable as

\[
a_n \frac{\beta^n}{n!} O_n(z),
\]

where \( O_n(z) \) is the \( n \)th differential coefficient of \( O(z) \). Thus the solution is found in the form

\[
I(\zeta) = a_0 O(\zeta) + \sum a_n \frac{\beta^n}{n!} O_n(\zeta). \tag{3.993}
\]

It is equivalent to the procedure followed in my first paper, and is of rather restricted application, in effect applying only to cases where the distortion of the original line is slight.

If \( \frac{1}{G(u)} \) cannot be expanded in the required form over the whole range \(-\infty, 0\), the method fails unless \( \int_0^\infty e^{-uzx} \frac{\phi(x)}{G(\beta x)} dx \) can be replaced by \( \int_0^r e^{-uzx} \frac{\phi(x)}{G(\beta x)} dx \) where \( r < R \), the radius of convergence of the expansion of \( \frac{1}{G(u)} \).

It is worth remarking here that the coefficients in this solution, and indeed the solution itself when valid, may be most readily obtained thus from the original equation (3.1). If it be legitimate to expand \( I(\zeta + \beta t) \) in a Taylor series, \textit{i.e.} in effect, if \( \beta \) be small compared to \( \zeta_0 \) where \( I(\zeta) = 0 \) if \( |\zeta| > \zeta_0 \), then

\[
O(\zeta) = \sum_{n=0}^{+\infty} \int_{-1}^{+1} I_n(\zeta) \frac{\beta^n}{n!} t^n g(t) dt
\]

\[
= \sum a_n \frac{\beta^n}{n!} I_n(\zeta),
\]

where

\[
a_n = a \int_{-1}^{+1} t^n g(t) dt,
\]

whence

\[
O_m(\zeta) = \sum \frac{a_m \beta^n}{n!} I_{m+n}(\zeta).
\]
And the solution of this infinite set of linear equations for \( I(\xi) \) in terms of \( O(\xi) \) and its differential coefficients is

\[
I(\xi) = \Sigma \mu_n \beta^n O_n(\xi),
\]

where the \( \mu \)'s and the \( \alpha \)'s are connected by the recurrence formula;

\[
\mu_{2n} + \mu_{2n-1} m_2 + \cdots + \mu_2 m_{2n-2} + m_{2n} = \alpha,
\]

where

\[
m_r = \frac{\alpha_r}{r!}.
\]

**Method IV.**—If \( \frac{1}{G(u)} \) has a suitable asymptotic expansion the equation may integrate out term by term and yield an asymptotic series for \( I(\xi) \) in terms of \( O(\xi) \) and its derivatives.

This is the most powerful mode of attack on the problem, and in the special case we are considering yields the most useful and comprehensive formulæ.

4. An Asymptotic Expansion for \( I(\xi) \).—Starting from equation (2.2) the function \( G(u) \) of § 3 is by (3.9)

\[
G(u) = \frac{5}{4\pi} \int_{-1}^{+1} e^{int} \left\{ \frac{8\sqrt{1-t^2} + 3\pi(1-t^2)}{10} \right\} dt
\]

\[
= \frac{1}{2u^3} \left\{ 2u^2 I_1(u) + 3u \cosh u - 3 \sinh u \right\}. \tag{4.1}
\]

This has the asymptotic expansion

\[
G(u) \sim \frac{e^u}{u \sqrt{2\pi u}} \left\{ 1 + \Sigma (-1)^n (4-1^2)(4-3^2) \cdots (4-2r-1^2) \right\} + \frac{3}{4} \frac{e^u}{u^3} \left\{ 1 - \frac{1}{u} \right\}. \tag{4.2}
\]

Substituting in (3.94), we obtain (on integrating first with respect to \( x \)) a series of integrals of the form \( \int_0^\infty x^n e^{-ax} dx \) immediately reducible to Gamma functions, and leading to a set of complex integrals of the form

\[
\frac{1}{2i} \int_C \frac{O(z)}{(u + \beta - z)^n} \, dz,
\]

where \( R(z) < u + \beta \). These are evaluated by deforming the contour \( C \) into one of Hankel's type, which we are at liberty to do if \( O(z) \) is zero for all sufficiently large values of \( z \), as is the case.

When \( n \) is even Cauchy's theorem applies directly; if \( n \) is odd we make use of the following theorem.†

* Carroll, *M.N.*, 90, 588, 1930, where a set of equations of this form is solved.
If \( f(x) \) is finite and single valued at all points of the positive real axis then

\[
\frac{1}{2i \sin \alpha \pi} \int_{H} (z) a^{\alpha-1} f(z) dz = \int_{0}^{\infty} t^{\alpha-1} \left\{ f(t) - f(o) - tf'(o) \ldots - \frac{t^{K}}{K!} f_{K}(o) \right\} dt
\]

\[
= a(a+1) \ldots (a+K) \int_{0}^{\infty} t^{a+K} f_{K+1}(t) dt, \quad (4.3)
\]

where \( f_{r}(t) \) is the \( r \)th differential coefficient of \( f \) with respect to \( t \), and \( K \) is a positive integer defined by

\[
K < | a | < K + 1
\]

and \( H \) is a contour starting and ending at \( +\infty \) encircling the origin once clockwise.

Applying this to our integrals we have, if \( a = u - z \),

\[
\int_{0}^{\infty} e^{-ax} \frac{G(\beta x)}{G(x)} dx = \int_{0}^{\infty} \frac{e^{-\frac{a}{x}}}{\beta} \frac{G(y)}{G(x)} dx
\]

\[
\sim \frac{\sqrt{2\pi}}{\beta} \int_{0}^{\infty} x^3 e^{-\left(\frac{a+1}{x}\right)^2} \left\{ 1 + \frac{3\sqrt{2\pi}}{4} \frac{1}{x^{\frac{1}{2}}} + \frac{3}{8} \frac{1}{x^2} - \frac{3}{2} \frac{\sqrt{2\pi}}{4} \frac{1}{x^{\frac{3}{2}}} - \frac{15}{128} \frac{1}{x^3} \ldots \right\} dx
\]

\[
= \frac{\sqrt{2\pi}}{\beta} \int_{0}^{\infty} e^{-\left(\frac{a+1}{x}\right)^2} x^3 \left\{ 1 + \frac{3\sqrt{2\pi}}{4} \frac{1}{x^{\frac{1}{2}}} + \left(\frac{9\pi+3}{8}\right) \frac{1}{x^2} + \frac{6 - 27\pi}{32} \sqrt{2\pi} \frac{1}{x^{\frac{3}{2}}} \right.
\]

\[
\left. + \frac{162\pi^4 - 128\pi^2 + 33}{128} \frac{1}{x^3} \ldots \right\} dx.
\]

Yielding—

1st Term:

\[
-8 \left(\frac{\beta}{2}\right)^\frac{3}{2} \int_{0}^{\infty} p^{\frac{3}{2}} O_3(p + u + \beta) dp.
\]

2nd Term:

\[
+ \frac{3\pi\beta}{2} O_1(u + \beta).
\]

3rd Term:

\[
+ \frac{9\pi + 3}{2} \left(\frac{\beta}{2}\right)^\frac{3}{2} \int_{0}^{\infty} p^{\frac{3}{2}} O_3(p + u + \beta) dp.
\]

4th Term:

\[
+ \frac{27\pi - 6}{16} \pi O(u + \beta).
\]

5th Term:

\[
- \frac{162\pi^4 - 128\pi^2 + 33}{64} \left(\frac{2}{\beta}\right)^\frac{3}{2} \int_{0}^{\infty} p^{\frac{3}{2}} O_3(p + u + \beta) dp,
\]

\[
+ \text{etc.} \quad (4.4)
\]

It should be noted that only values of \( O(\zeta) \) and its differential coefficients
are involved at points $\beta$ or more away from the origin, thus on reference to fig. 2 it is seen that only one-half of $O$ or $I$ is involved. If $u$ be $+ve$ the effect of $I(u)$ for $-ve$ values is zero for all points of $O(p)$ where $p > \beta$, so that it is unnecessary to assume any symmetry in $O$ or $I$; the two parts are evaluated separately if they differ.

Method V.—Another possibility, depending on the results of § 5.1, is to compute $\int_{-\infty}^{\infty} O(t) \cos u(t - \xi)dt$ and divide by the function in brackets in (5.13), thus obtaining $\int_{-\infty}^{\infty} I(t) \cos u(t - \xi)dt$ and hence (on completing the Fourier Integral) $I(\xi)$. This method is excellent in numerical tests (see § 6) and seems serviceable in practice (see M.N., 93, 516), but is likely to be troublesome if the observed curve is not accurately known, owing to the singularities introduced when the zeros of $\int_{-\infty}^{\infty} O(t) \cos utdt$ do not coincide with the zeros of $G(u\beta)$.

I expect this, however, to be the best and simplest practical solution of the problem.

5. The Determination whether an Observed Line is Distorted due to Rotation of the Star, and the Evaluation of the Rotational Speed.—In the first section of this paper it was stated that it could be discovered whether a given observed line is distorted by rotational effects, and also, if it is, that the rotational speed can be determined, all without knowledge, in addition to that of the observed profile.

The reason for this surprising possibility is that the observed line cannot (if it is distorted by rotation) have any arbitrary shape, but is in part determined by its mode of production; and I shall show how this factor may be disentangled from the factor due to the original (and, so far as we are concerned, arbitrary) form of the line.

To be strictly accurate, we cannot in fact prove that a given observed line profile is due to rotation, though we may be able to prove that it is not; it is of course possible that, by accident, a given observed line not due to rotation has in it just those elements of shape which rotation could have provided, but it will be appreciated from what follows that this is not a very likely contingency, and when we find agreement between two different determinations of rotational speed for one line, and a fortiori when two or more lines show the characteristics to be expected of lines produced by rotation with concordant values of the rotational speed, then we may feel convinced that the star is indeed rotating.

Put in its crudest possible form the test resembles that of Elvey and Struve, who recognised the feature to which they gave the name “dish shape” in lines broadened by rotation, but this rough eye-estimate of shape conveys but a scant impression of the analytical test to be described, and affords no direct value for $\beta$ without assuming the original line to be known completely, as does this method.

5.1. The Test for Rotational Distortion.—We can represent both the
original line \( I(\zeta) \) and the observed line \( O(\zeta) \) by means of Fourier Integrals, in the usual manner, thus

\[
\pi f(x) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cos u(t-x)f(t)dtdu. \tag{5.11}
\]

The part \( \int_{-\infty}^{\infty} \cos u(t-x)f(t)dt \) of the double integral represents the contribution to the function of the terms with a "frequency" \( u/2\pi \), i.e. considered as a function of \( u \) it gives the amplitude spectrum of the function.

Now if \( I(\xi) \) and \( O(\xi) \) be represented by Fourier Integrals in equation (2.2), the resulting spectrum of \( O(\xi) \) must be the same, whether derived directly from the observations or from \( I(\xi) \) by the right-hand side of (2.2). But the right-hand side of (2.2) gives a spectrum with zeros not merely at certain points fixed by \( I(\xi) \), but in addition at other points fixed not by \( I(\xi) \) but by \( \beta \) and the form of \( g(t) \), i.e. the law of darkening and the fact that rotation is at work leading to the particular form of equation (2.2). There may be other zeros depending on \( I(\xi) \) as I have said, but the point is we can calculate where the "rotational" zeros must come, and since the spectrum can be found directly from the observations \( O(\xi) \), we have only to find it and search among its zeros for ones to satisfy the given conditions.

The procedure is best described by simple illustration.

In the Fourier Integral for \( O(\xi) \) we have the expression

\[
\int_{-\infty}^{\infty} O(t) \cos u(t-\zeta)dt, \text{ and we are thus led to consider}
\]

\[
\int_{-\infty}^{\infty} O(t) \cos utdt. \tag{5.12}
\]

The other term, \( \int_{-\infty}^{\infty} O(t) \sin utdt \), vanishes if \( O(t) \) be symmetrical, but it is unnecessary to consider this integral, and we do not assume \( O(t) \) an even function, it is the evenness of \( g(t) \) that matters.

Now in virtue of (2.2) we have

\[
\int_{-\infty}^{\infty} O(t) \cos utdt = \frac{5}{4\pi} \int_{-\infty}^{\infty} \int_{-1}^{+1} I(t+\beta x) \left\{ \frac{8 \sqrt{1-x^2} + 3\pi(1-x^2)}{10} \right\} dx \cos utdt
\]

\[
= \frac{5}{4\pi} \int_{-\infty}^{+18} \sqrt{1-x^2} + 3\pi(1-x^2) \int_{-\infty}^{+1} I(t+\beta x) \cos utdt \, dx
\]

\[
= \frac{5}{4\pi} \int_{-\infty}^{+18} \sqrt{1-x^2} + 3\pi(1-x^2) \int_{-\infty}^{+1} I(t) \cos u(t-\beta x) \, dx
\]

\[
= \frac{5}{4\pi} \int_{-\infty}^{\infty} \int_{-1}^{+1} I(t) \cos ut \cos u\beta x \left\{ \frac{8 \sqrt{1-x^2} + 3\pi(1-x^2)}{10} \right\} dx \, dt
\]

\[
= \frac{5}{4\pi} \int_{-\infty}^{\infty} I(t) \cos ut \int \left\{ \frac{8\sqrt{1}(u\beta)}{u\beta} - \frac{12\pi \cos u\beta}{u^2\beta^2} + \frac{12\pi \sin u\beta}{u^2\beta^3} \right\}
\]

\[
= \frac{1}{2} \int_{-\infty}^{\infty} I(t) \cos ut \int \left\{ \frac{2J_1(u\beta)}{u\beta} - \frac{3 \cos u\beta}{u^2\beta^3} + \frac{3 \sin u\beta}{u^3\beta^3} \right\}. \tag{5.13}
\]
Now the expression between the brackets \{ \} in (5.13) has its first four zeros at

\[
u\beta = \begin{cases} 
4.147 \\
7.303 \\
10.437 \\
13.569
\end{cases}
\] (5.14)

so therefore also must the left-hand side. We can find the zeros of the left-hand side directly from the observations, and search among them (in favourable cases the search is very rapidly concluded) for a set to agree with (5.14) with a given single value of \( \beta \).

If the line is due to rotation these results must obtain, and conversely we can claim that, if they are found, the probability of rotation being actually the cause of the observed shape is rendered high, the higher the better the agreement and the greater the number of observed lines found to fit.

It will be seen in what follows that in certain circumstances (namely, when the distortion of the line is great) a good approximate value of \( \beta \) may be found otherwise, by merely assuming the original line to be nowhere negative, and if this value be used for a check of (5.14) and agreement found we have strengthened the case for rotational distortion as the cause of the observed shape of the line.

Numerical examples are given below after the next section.

It is to be observed that the position of the zeros in (5.14) depends on the law of darkening; for no darkening they are all approximately at values 0.3 smaller. This is not beyond the bounds of observational test.

5.2. Other Methods of Finding the Rotational Speed (Special Cases).—It is important to note in the first place that the magnitude of \( \beta \) in itself is not the deciding factor in the form of the solutions of the equations, but its magnitude in relation to the size and shape of the line considered. Thus the effect (apart from a scale factor) due to \( \beta \) on \( I(\xi) \) is the same as that due to \( m\beta \) on \( I(m\xi) \); a broad line requires much higher rotational speed to produce the same distortion in it than does a narrow one. We shall later on divide our discussion into two parts: \((a)\) when there is little distortion of the original line, and \((b)\) when the distortion is great. The results for the extreme cases can be developed to overlap somewhat and cover the whole range.

In Elvey’s * discussion of this problem there is a quotation from my first paper that is rather misleading in view of the point just mentioned. In my first paper I worked out the effect of rotation on a line of form \( I(\xi) = e^{-\alpha^2\xi^2} \) for various values of \( \beta \), within the range producing comparatively small distortion. For illustrative purposes a line at 5000 A.U. was chosen, a value \( 10^4 \) assigned to \( \alpha \), and the distorted curve \( O(\xi) \) worked out for \( \beta = 10^{-4} \) and \( 10^{-4}\sqrt{2} \). In a footnote to the illustrative example I remarked that the formula ceased to be valid near a velocity of 65 km./sec., i.e. \( \beta c = 65 \) km./sec. This is the remark quoted, but I wish to point out

that the remark is apropos the particular line chosen and not of general application.

The method followed in that discussion is effectively Method III of this paper and is valid so long as the Taylor expansion is valid. This may be allowable for any value of $\beta$, depending on $a$, i.e. on the width of the original line, as is pointed out on p. 555 of the paper; thus had I chosen a line at 5000 A.U. with $a = 10^{-3}$ the limiting value of $V$ would have been 650 km./sec.

If the width of the original line be $2\xi_0$, i.e. $I(\xi) = 0$, when $|\xi| > \xi_0$, we divide the discussion into

(a) $\beta \ll \xi_0$, i.e. small distortion. (The velocity may not be small; if $\xi_0$ is large $\beta$ may be large also.)

(b) $\beta \gg \xi_0$, i.e. great distortion. (If $\xi_0$ is small this may imply quite low velocities.)

Clearly, it can happen that in the spectrum of any given star the effects of rotation may vary widely from one line to another, depending on the form of the original line.

5.3. Small Distortion.—Equation 1 (a) allows (see § 3, p. 486) in this case a solution of the form

$$I(\xi) = O(\xi) - \frac{2}{3} \xi_0 \beta^2 O_8(\xi) + \cdots$$  

(5.31)

Near the wings of the line, when $O_8(\xi)$ and higher differential coefficients may be neglected, we have only the first two terms of the right-hand side of (5.31), and if we assume that at the wings of the observed line the original line is negligible, we find

$$O_8(\xi) = \frac{80}{9\beta^2} O(\xi),$$

that is to say

$$O(\xi) = Ce^{-\xi} \sqrt{\frac{80}{9}}$$  

(5.32)

Thus the wings of the observed line may be roughly exponential in form. This cannot be strictly true as it is inconsistent with the more accurate expression (3.93) or (3.94), but it should yield a fair approximation.

It is useless as a means of finding $\beta$ as a rule, since there is, first of all, no guarantee that the form is due to rotation, and, secondly, a few trials convince one that the logarithmic differential of $O(\xi)$ is not nearly constant at the wings. It is often constant over a fair range, but it is hard to justify the selection of this value rather than any later one.

The result (5.32) is perhaps of some interest in providing a possible explanation for an observation of Elvey's * that the profiles of observed lines likely to be deformed by rotation are exponential in form, but his result is obtained by graphical plotting of logarithms of the observed intensities and cannot be taken as establishing anything more than a very rough linearity of the logarithmic derivative of $O(\xi)$.

In any event if it be shown that a given observed line be strictly exponential

* * Ap. J., 70, 141, 1929.
in form, then from (2.2) and its solution it follows that so must the original line be, and the explanation is still to be sought. I am merely pointing out a possible reason for Elvey's approximate result, rather stronger than the well-known fact that nearly any observations when plotted logarithmically are approximately linear.

However, the slope of the observed line at the wing may yield a value of $\beta$. Thus, if the observed line can be represented by $Ae^{-\alpha \xi}$ at the wings, then

$$\beta = \frac{1}{2} \frac{8\alpha}{\sqrt{9}}. \tag{5.33}$$

In practice it is not very easy to determine $\alpha$ in the wings of the line, where the observations are few and not very accurate, and the following method has been found more satisfactory than a direct attempt to find $\alpha$ by plotting $\log O(\xi)$ against $\xi$.

Near the centre $O(\xi)$ must be of roughly parabolic shape, and near the wings must tend to exponential form under these circumstances. If $O(\xi)$ be symmetrical, as it commonly is within experimental error or as a first approximation, then a curve having the required properties is

$$O(\xi) = O_0 e^{\xi - \frac{4}{3}\xi^2 + \alpha^2 \xi},$$

since when $\xi$ is small $\left(\frac{a}{b} \ll 1\right)$ this is

$$O = O_0 e^{-\frac{1}{2} \frac{a^2}{b^2}}$$

and when $\xi$ is large $\left(\frac{a}{b} \gg 1\right)$ it is

$$O = O_0 e^{\alpha^2 e^{-\frac{a^2}{b^2}}}.$$

One further point of interest is that we can place some sort of upper limit on the value of $\beta$ by using the fact that $I(\xi)$ cannot exceed unity. This upper limit is not of great value when $O(\theta)$ is small, or rather $O(\theta)/I(\theta)$, but this mode of attack is generally unsuitable then; we have assumed the distortion small, so that $I(\theta)$ cannot much exceed $O(\theta)$, and if any value $I_0$ be assigned to $I(\theta)$, then we must have

$$I_0 > O(\theta) - \frac{3}{2} \frac{\beta^2}{\alpha^2} O_2(\theta).$$

It is instructive to examine this method from a somewhat different point of view. Consider for simplicity a star with no darkening factor, then the function $G(\beta x) = \frac{2}{\pi} \int_{-1}^{1} e^{\beta x \sqrt{1 - t^2}} dt$ is $2I_1(\beta x)$ and we obtain our solution by expanding in ascending powers of $\beta x$, but this cannot be done if $\beta x$ exceeds the first zero $y_1$ of $I_1(z)$ in value, since $\frac{z}{I_1(z)}$ has a pole when $z = \pm iy_1$, and the expansion diverges. Thus it is only when the range of integration in (3.93) is less than $y_1/\beta$ effectively that the procedure is legitimate, i.e. $\frac{\beta x f(x)}{2I_1(\beta x)}$...
or $\psi(x)$ must be negligible when $x$ is $\frac{\sqrt{1 - t^2}}{\beta}$, or in effect $e^{ux}I(u)$ is negligible when $x > \frac{\sqrt{1 - t^2}}{\beta}$; these restrictions make the method of little value.

5.4. Great Distortion.—Naturally when the effect of rotation is great it should be then that the value of $\beta$ is most easily and accurately determined. We shall now show how this may be effected.

We start with

$$O(\zeta) = \frac{5}{4\pi} \int_{-1}^{1} I(\zeta + \beta t) \left( \frac{8\sqrt{1 - t^2} + 3\pi(1 - t^2)}{10} \right) dt$$

(5.40)

and

$$I(\zeta) = 0, \mid \zeta \mid \geq \zeta_0 > 0 \text{ where } \beta > \zeta_0;$$

we can rewrite (5.40) as

$$O(\zeta) = \frac{5}{4\pi\beta} \int_{-\beta}^{\beta} I(\zeta + u) \left( \frac{8\sqrt{1 - u^2/\beta^2} + 3\pi(1 - u^2/\beta^2)}{10} \right) du,$$

(5.41)

and since $u < \beta$ we can expand $\sqrt{1 - u^2/\beta^2}$, obtaining for $\zeta = 0$ (the central value)

$$h = O(\zeta) = \frac{5}{4\pi\beta} \int_{-\beta}^{\beta} I(u) \left( \frac{8 + 3\pi}{10} \frac{4 + 3\pi u^2}{10} \beta^2 \ldots \right) du$$

(5.42)

$$= \frac{8 + 3\pi}{4\pi} \frac{A}{\beta} \int_{-\beta}^{\beta} u^2 I(u) du + \ldots$$

(5.43)

since $\beta > \zeta_0$, where $A$ is the area of $I(\zeta)$ or $O(\zeta)$, and $h$ and $A$ are known directly from the observation. So we have at once the following approximations for $\beta$:

1st Approximation:

$$\beta_1 = \frac{A}{h} \frac{3\pi + 8}{8\pi},$$

(5.44)

2nd Approximation:

$$h = \frac{A}{h} \frac{3\pi + 8}{8\pi} \int_{-\infty}^{+\infty} u^2 I(u) du.$$  

(5.45)

We have seen in § 4 how this may in turn be found in terms of $O(\zeta)$.

Returning to (5.41) we see that the integrand is the product of $I(\zeta + \beta t)$ and $\frac{8\sqrt{1 - t^2} + 3\pi(1 - t^2)}{10}$ and, referring to fig. 1, the effect of altering $\zeta$ will be small until $\zeta + \zeta_0 = \beta$, thus the central part of the observed profile will be broad and flat, i.e. we are considering the Dish-shaped Profiles. Analytically the same result is obtained on expanding $I(\zeta + u)$ in ascending powers, or better by differentiating (5.41) twice with respect to $\zeta$ and integrating by parts, when we find (5.42) and the new result that $O(\zeta)$ and $O''(\zeta)$ are nearly constant when $\zeta$ is small. So that for the Dish-shaped Profiles a good approximation to $\beta$ is given by (5.44).
We shall see (§ 6) that (5.44) gives a useful value in less extreme cases.

An Upper Limit for $\beta$.—We have by (5.41), since $\sqrt{1 - \frac{u^2}{\beta^2}} < 1$,

$$ O(\xi) < \frac{2}{\pi \beta} \int_{-\beta}^{\beta} I(\xi + u) du, $$

and since $I(\xi) > 0$ (all $\xi$), the right-hand side of this expression is

$$ < \int_{-\infty}^{\infty} I(\xi) d\xi. $$

Thus

$$ h = O(o) < \frac{A}{\beta} \cdot \frac{3\pi + 8}{8\pi}, $$

or

$$ \beta < \frac{A}{h} \cdot \frac{3\pi + 8}{8\pi}. \quad (5.46) $$

This is then an upper limit for $\beta$, whether the distortion be great or small. The previous equations provide for closer delimitation of $\beta$ if required.

This result is of interest in that it provides a value of $\beta$ for use in conjunction with (5.14), and while, where the distortion is large and the value of $\beta$ found by (5.44) and (5.45) is a good approximation, this inequality is not specially needed, it is quite useful in other cases.

We can now see that the method used by Struve and Elvey is likely to give good approximate values of $\beta$, providing the distortion is fairly large, since under these circumstances the form of $O(\xi)$ does not depend markedly on the form of $I(\xi)$, but primarily on its area, i.e. if the area of the assumed $I(\xi)$ is correct $h$ is determined mainly by that, and we can regard their method as a similar but better approximation than (5.44) or (5.45).

6. Numerical Tests and Illustrations.—In order to examine the relative
accuracy and suitability of various methods described, and to illustrate the application of these results in practice, I have computed from an arbitrary original line the "observed" shapes corresponding to two different rotational speeds, one producing moderate or small distortion, the other producing great distortion. These two "observed" curves are used in illustration; the "calculated" values of $\beta$ and the original obtained from them can then be compared with the known values.

To simplify the labour of the numerical work a star with no darkening factor has been used. This makes no difference to the *modus operandi*;
and the appropriate formulæ, where they differ from those already obtained will be stated. They differ little in effect from the formulæ, taking darkening into account, and may be used instead for approximate calculations. I am specially indebted to Miss L. J. Ingram, M.A., for help in this part of the work; without her painstaking and very accurate assistance the appearance of this paper would have been greatly delayed owing to the mere length of the numerical work, a large part of which does not appear in the final results at all, especially the thorough tests on actual stars of various methods.

### Table II

**Moderate Distortion**

The "Observed" form \( O(\xi) \) of the line produced from \( I(\xi) \), Table I, by a rotational speed of 100 km./sec., or \( \beta = \frac{1}{10} \times 10^{-4} \)

<table>
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<tr>
<th>( \xi \times 10^4 )</th>
<th>( O(\xi) \times 10^4 )</th>
<th>( \xi \times 10^4 )</th>
<th>( O(\xi) \times 10^4 )</th>
<th>( \xi \times 10^4 )</th>
<th>( O(\xi) )</th>
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<td>3178</td>
<td>8.50</td>
<td>297</td>
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<td>3178</td>
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</table>

\[
A = \int_{-\infty}^{\infty} O(\xi) d\xi = 6.244
\]

I am also indebted to her for careful checking of the whole of the equations in this paper.

The line used as an original, \( I(\xi) \), has a width of about \( 2 \times 10^{-3} \) in \( \xi \), i.e. about 10 A., if centred at 5000 A.; the two rotational speeds are 100 km./sec., i.e. \( \beta = \frac{1.5}{3} \times 10^{-4} \), and 300 km./sec., i.e. \( \beta = 10 \times 10^{-4} \).

It is an obvious convenience to work the numerical calculations with \( 10^{-4} \) as a unit, so that \( \beta \) is regarded as \( \frac{1.5}{3} \) or 10, and the line as of width 20.

Tables I, II and III give the values of \( I(\xi) \) and \( O(\xi) \) for these lines, and fig. 2 shows them plotted all on the same scale. It is worth remarking that the striking difference between the two curves of \( O(\xi) \) for \( V = 100 \text{ km./sec.} \) and \( V = 300 \text{ km./sec.} \) is not so apparent in practice, since in the absence of any knowledge of the true value of \( I(\xi) \), especially of \( I(\theta) \), the correct scale is unknown, and \( O(\xi)_{\beta=10} \) is not very different to the eye from \( O(\xi)_{\beta=10} \)
Fig. 2.—The original line \( I(\xi) \), and its "observed" distorted forms \( O(\xi) \) corresponding to rotational speeds of 100 and 300 km/sec., used for tests of the methods developed.
when the former is plotted on half the scale in height and double the scale in width, as may be seen on comparing figs. 3 and 4.

Finally, the figures used correspond closely to actual cases, since, although the values of $\nu$ chosen sound very high, the line is rather wide, and by the rule mentioned on p. 491 the results apply equally to a line like $I(\zeta)$ but plotted on, say, half the scale of $\zeta$, i.e. of width 5 A.U., if the velocities, and so $\beta$, be halved also.

**Table III**

The "Observed" form $O(\zeta)$ of the line produced from $I(\zeta)$, Table I, by a rotational speed of $V = 300$ km./sec., $\beta = 10^{-4}$

<table>
<thead>
<tr>
<th>$\zeta \times 10^4$</th>
<th>$O(\zeta)$</th>
<th>$\zeta \times 10^4$</th>
<th>$O(\zeta)$</th>
<th>$\zeta \times 10^4$</th>
<th>$O(\zeta)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>O-O</td>
<td>3806</td>
<td>0-0</td>
<td>2459</td>
<td>14-0</td>
<td>147</td>
</tr>
<tr>
<td>O-5</td>
<td>3800</td>
<td>0-5</td>
<td>2252</td>
<td>14-5</td>
<td>103</td>
</tr>
<tr>
<td>1-0</td>
<td>3782</td>
<td>1-0</td>
<td>2030</td>
<td>15-0</td>
<td>69</td>
</tr>
<tr>
<td>1-5</td>
<td>3752</td>
<td>1-5</td>
<td>1800</td>
<td>15-5</td>
<td>44</td>
</tr>
<tr>
<td>2-0</td>
<td>3708</td>
<td>2-0</td>
<td>1562</td>
<td>16-0</td>
<td>26</td>
</tr>
<tr>
<td>2-5</td>
<td>3651</td>
<td>2-5</td>
<td>1329</td>
<td>16-5</td>
<td>14</td>
</tr>
<tr>
<td>3-0</td>
<td>3580</td>
<td>3-0</td>
<td>1106</td>
<td>17-0</td>
<td>6</td>
</tr>
<tr>
<td>3-5</td>
<td>3493</td>
<td>3-5</td>
<td>905</td>
<td>17-5</td>
<td>3</td>
</tr>
<tr>
<td>4-0</td>
<td>3391</td>
<td>4-0</td>
<td>728</td>
<td>18-0</td>
<td>1</td>
</tr>
<tr>
<td>4-5</td>
<td>3273</td>
<td>4-5</td>
<td>579</td>
<td>18-5</td>
<td>0</td>
</tr>
<tr>
<td>5-0</td>
<td>3140</td>
<td>5-0</td>
<td>455</td>
<td>19-0</td>
<td>0</td>
</tr>
<tr>
<td>5-5</td>
<td>2993</td>
<td>5-5</td>
<td>353</td>
<td>19-5</td>
<td>0</td>
</tr>
<tr>
<td>6-0</td>
<td>2830</td>
<td>6-0</td>
<td>270</td>
<td>20-0</td>
<td>0</td>
</tr>
<tr>
<td>6-5</td>
<td>2652</td>
<td>6-5</td>
<td>202</td>
<td></td>
<td></td>
</tr>
<tr>
<td>7-0</td>
<td>2459</td>
<td></td>
<td>14-0</td>
<td>147</td>
<td></td>
</tr>
</tbody>
</table>

$$A = \int_{-\infty}^{\infty} O(\zeta) \, d\zeta = 6.239$$

6.1. The Test for Rotation, and the Determination of the Speed.—Before applying the test for rotation it is useful to find the upper limit to $\beta$, and incidentally to see the degree of approximation of formulæ.

In this case (no darkening) they become

$$h < \frac{2A}{\pi \beta} \quad (6.11)$$

and

$$h < \frac{2A}{\pi \beta} - \frac{1}{\pi \beta^3} \int_{-\beta}^{\beta} u^2 I(u) \, du$$

$$= \frac{2A}{\pi \beta} \left( \frac{2}{\pi \beta} \right)^{3/2} \int_{0}^{\infty} p^{3/2} O(p + \beta) \, dp. \quad (6.12)$$

Taking first the case of great distortion, when these formulæ should yield a good approximation (i.e. we can replace $<$ by $=$), we get, using Table III, from (6.11)

$$\beta = 10.42 \times 10^{-4}$$

and from (6.12)

$$\beta = 10.002 \times 10^{-4}$$
or $V = 300.1$ km/sec., a very close approximation. From Table II, the case of small distortion, we get from (6.11) $\beta = 5.21 \times 10^{-4}$, rather high compared with the correct value $3.33 \times 10^{-4}$, and on using (6.12) we find little or no change. This is to be expected since the formulae can hardly apply, except as an upper limit, since $2\beta$ is of the same magnitude as the width of $I(\zeta)$ or smaller, and so $\int_{-\beta}^{\beta} I(\zeta) d\zeta$ cannot be taken as $A$ the area of $I(\zeta)$.

Turning now to the test for rotational distortion, by (5.12) and (2.1), the zeros are now simply at the zeros of $J_1(u\beta)$, i.e.

$$u\beta = \begin{cases} 
3.832 \\
7.016 \\
10.174 \\
13.324
\end{cases} \quad (6.13)$$

For the line showing great distortion we find zeros of $\int_{-\infty}^{\infty} O(t) \cos ut dt$ at

$$u = \begin{cases} 
0.382 \\
0.702 \\
1.021 \\
1.339
\end{cases} \quad \text{yielding respectively } \beta \times 10^4 = \begin{cases} 
10.01 \\
9.99 \\
9.99 \\
9.97
\end{cases}$$

in excellent agreement with the true value $\beta = 10 \times 10^{-4}$.

The integral is small when $u$ is large, and it is only in cases of great distortion that one can expect to press the results as far as the fourth zero.

Fig. 3 shows the graph of $O(\zeta)$ and of $\int_{-\infty}^{\infty} O(t) \cos ut dt$. It is noteworthy that there is no confusion caused by "natural" zeros, i.e. ones not due to the rotational element of the shape of $O(\zeta)$.

In the other example shown in Fig. 4, small distortion, from Table II, we get by (6.12) zeros at

$$u\beta = \begin{cases} 
1.13 \\
2.21 \\
2.77
\end{cases} \quad \text{yielding respectively } \beta \times 10^4 = \begin{cases} 
3.38 \\
3.17 \\
3.67
\end{cases}$$

The agreement with the correct value of $\beta (3.33)$ is good. It could, of course, be made much better by more elaborate numerical integration, but it is more instructive to take the figures as they stand for comparison with the results for $\beta = 10$. We see that, first of all, rather less accuracy is to be expected, and, secondly, owing to the much larger values of $u$ involved, at most three zeros can be picked up with certainty. Again no trouble is experienced with "natural" zeros, and we may anticipate little trouble of this kind with actual profiles.

A word of warning about the evaluation of $\int_{-\infty}^{\infty} O(t) \cos ut dt$ may be welcome. In numerical work this integral is not in fact evaluated, but
Fig. 3.—The test for stellar rotation.

The observed form $O(\zeta)$ and $\int_{-\infty}^{\infty} O(t) \cos \nu dt$ showing zeros of the latter, when $V = 300 \text{ km./sec.}$ (Great distortion.)
use is made of some formula of mechanical integration, Simpson's rule, Weddle's rule or Gregory's formula being suitable and convenient. Thus in fact we evaluate \[ \sum_{r=-n}^{2n} O(rw) \cos urw \cdot w \] where \( w \) is the interval between successive ordinates, and \( O(rw) = O(-rw) = 0 \) if \( r \geq n \), but this procedure necessarily yields a periodic function with period, in \( u \) of \( 2\pi/w \), whereas the integral is not periodic. It is therefore necessary to choose \( w \) so small that \( \frac{2\pi}{w} \) is considerably larger than twice the last zero to be found, otherwise the value obtained may be considerably in error.

To sum up, we see that the analysis of line shape gives definite and accurate results, and the approximate formulae for \( \beta \) are accurate when the distortion is great; when it is small the value of \( \beta \) found is considerably too large. In any case then we must reject values of \( \beta \) inconsistent with the upper limit calculable by (6.11) or (6.12).

With sufficiently good determinations of \( O(\xi) \) for lines showing great distortion it should be possible to test whether the results agree more closely with (6.13) or (5.14), i.e. we can gain some idea of the law of darkening. It is doubtful whether present measurements are sufficiently accurate for the purpose, but the point is discussed in the following paper.

6.2. The Calculation of the True Form of the Line.—We will take first of all the case of great distortion, \( \beta = 10 \) (Table III).

It is at once clear that (5.31) is quite inapplicable; such an expansion is invalid, as will be seen below, where Method I of \( \S \) 3 is applied.

The asymptotic expansion here works to perfection. The formula for no darkening, corresponding to (4.4), is

\[ I(u) = \]

1st Term:

\[ -4 \left( \frac{\beta}{2} \right)^{\frac{3}{2}} \int_{0}^{\infty} p^{2} O_{3}(p + u + \beta) dp; \]

2nd Term:

\[ \frac{3}{4} \left( \frac{\beta}{2} \right)^{\frac{1}{2}} \int_{0}^{\infty} p^{3} O_{4}(p + u + \beta) dp; \]

3rd Term:

\[ - \frac{33}{128} \left( \frac{2}{\beta} \right)^{\frac{1}{2}} \int_{0}^{\infty} p^{2} O_{5}(p + u + \beta) dp; \]

4th Term:

\[ \frac{249}{2048} \left( \frac{2}{\beta} \right)^{\frac{3}{2}} \int_{0}^{\infty} p^{4} O(p + u + \beta) dp, \]  

(6.21)

where from Table III we find the original as given in Table IV. The agreement is practically perfect, and, on the scale of fig. 2, the calculated and true values of \( I(\xi) \) are indistinguishable. The accuracy far exceeds the accuracy to which the observational data are known. The method of fitting a curve of the form \( e^{b-\sqrt{a^{2}-x^{2}}} \) cannot be applied, as it is readily demonstrable that no curve of this form will fit \( O(\xi) \) even moderately well.
The remaining method is Method I of § 3. $O(\zeta)$ can be represented with considerable accuracy by

$$O(\zeta) = \frac{O_0}{(1 + 0.02025425\rho^2)(1 - 0.0121647\rho^2 + 0.00146453\rho^4)}$$

leading to the integrals from $0$ to $\infty$ of expressions from the complex integral as follows:

$$I(u) = \int_0^\infty e^{-ut} \frac{10t}{2I_1(10t)} (3.588 \sin 7.027t - 4.1639 \sinh 7.880t \cos 4.533t$$
$$+ 1.6767 \cosh 7.880t \sin 4.533t) \, dt. \quad (6.22)$$

The resulting curve for $I$ is shown in fig. 5. It is a fair approximation, running too high in the centre and at the wings, and being too low in between. The inapplicability of (5.31) can be seen from (6.22), where the integrand is not negligible until $t$ is much larger than $0.383$, or $\gamma_0$ the first zero of $I_1(x)$.

Of course a polynomial of higher degree would yield a better result, but, as can be seen, the labour of the computation becomes excessive, especially as the asymptotic expansion yields accurate results comparatively easily.

The real difficulty to be expected in practice is readily seen. As we saw earlier (p. 495), the shape of the central part of a greatly distorted line does not much depend on the shape of the original line—it is the wings that matter; but this means that to find $I(\zeta)$ accurately we must know their shape well; indeed, as Table IV shows, so well that the third differential coefficient may be well known, since it occurs in the first term of (6.21), and contributes most largely to $I(\zeta)$.

Approximations to the shape will naturally lead to results similar to that given by (6.22), so that little is gained by using specially smoothed data.
Thus we cannot hope to find $I(\xi)$ at all accurately from such measurements as exist at present if $O(\xi)$ is much distorted; (6.22) may, however, yield a quite useful rough approximation.

Taking now the case of small distortion, $\beta = \frac{1}{2} \beta$ (Table II), we find similar results, which to save space will not be given in detail.

The asymptotic expansion works well but does not converge rapidly, and even with a fifth term added to (6.21) is a few per cent. out. In illustration the value for $I(\alpha)$ only will be quoted; for the later values, the larger $u$ is, the better the agreement with the true value $I(u)$. We find instead of 1.000 for $I(\alpha)$ the value 0.936, about 6 per cent. too low.

An alternative formula for $I(\alpha)$ obtained by differentiating (6.21) with respect to $u$, and setting $u=0$, gives (since $I'(\alpha)=0$), on substituting from (6.21),

$$
I_0 = \frac{3}{2} \left( \frac{\beta}{2} \right)^{\frac{1}{2}} \int_0^\infty p^3 O_3(p+\beta)dp - \frac{3}{16} \left( \frac{2}{\beta} \right)^{\frac{1}{2}} \int_0^\infty p^5 O_1(p+\beta)dp + \text{etc.}
$$

This is useful since $O_3(\xi)$ is not required, and yields in this case the slightly better approximation 1.020 for $I(\alpha)$, i.e. only about 2 per cent. in error.

A simple test for the applicability of the asymptotic expansion is obtained by integrating (6.21) with respect to $u$ from 0 to $\infty$, thus obtaining $A/2$, the point being that this quantity is always known in practice whereas $I(\alpha)$ is not (and though it may be known not to exceed some given value, this as a rule is much too great an upper limit to be useful). Thus in this case we find the following results (Table V), with the given values of $\beta$. It is clear that the series converges very slowly, but four terms seem to be about sufficient for practical purposes, i.e. accuracy to 5 per cent. or so.

| Table V |
|------------------|------------------|------------------|
| Contributions from the Different Terms of the Asymptotic Expansion to $A/2$ |     |     |     |
| $\beta \times 10^4$ | 3.0 | 3.5 | 4.0 |
| 1st term | 1.885 | 2.123 | 2.248 |
| 2nd ,, | 0.622 | 0.550 | 0.474 |
| 3rd ,, | 0.283 | 0.203 | 0.145 |
| 4th ,, | 0.150 | 0.086 | 0.051 |
| Total | 2.941 | 2.962 | 2.918 |

$A/2 = 3.122$

It must be remembered that this is a fairly severe case for this method. An attempt to apply Method I is abortive, the difficulty being that the integrands in the corresponding expression to (6.22) oscillate violently over a large range. The integration can of course be performed and probably the result would be satisfactory, since the first term only (which converges quickly) gives $I(\alpha)$, only about 10 per cent. too large, and the other two terms should yield a small negative contribution, but as a practical method for small distortion this one seems useless.
It is clear that the method of simple expansion giving 3.993 cannot apply, and indeed we get, on trying it, only 0.857 for \( I(0) \), much too small a value.

There seems to be a real difficulty in computing \( I(\xi) \) from \( O(\xi) \) even when \( \beta \) is known correctly. Theoretically the problem is solved, but the practical use of the results involves approximations of some form or other, e.g. approximate functional representation, or the termination at some convenient point of a series, and the practical difficulty arises in the following way. When \( \beta \) is large, or rather when the distortion is great, the computations are readily carried out and yield accurate results; when the distortion is small, the numerical work is only carried out with difficulty and the approximations are poorer: but unfortunately it is precisely in the former case that the observational material is poor and not able to yield (at present) a good determination of \( O(\xi) \), which we have seen must, for this purpose, be well known in the wings. When the observed form of \( O(\xi) \) is likely to be well known (small distortion) we lack a satisfactory calculus. I have some hopes of other lines of attack, but it does not seem wise to hold up the announcement of these results meantime. Clearly, were these properties of the functions of the form \( f(a) = \int_0^\infty e^{-ax} \frac{x}{I_1(x)} \, dx \) known we might expect to use them profitably, but they are obviously transcendentals at least more complicated than the logarithmic derivative of the Gamma function, and at present nothing seems known about them, and I have not been able to do more than this paper shows.

Method V here shows its power. It is in effect a modification of
Schwarzschild’s method. We here, in this rather difficult case of small distortion, find excellent agreement between the true and computed \( I(\xi) \) as follows:

<table>
<thead>
<tr>
<th>( J = 0 )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>True</td>
<td>100</td>
<td>88</td>
<td>63</td>
<td>43</td>
<td>30</td>
<td>20</td>
<td>12</td>
<td>5</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Computed</td>
<td>99</td>
<td>87</td>
<td>63</td>
<td>43</td>
<td>30</td>
<td>19</td>
<td>11</td>
<td>6</td>
<td>2</td>
<td>0</td>
</tr>
</tbody>
</table>

It is clear that difficulty will only arise over the uncertainty in the positions of the zeros. Tests on actual stars will decide the utility of the method, but all the indications are favourable.

In conclusion I wish to express my indebtedness to Dr. Paul White for his assistance in discussion and to Dr. L. J. Comrie for kindly supplying tables of \( J_j(x) \) at intervals of 0.001 in \( x \) in the neighbourhood of the zeros of the function in \((5.13)\).

**Summary**

The effect of rotation of a star on the form of its spectrum lines as observed is discussed. The observed line profile depends on the “true” or original form, on the law of darkening (slightly) and on the star’s rotational speed.

The observed form of a line is not entirely arbitrary but owes its shape in part to its mode of production. It is shown how, by a suitable analysis of this observed shape, it can be determined whether it is distorted by stellar rotation, and how, if the star be rotating, its speed can be found directly from the observed line without assumptions concerning the true form of the line.

The integral equation connecting the true and observed profiles is solved and methods of practical use of the solution discussed, whereby the true form can be computed from the observed form and the rotational speed.

Various special cases are discussed, and a simple formula giving an upper limit for the rotational speed is derived.

It is pointed out that high rotational speed in itself is not necessary to produce great distortion of a line; it depends also on the width of the original line. The dish-shaped profiles of Elvey, Shajn and Struve are examined and shown to be characteristic of great distortion.

Numerical illustrations are given, and indications of the accuracy attainable by various methods.

The possibility of testing a law of darkening towards the limb in a rotating star by these methods is pointed out.