Dr. J. C. P. Miller, On a Special Case

Of the remaining 4 stars, SV And is far outside the relation shown by the others. These three, all N stars, seem to constitute a group for which $p$ is practically constant (mean value = 11.4) between $P_1 200$ and 532 days.

VY And with $P_1 = 149^d$ is between the two groups, and a detailed study of its light-variation seems very desirable.

These statements have not been made in support of any theory; I have only made them partly as a warning against premature conclusions and partly to urge the necessity of further researches.

Table I has only 4 stars with three waves. This number is too small for definitive conclusions as to $q$. Considering, however, that for RV Tau and SV And $P_3$ has been derived from good, homogeneous material, and that also for $\mu$ Cephei both $P_2$ and $P_3$ seem to be well established, I think it worth while to draw attention to the fact that $q$ seems to have a constant value which in a preliminary way may be fixed at 6. This result, as well as the constant value of $p$ for the larger values of $P_1$ ($p = 2q$?), may have some meaning with respect to the theory of stellar pulsations.

I should like to conclude this note with the remark that reliable curves obtained from homogeneous material, running over a long interval of time, may clear up many points with regard to the complexity of stellar variation. Undoubtedly such material is already present in the manuscripts of a few single observers and in great wealth in the volumes of the B.A.A., only waiting to be properly handled.

University Observatory, Utrecht:
1934 June 26.

ON A SPECIAL CASE IN THE DETERMINATION OF PROBABLE ERRORS.

J. C. P. Miller, Ph.D.

§ 1. In determining lunar diurnal variations in geophysical elements (such as barometric pressure or the magnetic field) it is convenient first to obtain the solar diurnal harmonic components of the variation from groups of days, each corresponding to a particular age of the moon. The resulting values of the amplitude and phase of any component may be represented by vectors, there being one such vector for each group of days. These components include a solar part which we take to be the same in amplitude and phase in all the groups; this is determined from the mean of the results for all groups.

When this constituent is removed, the results include a true lunar variation together with a part of accidental origin. The lunar part is supposed to be of the same constant amplitude in each group, while its phase varies regularly from group to group in a way which is known from theory.

When the amplitude and phase of the lunar part have been found, there remains the problem of determining the magnitude of the accidental part of the component variation, for it is from this that we must infer the probable
error of the lunar diurnal variation itself. The problem can be expressed in simple mathematical form as follows:

§ 2. Consider a plane set of points \( P_m(m = 1 \text{ to } m = n) \) which specify a set of vectors \( \mathbf{R}_m \) represented by the lines \( OP_m \) drawn from an origin \( O \).

These vectors are supposed made up of two parts

\[
\mathbf{R}_m = \mathbf{L}_m + \mathbf{A}_m,
\]

where \( \mathbf{L}_m \) is a vector \( OL_m \) corresponding to the lunar diurnal variation, of constant amplitude \( L \) and of phase \( l_m \) which is known but not constant, while \( \mathbf{A}_m \) is a vector \( L_mP_m \), of amplitude \( A_m \) and phase \( l_m + \theta_m \), where \( \theta_m \) is uncorrelated with \( l_m \). Let \( \mathbf{R}_m' \) and \( \mathbf{A}_m' \) denote vectors equal to \( \mathbf{R}_m \) and \( \mathbf{A}_m \) except that their phases are reduced by the amount \( l_m \). Then

\[
R_m = R_m', \quad A_m = A'_m
\]

and

\[
\mathbf{R}_m' = \mathbf{L} + \mathbf{A}_m',
\]

where \( \mathbf{L} \) is now a constant vector of length \( L \) and zero phase. It is supposed that the vectors \( \mathbf{A}_m \) have a Gaussian distribution, so that the number whose amplitudes lie between \( r \) and \( r + dr \), and whose phases lie between \( \theta \) and \( \theta + d\theta \), is given by

\[
\frac{h^2}{\pi} n e^{-h^2r^2} d\theta dr.
\]

Then the mean value \( \langle A \rangle \) of \( A_m \) is given by

\[
A = \frac{h^2}{\pi} \int_0^\infty \int_0^{2\pi} e^{-h^2r^2} r^2 d\theta dr = \frac{\sqrt{\pi}}{2h} = .8862/h,
\]

and the probable error \( r_0 \) of any one of the vectors \( OP_m \) is given by

\[
1 - e^{-h^2r_0^2} = \frac{1}{2} \quad \text{or} \quad r_0 = .8326/h,
\]

so that

\[
r_0 = .9394A.
\]

§ 3. It is desired to determine \( A \) from the mean \( \langle R \rangle \) of the amplitudes \( R_m \), these being much more readily evaluated than \( A_m \). The relation between \( A \) and \( R \) must clearly involve \( L \), and in the first instance \( R \) is obtained as a function of \( A \) and \( L \). This functional relation must be inverted in order to obtain \( A \) as a function of \( L \) and \( R \). The relation found is of the form

\[
\frac{r_0}{R} = \frac{.9394A}{R} = f(L/R).
\]

The function \( f(L/R) \) is tabulated in Table I; it varies rapidly (tending to zero) when \( L/R \) is nearly unity, and in this case \( A \) cannot in practice be found with sufficient accuracy from this relation.

If the phases \( l_m \) are known it is, of course, possible to measure the individual distances \( A_m \) and to obtain their mean directly. In the actual applications to lunar diurnal periodicities the work can be slightly simplified...
when $L/R \approx 1$, because of the fact that the successive phases $l_m(m = 1$ to $12)$ in those applications differ by amounts which are nearly equal to $30^\circ$. This case is briefly considered in § 6, and a function $f'$ (corresponding to $f$) is tabulated in Table II.

In order to facilitate the evaluation of individual $A_m$'s in cases where this is desired, we give $\sin l_m$, $\cos l_m$ in Table III, taking $l_0 = \pi$.

§ 4. Clearly

$$R_m^2 = L^2 + A_m^2 + 2L A_m \cos \theta_m,$$

and consequently

$$R = \frac{\hbar^2}{\pi} \int_0^{2\pi} e^{-\lambda^2 r^2} (L^2 + r^2 + 2L r \cos \theta)^{1/2} r d\theta dr,$$

or putting $r = L \rho$, $\rho = \lambda/L$, $R = AS$

$$S = \frac{(2\lambda)^3}{\pi \lambda^3} \int_0^{2\pi} e^{-2\rho^2 \rho} (1 + 2\rho \cos \theta + \rho^2)^{1/2} d\theta d\rho.$$

As $\lambda \equiv \frac{\sqrt{\pi}}{2} \frac{L}{A} \to 0$, $R \to A$ and $S \to 1$.

As $\lambda \to \infty$,

$R \to L$ and $S \to 2\lambda/\sqrt{\pi}$.

Dr. R. J. Schmidt and I have succeeded in showing that

$$\frac{d^2S}{d\lambda^2} + (2\lambda^2 + 1) \frac{dS}{d\lambda} - 2\lambda S = 0.$$

To see this, note that

$$\int_0^{2\pi} \frac{(1 + \rho \cos \theta)(\rho + \cos \theta)}{(1 + 2\rho \cos \theta + \rho^2)^{3/2}} d\theta = \int_0^{2\pi} \cos \theta (1 + 2\rho \cos \theta + \rho^2) + \rho \sin^2 \theta}{(1 + 2\rho \cos \theta + \rho^2)^{3/2}} = 0, \quad (4.1)$$

integrating the second term on the right by parts.

Put

$$u = \int_0^{2\pi} \frac{(1 + 2\rho \cos \theta + \rho^2)^{1/2}}{d\theta} = \int_0^{2\pi} \frac{(1 + 2\rho \cos \theta + \rho^2)^2}{d\theta} = \int_0^{2\pi} \frac{(\rho^2 - 1)^2}{d\theta}.$$

Then

$$\frac{du}{d\theta} = \int_0^{2\pi} \frac{(\rho + \cos \theta) d\theta}{(1 + 2\rho \cos \theta + \rho^2)^{1/2}} = \int_0^{2\pi} \frac{(\rho^2 - 1)(\rho + \cos \theta)}{(1 + 2\rho \cos \theta + \rho^2)^{3/2}} d\theta,$$

$$\frac{d^2u}{d\theta^2} = \int_0^{2\pi} \frac{d\theta}{(1 + 2\rho \cos \theta + \rho^2)^{1/2}} - \int_0^{2\pi} \frac{(\rho + \cos \theta)^2}{(1 + 2\rho \cos \theta + \rho^2)^{3/2}} d\theta$$

$$= \int_0^{2\pi} \frac{2 + \rho^2 + \rho}{(1 + 2\rho \cos \theta + \rho^2)^{3/2}} d\theta,$$

using (4.1) to eliminate $\cos^2 \theta$ from the numerator in each case.

Combining the above relations, we have
\[
(r^a - \rho) \frac{d^2u}{d\theta^2} - (r^a + 1) \frac{du}{d\theta} + pu = 0. \tag{4.2}
\]

Now
\[
\lambda \int_0^\infty \rho^n f(\rho) e^{-\lambda r^a} d\rho = \left[ -\frac{\lambda^{a-2} n^{-1}}{2} f(\rho) e^{-\lambda r^a} \right]_0^\infty + \frac{1}{2} \lambda^{a-2} \int_0^\infty \frac{d\{f(\rho) \rho^{n-1}\}}{d\rho} e^{-\lambda r^a} d\rho,
\]
so that
\[
S = \frac{2\lambda^3}{\pi^{3/2}} \int_0^\infty \rho u e^{-\lambda r^a} d\rho = \frac{\lambda}{\pi^{3/2}} \left( 1 - \int_0^\infty \frac{du}{d\rho} e^{-\lambda r^a} d\rho \right),
\]
\[
\frac{dS}{d\lambda} = \frac{6\lambda^2}{\pi^{3/2}} \int_0^\infty \rho u e^{-\lambda r^a} d\rho - 4\lambda^4 \int_0^\infty \rho^n u e^{-\lambda r^a} d\rho
\]
\[= \frac{2\lambda^3}{\pi^{3/2}} \int_0^\infty \left( \rho u - \rho^3 \frac{du}{d\rho} \right) e^{-\lambda r^a} d\rho
\]
\[= \frac{1}{\pi^{3/2}} \left( 1 - \int_0^\infty \frac{d^2u}{d\rho^2} e^{-\lambda r^a} d\rho \right)
\]
\[
\frac{d^2S}{d\lambda^2} = -\frac{2\lambda^3}{\pi^{3/2}} \int_0^\infty \rho^2 u e^{-\lambda r^a} d\rho.
\]

From (4.2) and (4.3) we find, as required,
\[
\lambda \frac{d^2S}{d\lambda^2} + (2\lambda^2 + 1) \frac{dS}{d\lambda} - 2\lambda S = 0
\]
with the boundary conditions
\[
S = 1 \text{ when } \lambda = 0,
\]
\[
\frac{dS}{d\lambda} \to 2/\sqrt{\pi} \text{ as } \lambda \to \infty.
\]

§ 5. From (4.4) we find
\[
S = 1 + \frac{1}{2} \lambda^2 - \frac{1}{3} \lambda^4 + \frac{1}{6} \lambda^6 - \ldots + (-1)^{r-1} \frac{1 \cdot 3 \cdot 5 \ldots (2r-3)}{2^r \cdot r!} \lambda^{2r} + \ldots \tag{5.1}
\]
which is convergent for all \( \lambda \), but not convenient for large \( \lambda \), for which case we have the asymptotic expansion
\[
S = \frac{2\lambda}{\sqrt{\pi}} \left( 1 + \frac{1}{4\lambda^2} + \frac{1}{2 ! (2\lambda)^4} + \frac{1 \cdot 3}{3 ! (2\lambda)^6} + \ldots \right). \tag{5.2}
\]

These expressions are both of the form
\[
\frac{R}{L} = \frac{L}{A} = \phi \left( \frac{L}{A} \right).
\]

\( L \) and \( R \) are supposed known and we wish to find \( A \), so (5.1) and (5.2) must be converted to the form
\[
\frac{L}{A} = F \left( \frac{L}{R} \right),
\]
\[
\frac{r_0}{R} = \frac{2.9394 L}{R} = 2.9394 \left( \frac{L}{R} \right) f(L/R)
\]

which can then readily be found.
From (5.1) and (5.2) we find
\[ \lambda = \frac{\sqrt{\pi}}{2} \cdot \frac{L}{A} = \nu + \frac{1}{2} \nu^2 + \frac{7}{8} \nu^3 + \frac{33}{32} \nu^4 + \ldots \]  
and
\[ \frac{1}{4 \lambda^2} = \frac{A^2}{\pi L^2} = \sigma - \frac{1}{2} \sigma^2 - \sigma^3 - \frac{5}{4} \sigma^4 - \frac{5}{3} \sigma^5 - \ldots \]
where \[ \nu = \frac{\sqrt{\pi}}{2} \cdot \frac{L}{R}, \quad \sigma = \frac{R}{L} - 1. \]

To fill the gap between (5.3) and (5.4) we use (5.1) and interpolate. The function \( f(L/R) \) is tabulated in Table I.

§ 6. In § 3 we noted that for \( \frac{L}{R} \approx 1 \), Table I does not give a sufficiently accurate value for \( A \).

When we are dealing with lunar periodicities, the phases \( l_m \) are as follows, apart from a possible constant added to each:

<table>
<thead>
<tr>
<th>( m )</th>
<th>( l_m )</th>
<th>( l_m'' = l_m - 30^\circ (m - 7) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>144.9</td>
<td>-5.1</td>
</tr>
<tr>
<td>3</td>
<td>115.9</td>
<td>-4.1</td>
</tr>
<tr>
<td>4</td>
<td>87.0</td>
<td>-3.0</td>
</tr>
<tr>
<td>5</td>
<td>58.0</td>
<td>-2.0</td>
</tr>
<tr>
<td>6</td>
<td>29.0</td>
<td>-1.0</td>
</tr>
<tr>
<td>7</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>8</td>
<td>-29.0</td>
<td>1.0</td>
</tr>
<tr>
<td>9</td>
<td>-58.0</td>
<td>2.0</td>
</tr>
<tr>
<td>10</td>
<td>-87.0</td>
<td>3.0</td>
</tr>
<tr>
<td>11</td>
<td>-115.9</td>
<td>4.1</td>
</tr>
<tr>
<td>12</td>
<td>-144.9</td>
<td>5.1</td>
</tr>
</tbody>
</table>

Phase changes which are multiples of \( 30^\circ \) are very simple to apply numerically. In the third column we give the residuals after the phase of \( L_m + A_m \) has been reduced by \( 30^\circ (m - 7) \).

Let \( L_m'' + A_m'' \) denote the vector \( L_m + A_m \) after this phase change has been made. \( L_m'' \) has amplitude \( L \) and phase \( l_m'' \) and \( A_m'' \) has amplitude \( A_m \) and phase \( \theta_m + l_m'' \). The mean \( L'' \) of the vectors \( L_m'' \) has amplitude
\[ L'' = \frac{1}{2} \Sigma L \cos l_m'' = 0.99856L \]
and phase
\[ l'' = \frac{1}{2} \Sigma l_m'' = 0. \]

We wish to express \( A \) in terms of the mean \( R' \) of distances measured from the end point of the mean vector \( L'' \).

Denote by \( L_m' \) the amplitude of the vector \( L_m - L_m'' \). We then have
For each $L'_m$ we have a series

$$S'_m = 1 + \frac{k^2}{2} \lambda^2 - \frac{k^4}{4} \lambda^4 + \frac{k^6}{6} \lambda^6 - \cdots$$

where $k_m = L'_m/L$.

Whence

$$\frac{R'}{A} = S' = \frac{1}{12} \sum_{m=1}^{12} S'_m = 1 + \left( \frac{1}{12} \sum_{m=1}^{12} k^2_m \right) \lambda^2 - \left( \frac{1}{12} \sum_{m=1}^{12} k^4_m \right) \lambda^4 + \cdots$$

$$= 1 + 0.021439 \lambda^2 - 0.051005 \lambda^4 + 0.081102 \lambda^6 - 0.111121 \lambda^8 + 0.141126 \lambda^{10}.$$

Converting to express $L/A$ in terms of $L/R'$ as before, we have

$$\lambda = \frac{\sqrt{\pi} L}{2 A} = \frac{R'}{A} + 0.021002 \left( \frac{L}{R'} \right)^3 - 0.051716 \left( \frac{L}{R'} \right)^5 + 0.083419 \left( \frac{L}{R'} \right)^7$$

$$- 0.11584 \left( \frac{L}{R'} \right)^9 + \cdots$$

From this we derive $f'(L/R')$, given in Table II, such that

$$\frac{r'_0}{R'} = f'(L/R').$$

$r'_0$ will, of course, be equal to $r_0$ when both are reliable.

§ 7. Comparison of Probable Errors given by

$$r_0 = Rf(L/R) \quad \text{and} \quad r'_0 = R'f'(L/R').$$

In forty cases, in which $L/R$ lies between $0.45$ and $0.93$, probable errors were obtained by using both relations. We suppose in each case that $r'_0$ is the value to be preferred, and express $|r_0 - r'_0|$ as a percentage of $r'_0$. We can then infer a maximum disagreement, which is a function of $L/R$.

<table>
<thead>
<tr>
<th>$L/R$</th>
<th>0.6</th>
<th>0.75</th>
<th>0.8</th>
<th>0.9</th>
</tr>
</thead>
<tbody>
<tr>
<td>Max $\frac{</td>
<td>r_0 - r'_0</td>
<td>}{r'_0}$</td>
<td>8%</td>
<td>15%</td>
</tr>
</tbody>
</table>

It is evident that $r_0$ is not safe if $L/R$ exceeds about $0.75$. Comparison of Table I and Table II suggests that $r'_0$ should be reliable for $L/R' < 15$, and so anywhere within the limits of Table II.