Synthetic Aperture Radar Remote Sensing of Shear-Driven Atmospheric Internal Gravity Waves in the Vicinity of a Warm Front

DUSTIN J. SWALES
I. M. Systems Group, Rockville, Maryland

GEORGE S. YOUNG
Department of Meteorology, The Pennsylvania State University, University Park, Pennsylvania

TODD D. SIKORA
Department of Earth Sciences, Millersville University, Millersville, Pennsylvania

NATHANIEL S. WINSTEAD
Applied Physics Laboratory, Johns Hopkins University, Baltimore, Maryland

HAMPTON N. SHIRER
Department of Meteorology, The Pennsylvania State University, University Park, Pennsylvania

(Manuscript received 9 September 2011, in final form 19 January 2012)

ABSTRACT

The synthetic aperture radar ocean surface signature of atmospheric internal gravity waves in the vicinity of a synoptic-scale warm front is examined via a classic Kelvin–Helmholtz velocity profile with a rigid lower boundary and a sloping interface. The horizontal distance that the waves extend from the surface warm front is consistent with a bifurcation along the warm frontal inversion from unstable to neutral solutions. Similarity theories are derived for the wave span and the location of maximum growth rate relative to the surface front position. The theoretical maximum wave growth rate is demonstrated to occur near this bifurcation point and, hence, to explain the observed pattern of wave amplitude. Finally, a wave crest-tracing procedure is developed to explain the observed acute orientation of waves with respect to the surface warm front.

1. Introduction

Warm fronts are often considered to be one of the more benign aspects of baroclinic cyclones. Modern satellite remote sensors, however, tell a different story with warm fronts supporting a variety of interesting mesoscale phenomena (Young et al. 2005). Using ocean surface images from satellite-borne synthetic aperture radar (SAR) and a range of other data sources, Young et al.’s (2005) examined 22 warm fronts, primarily in the Gulf of Alaska. One of the more striking findings was that, in conjunction with the cold conveyor belt (Carlson 1980), many of the warm fronts exhibited wind shear–driven structures on the mesoscale. Young et al.’s (2005) Fig. 7 shows the SAR signature of meso-β-scale vortices, apparently driven by horizontal wind shear instability, a phenomenon that occurred for four of the 22 warm fronts examined. Their Fig. 8 shows the SAR signature of interest in the current study, atmospheric internal gravity waves (IGWs), apparently driven by vertical wind shear across the warm frontal inversion, a phenomenon that occurred for three of the 22 warm fronts examined. The goal of the current study is to use a SAR image to validate an extension of existing IGW theory (Lindzen 1974; Lindzen and Rosenthal 1976) to the...
sloping-inversion warm front situation. SAR serves as the verification sensor for the current study because it provides the high-resolution ocean surface roughness/near-surface wind speed mapping capability required (e.g., Young et al. 2008).

In the SAR Marine User’s Manual (Jackson and Apel 2004), Holt (2004) presents an overview of SAR imaging of the ocean surface. Spaceborne SAR image pixels have dimensions on the order of tenths of meters to tens of meters (e.g., pixel size for RADARSAT-2 can be viewed at http://gs.mdacorporation.com/includes/documents/RS-2_Product_Details.pdf) and represent a normalized radar cross section (NRCS), which is a measure of backscatter. SAR NRCS returned from the surface is a function of radar wavelength, radar polarization, imaging geometry, and surface roughness elements having scales similar to the radar wavelength (e.g., approximately 5.6 cm for the RADARSAT-1 SAR). Thus, for a given set of radar characteristics, SAR NRCS returned from the ocean surface is mainly a function of near-surface wind speed and direction because the centimeter-scale surface roughness elements tend to be wind driven. At incident angles for SAR products commonly used for marine meteorological studies (e.g., 20° to 46° for the RADARSAT-1 ScanSAR Wide B beam mode) and at wind speeds less than about 25 m s⁻¹, SAR NRCS returned from the ocean surface increases with near-surface wind speed. Further, for a given over-ocean near-surface wind speed, minima in SAR NRCS occur when the radar look direction is perpendicular to the near-surface wind direction. Readers are referred to Young et al. (2008) for a concise review.

Because of the intimate relationship between SAR NRCS and the over-ocean near-surface wind vector, researchers have devoted considerable effort to employing spaceborne SAR as a scatterometer. Monaldo et al. (2004) provide a recent review of those efforts and Beal (2005) provides a marine meteorology user’s guide for SAR-derived wind speed (SDWS) images. SDWS images are typically valid at 10 m above sea level and, like SAR NRCS images, possess high resolution.

Because SDWS images possess both high resolution and swath widths of hundreds of kilometers, a plethora of marine meteorological signatures have been documented within them. The reader is referred to Young and Winstead (2005) for a review. The same is true for parent, but radiometrically corrected, ocean surface SAR NRCS (hereafter, simply SAR) images [see Sikora et al. (2006) for a review]. In general, the SDWS signature and SAR signature of any one marine meteorological phenomenon are similar.

The SDWS and SAR signatures of the mountain lee wave variety of IGW are discussed in Young and Winstead (2005) and Li (2004), respectively. Recall that the signatures of IGWs suspected to be driven by vertical wind shear across frontal inversions have been reported in SAR images (Young et al. 2005). As Young and Winstead (2005) point out, mountain lee wave and vertical wind shear–driven IGW signatures are similar. Both reveal a packet of elongated oscillations (in near-surface wind speed or NRCS), the pattern of which is reminiscent of the cloud signature of IGWs that is often present within visible satellite images (e.g., Grubišić and Billings 2008). In general, the IGW SDWS and SAR signatures are less mottled than the signatures of atmospheric roll vortices (Young and Winstead 2005; Sikora and Ufermann 2004), mimicking the same contrast observed within visible satellite images. An exception to this observation may be atmospheric roll vortices generated by inflection point instability (e.g., Alpers and Brümmer 1994).

The SAR and SDWS images found in the literature and those used in the current study imply that frontal IGWs can cause appreciable variations in the near-surface wind speed, with wavelengths in the meso-α scale and wave spans extending for meso-β-scale distances on the cold side of the surface front. Thus, IGWs can impact marine activities, such as small craft operations, by imposing appreciable wind variations on relatively small time and space scales.

The focus of the current study is on a bandpass-filtered SAR image (Fig. 1) containing the signature of IGWs in the vicinity of a synoptic-scale warm front. The original SAR data used to construct Fig. 1 were first smoothed from a pixel spacing of 50 m to a pixel spacing of 500 m. They were then high-pass filtered using a filter weight pattern following a Gaussian distribution with a standard deviation of 10 pixels and a maximum extent of 40 pixels. The goal of the bandpass filtering was to highlight the IGW signature described below.

The original SAR image chosen for the current study was obtained from the Johns Hopkins University Applied Physics Laboratory (JHUAPL) after being processed by the Alaska Satellite Facility. It is from the SAR onboard RADARSAT-1. RADARSAT-1 SAR is right-looking with horizontal–horizontal polarization. The SAR image is from the ScanSAR Wide B beam mode.

Figure 1 is shown as SAR rather than SDWS because, as discussed in Beal (2005), near-surface wind directions are an a priori requirement for the generation of SDWS. In the case of Fig. 1, none of the available archived near-surface wind direction analyses diagnosed the synoptic-scale warm front position with sufficient accuracy to support SDWS retrieval in that zone of sharp streamline curvature.

Within Fig. 1, the black curve of small NRCS arcing across the bottom half of the image is due to the wind lull
The signatures of the IGWs are the oscillations just to the north, the cold side, of the surface warm front signature (Fig. 1). That IGW signature is quite similar to that previously reported in the literature. Given that the IGW case described herein is well removed from topographic barriers and is instead found in the vicinity of a synoptic-scale warm front, the working assumption is that the IGWs are vertical wind shear driven. Young et al. (2005) report that IGWs are most frequently associated with occluded fronts, followed by warm fronts, and then secluded fronts. The focus herein is IGWs in the vicinity of a warm front because this situation is expected to be less complex to model.

A warm frontal region can be modeled analytically using a two-layer system with different velocities in each layer. Vertical wind shear–driven instability can arise at...
the interface between the two layers of fluid (Helmholtz 1868; Nielsen 1992). That instability is commonly referred to as parallel shear instability or Kelvin–Helmholtz (K-H) instability. The corresponding unstable waves grow by extracting energy from the background flow and, in turn, act to reduce the vertical gradient in vertical wind shear (Drazin and Howard 1966).

At small wavenumbers in an infinite domain, neutral solutions, for which wave growth rate equals zero, exist for K-H instability in the form of IGWs that propagate energy away from the vertical wind shear zone (Lindzen 1974). The amplitudes of the IGWs decay with vertical distance from the vertical wind shear zone. In the presence of a rigid lower boundary, however, IGWs can be reflected back toward the vertical wind shear zone, thereby acting to enhance the K-H instability (Lindzen 1974). This reflection leads to unstable solutions, for which wave growth rate is positive, for the IGWs. Therefore, in the presence of a rigid lower boundary such as the sea surface, wavenumbers that would lead to neutral waves for an infinite domain instead lead to unstable waves (Lindzen and Rosenthal 1976). A sudden change in the Brunt–Väisälä frequency at some level away from the vertical wind shear zone can cause similar results (Pellacani et al. 1978). Such a region inhibits the vertical propagation of IGWs in a similar to that seen with a rigid lower boundary (Pellacani et al. 1978). For an extensive review of K-H instability, please see Drazin and Howard (1966).

The IGW signatures present in Fig. 1 extend a limited distance from the surface warm front. Moreover, the amplitude of several of those signatures increases with distance from the surface warm front (Fig. 1 inset). Finally, many of the signatures possess an acute orientation with respect to the surface warm front (Fig. 1 inset). An explanation of these observations is the focus of the current study and is addressed below via linear theory modeling.
2. Procedures

a. SAR image parameters

The average wavelength ($\lambda = 4.5\ \text{km}$), average angle of the IGWs with respect to the surface warm front ($\theta = -70^\circ$), and average wave span ($S = 36\ \text{km}$, the horizontal distance the waves extend in the direction perpendicular to the surface warm front) were calculated via photogrammetry using reference points along the wave crests within the inset of Fig. 1. Figure 3 provides a plan view illustration of those IGW parameters.

b. Linear theory model

Frontal zones represent broad, sloping, quasi-two-dimensional sheets of static stability and vertical wind shear (Martin 2006). Thus, as described above, IGWs may develop at the frontal zone and propagate vertically and horizontally. Such propagation can be represented by a two-layer semi-infinite Boussinesq fluid with a K-H velocity profile, a sloping interface, and a rigid lower boundary (Fig. 4). The corresponding K-H stability is determined by the sign of the imaginary part $c_i$ of the complex phase speed $c$. For $c_i > 0$ there is instability and growth, for $c_i < 0$ there is stability and decay, and for $c_i = 0$ there are neutral solutions. The complex phase speed as a function of the horizontal distance from the surface warm front is determined by solving the dispersion relationship numerically.

The equations employed herein are based on those in Lindzen (1974) and Lindzen and Rosenthal (1976) and so only a brief review is provided. The model is a two-layer, two-dimensional ($x-z$) semi-infinite Boussinesq fluid with a constant Brunt–Väisälä frequency $N$ throughout. This approximation to the $N$ profile was undertaken for two reasons. First, observational data on thermodynamic profiles are absent for the case described herein. Second, the warm front environments in which IGWs have been observed are quite stable (warm air over cool water), so the effects of an underlying convective boundary layer need not be considered. Decomposing the Boussinesq equation set into a base state and a perturbation about that base state yields the following equation for the perturbation vertical velocity $w'$:

$$\left(\frac{\partial}{\partial t} + U_o \frac{\partial}{\partial x}\right)^2 \left(\frac{\partial^2 w'}{\partial x^2} + \frac{\partial^2 w'}{\partial z^2}\right) + N^2 \frac{\partial^2 w'}{\partial x^2} = 0.$$

Within (1), $U_o$ is the base state horizontal wind. Assuming an appropriate plane parallel wave solution, and a mean vertical velocity of zero,

$$w' = \text{Re}[w(z)e^{i(kx-\omega t)}],$$

where $k$ is the horizontal wavenumber and $\omega$ is the frequency of oscillation, yields

$$\frac{d^2 w}{dz^2} + \left[\frac{N^2}{(U_o - c)^2} - k^2\right] w = 0.$$

Equation (3) describes vertical velocity away from the frontal zone ($z = 0$). In the standard two-layer K-H model, the velocity profile has the form

$$U_o = +U \quad z > 0$$
$$U_o = -U \quad z < 0.$$

This velocity profile is employed herein because it results in a simpler derivation and dispersion relation than would a three-layer model (e.g., Chandrasekhar 1961) or a model with a continuous vertical wind shear profile (e.g., Drazin and Howard 1966). This simple velocity profile represents the initial 2D wind field of a warm front region. In this study we are assuming that the velocity field does not change over time. However, it has been previously observed that the vertical wind shear...
zone can evolve into a shear layer, due to the overturning caused by the Kelvin–Helmholtz instability. Furthermore, one would expect that the overturning would cause the shear layer to evolve front such that the Richardson number decreases with distance from the surface warm front (Nielsen 1992). Lack of in situ observations of vertical stability over the high-latitude ocean precludes the quantification of any of these secondary effects, hence this analysis uses the simplest applicable wind and stability profiles.

Substituting (4) into (3) results in two second-order linear homogeneous differential equations:

\[
\frac{d^2w_1}{dz^2} + l_1w_1 = 0, \quad z > 0, \tag{5}
\]

\[
\frac{d^2w_2}{dz^2} + l_2w_2 = 0, \quad z < 0, \tag{6}
\]

in which

\[
l_1 = \left[ \frac{N^2}{(U - c)^2} - k^2 \right]^{1/2} \quad \text{and} \quad l_2 = \left[ \frac{N^2}{(U + c)^2} - k^2 \right]^{1/2}.
\]

Subscripts 1 and 2 represent the upper and lower layer, respectively. Equations (5) and (6) are known as the Taylor–Goldstein equations, which govern vertical velocity in stratified and parallel flows (Kundu 2008).

Equations (5) and (6) can be solved simultaneously given appropriate boundary conditions (Lindzen 1974). For a unique solution, the system requires four boundary conditions. At the interface between the two layers of fluid,

\[
\frac{w_1}{U - c} = -\frac{w_2}{U + c} \quad \text{and} \quad (U - c)\frac{dw_1}{dz} = -(U + c)\frac{dw_2}{dz}. \tag{7}
\]

To avoid a trivial solution, the vertical velocity must vanish at the lower rigid boundary (\(z = -H\), where \(H\) is fluid depth). Finally, because wave amplitude decays with vertical distance from the vertical wind shear zone, the radiation condition is invoked at the upper boundary. This condition allows one to discard solutions in which the vertical velocity grows exponentially with vertical distance from the vertical wind shear zone.

For solutions that decay away from the vertical wind shear zone, \(l_1^2 < 0\) and \(l_2^2 < 0\). Thus, the solutions for the K-H model with a rigid lower boundary are

\[
w_1 = A_1 e^{-n_1z} \quad \text{and} \quad w_2 = -A_2 e^{2n_1H + n_2z} + A_3 e^{-n_2z}. \tag{9b}
\]

Here, \(n_1\) and \(n_2\) are the complex conjugates of \(l_1\) and \(l_2\), respectively, and \(A_1\) and \(A_2\) are arbitrary constants. Continuity of the two interfacial boundary conditions, invoking the radiation condition in the upper layer and applying a lower rigid boundary in the lower layer, yields the dispersion relationship:

\[
(U - c_r - ic_i)^2 n_1 (1 - e^{2n_1H}) = (U + c_r + ic_i)^2 n_2 (1 + e^{2n_2H}). \tag{10}
\]

This equation describes the wave growth and decay at the interface region due to the reflection of vertically propagating waves by the lower boundary. Within (10), \(c_r\) is the real part of the complex phase speed.

Up to this point, the model is based on that described in Lindzen and Rosenthal (1976). To model the evolution of waves in a warm frontal zone, a sloping interface is incorporated. The growth rate of the interfacial waves due to reflection of the downward propagating IGWs by the rigid lower boundary back up to the vertical wind shear zone is proportional to \(c_i\). The point at which \(c_i\) goes to zero on this interface is a function of \(H\). Thus, finding \(H\) at which \(c_i\) becomes zero is equivalent to finding the point on the interface at which this surface reflection no longer affects the vertical wind shear zone. The frontal slope can be used to find the distance of this point from the surface warm front.

Applying the sloping interface as a horizontally heterogeneous boundary condition to the two-layer model results in a system that cannot be treated analytically to derive a dispersion relation. One can, however, exploit the linearity of the dispersion relationship with respect to a slowly varying \(H\). That is, the dispersion relationship can be solved independently at discrete distances from the front, each associated with a unique value of \(H\), along the sloping frontal zone. This approximation is valid as long as the frontal slope \(m\) is small (e.g., \(1/200\)), as is typical for warm fronts (Neiburger et al. 1971).

Recall that (10) only applies to the vertical component of wave propagation. The steeper the front, the greater is the horizontal component of wave propagation. Thus, for steep fronts (e.g., the leading portion of a cold front) (10) is not valid. For those situations, the horizontal component of wave reflection needs to be taken into account, both in the dispersion relationship and when solving the resulting system simultaneously for all distances from the surface front. In cases where this is necessary, a mesoscale model capable of fully resolving the frontal zone would provide a more tractable approach.
An iterative algorithm known as the two-dimensional false position method (Acton 1990) is used to find the roots of (10) numerically. That algorithm finds the zeros of the intersection of the real and imaginary parts of (10). While this method exhibits only linear convergence, it is the traditional method for the type of problem addressed herein and it resulted in convergence in all cases examined. Performing this scheme over a range of \( k, N, \) and \( U \) yields the transect of \( c \) in the direction away from the front expressed as a function of \( H \). The \( U \) and \( N \) ranges were selected based on synoptic analysis of warm front regions and the \( k \) ranges were chosen based on analyses of SAR images of warm front regions. The values of \( H \) are related to the horizontal distance from a surface warm front by assuming a constant \( m = 1/200 \).

Given the above arguments, the average wave span identified for Fig. 1 is hypothesized to be associated with bifurcation from unstable to neutral solutions (Kundu 2008) as the altitude of the vertical wind shear zone increases. Neutral wave solutions exist beyond the bifurcation points, while IGWs are observed between the bifurcation points and the surface warm front. It is also hypothesized that the growth rate of the IGWs reaches a maximum near the bifurcation points, thus explaining their observed increase in amplitude with distance from the surface warm front in Fig. 1.

3. Results

Improvement in computer power since Lindzen addressed the IGW problem allows rapid solution of (10) across the physically reasonable parameter space. Thus, it is now possible to determine similarity relationships for key features of IGW behavior. Below, we focus on the wave span and distance from the surface warm front at which maximum growth rate occurs because these features are captured readily by SAR.

a. Linear theory model

1) THEORETICAL WAVE SPAN

Recall that the argument behind the wave span hypothesis is that, at some critical fluid depth \( H_{c_1=0} \), reflection by the sea surface back to the vertical wind shear zone is no longer sufficient to cause growth because the amplitude of the vertically propagating wave decays below that critical threshold. Therefore, for \( H > H_{c_1=0} \), waves would not interact with the lower boundary, resulting in neutral solutions. The corresponding wave span is related to the depth of the fluid at the bifurcation point by assuming a constant \( m \); hereafter this distance is referred to as the theoretical wave span \( S_t \):

\[
S_t = mH_{c_1=0}.
\]  

Recall that a constant \( m = 1/200 \) is employed herein.

Figure 5 shows \( S_t \) for a wavelength of 4.5 km in \((U, N)\) space. Examining Fig. 5, one immediately can notice linearity in the contours. Also, but not shown, plots for a range of other wavelengths exhibit linear contours. This linearity of the contours and the resemblance between the contour plots of \( S_t \) for different wavelengths suggest the existence of a dynamic similarity relationship between the \( S_t \) and the parameters \( U, N, \) and \( k \). To find this relationship, the following dimensionless quantities are formed:

\[
\hat{k} = \frac{kU}{N},
\]

\[
\hat{S}_t = \frac{S_tN}{U},
\]  

where \( \hat{k} \) is the dimensionless wavenumber and \( \hat{S}_t \) is dimensionless theoretical wave span (Lindzen and Rosenthal 1976). Plotting these dimensionless variables over a range of \( S_t, U, N, \) and \( k \) yields graphical confirmation of the relationship (Fig. 6). A power-law fit performed on the curve observed within Fig. 6 results in an empirical approximation for \( \hat{S}_t \) as a function of \( \hat{k} \),

\[
\hat{S}_t = 0.23(\hat{k})^{-5/3}.
\]

The \( r^2 \) value for this fit is 0.995, indicating that the relationship is indeed a power law. Dimensionalizing (14)

---

1 The slight kinks in the contours are due to the fact that solutions are numerical and hence there is not infinite precision.
yields a direct relationship between \( S_t \) and the parameters \( U \), \( N \), and \( k \):

\[
S_t = 0.23(m)\left(\frac{U}{N}\right)^{-2/3} (k)^{-5/3}.
\] (15)

The SAR-observed average wave span (36 km) falls in the upper left quadrant of \((U, N)\) space in Fig. 5. To test the validity of (15), North American Regional Reanalysis (NARR) (Mesinger et al. 2006) data corresponding to 0300 UTC 16 March 2004 and the apex of the warm front found within Fig. 1 (56.8°N, 142.5°W) were examined. The corresponding \( U \) is approximately 2.5 m s\(^{-1}\) and the corresponding \( N \) is approximately 0.01 s\(^{-1}\), matching the upper left quadrant of Fig. 5 quite well.

2) LOCATION OF MAXIMUM GROWTH RATE

A dynamic similarity can also be found for the relation between the location of the maximum growth rate, expressed as distance from the front, and \( U \), \( N \), and \( k \). The corresponding nondimensional quantities are \( \tilde{k} \) and \( \hat{H}_{\omega_{\max}} \)

\[
\hat{H}_{\omega_{\max}} = \frac{H_{\omega_{\max}} N}{U},
\] (16)

where \( H_{\omega_{\max}} \) is the fluid depth at the distance from the surface front where maximum growth rate occurs. Examining the relationship between \( \tilde{k} \) and \( \hat{H}_{\omega_{\max}} \) (Fig. 7), the points obey a power law of the form

\[
\hat{H}_{\omega_{\max}} = 0.013(\tilde{k})^{-5/3}.
\] (17)

The \( r^2 \) value for this fit is 0.997, demonstrating that the relationship is indeed a power law. To express (17) in terms of the horizontal distance perpendicular from the surface warm front at which the maximum growth rate \( L_{\omega_{\max}} \) occurs, the left side of (17) is replaced by \( mL_{\omega_{\max}} \). Solving for \( L_{\omega_{\max}} \) yields an expression in terms of the parameters \( m, U, N, \) and \( k \):

\[
L_{\omega_{\max}} = 0.19m\left(\frac{U}{N}\right)^{-2/3} (k)^{-5/3}.
\] (18)

Because both \( L_{\omega_{\max}} \) and \( S_t \) obey power laws of the same quantity, there is a linear dependence between the two:

\[
L_{\omega_{\max}} = \frac{19}{22}S_t.
\] (19)

This relationship implies that the maximum growth rate occurs 19/22 (86%) of the theoretical wave span. This relationship explains why the amplitude of many of the IGW signatures within Fig. 1 reaches a maximum near their away-from-front end. Figure 8 quantifies that assessment for the inset within Fig. 1. Therein, the along-front mean of the filtered NRCS is presented to verify the surface warm front location (the minimum) and the along-front mean of the filtered NRCS standard deviation is presented as a proxy for IGW amplitude. The maximum amplitude is thus 43 pixels north of the front while the wave span is approximately 55 pixels.

b. Crest tracing

IGW signatures observed within SAR images often are not perpendicular to the surface warm front but rather are oriented at angle \( \theta \) to the front (e.g., see Fig. 1 inset), although the thermal wind is expected to be parallel to the front. This behavior is hypothesized to
result from the refraction experienced by an IGW traveling along a sloping front (Howe 2007). A wave crest-tracing algorithm is employed to explore this hypothesis using the transect of \( c \) in the cross-frontal direction (i.e., the functional dependence of phase speed on distance from the front), expressed as a function of \( H \).

The wave crest-tracing algorithm proceeds as follows. One first chooses a discrete number of points along an initial wave crest oriented perpendicular to the surface warm front (Fig. 9). Then for some small time interval \( \Delta t \), where \( \Delta t \ll \) period of oscillation (i.e., \( 1/N \)), one allows the wave to propagate in the \( x \) direction. If \( c \) changes with \( H \) and, therefore, with horizontal distance from the surface warm front, the wave refracts. Therefore, at later times the crest is no longer perpendicular to the front, but rather it is bent at some angle \( u \) that itself can vary with horizontal distance from the front. Thus, the two-dimensional linear theory model is also rotated so as to be locally perpendicular to the wave front.

A piecewise quadratic fit is then performed on the new points to determine the slope of the tangent line at each discretization point along the new crest. From this slope, the angle of refraction at each discretization point (i.e., the angle of the wave crest relative to the front) can be determined. This angle, which varies with time and with distance from the front, is used to determine the local orientation of the two-dimensional model. Using the angle of refraction, one can compute the new wavenumber at each discretization point:

\[
k = \sqrt{k_x^2 + k_y^2} = \sqrt{1 + \tan \theta}.
\]

In (20), \( k_x \) remains constant, while \( k_y \) increases as a function of \( \theta \).

Likewise, to compute \( c \) at the next time step, one must first compute the component of wind speed perpendicular to the new wave crest. This step is necessary because the propagation of the wave is dependent on that component wind speed. The wind speed perpendicular to the crest is

\[
U_P = U_o \cos(\theta).
\]

Using the new wind speed and \( k \), one can compute a new complex phase speed \( c_N \) at each discrete point along the new wave crest using the two-dimensional false point method (Acton 1990). The \( x \) and \( y \) components of the new complex phase speed are used to determine the new position of the wave crest. The \( x \) and \( y \) components of the new complex phase speed are

\[
c_{Nx} = c_N \cos \theta, \tag{22}
\]

\[
c_{Ny} = c_N \sin \theta. \tag{23}
\]
The new positions of the wave crest discretization point are

\[
x_N = x_o + c_{ Nx} \Delta t, \tag{24}
\]
\[
y_N = y_o + c_{ Ny} \Delta t, \tag{25}
\]

where \(x_o\) and \(y_o\) are the original positions of the discretization point. This process is iterated in time until a steady state is reached (i.e., when no further refraction occurs) and thus may be expressed explicitly as

\[
\theta_{t-\Delta t} - \theta_{t} = 0. \tag{26}
\]

The initial transect of the complex phase speed is from a point in the upper left quadrant of \((U, N)\) space in Fig. 5. This point is chosen because, we recall, the SAR-observed wave spans lies near this region of \((U, N)\) space. The real part of this initial transect varies with horizontal distance from the surface warm front, as illustrated in Fig. 10. Given this initial transect, the wave will refract toward the surface warm front, much as an ocean swell refracts when approaching a beach. Because the SAR data found within Fig. 1 represent a snapshot at some point during IGW evolution, one cannot compare a theoretical final (i.e., asymptotic) steady-state value of \(\theta\) with that computed from Fig. 1.

It is worth noting that while Fig. 1 shows a similar alignment of the wave crests relative to the front, it does not show the curvature which this linear theory predicts. Thus, while answering one question, it raises another: what process maintains the straightness of the observed wave crests?

4. Conclusions

SAR images of the ocean surface often reveal the signature of vertical wind shear-driven atmospheric internal gravity waves (IGWs) on the immediate cold side of warm fronts. A two-layer semi-infinite Boussinesq fluid with a standard Kelvin–Helmholtz velocity profile and a sloping interface is used to model vertical wind shear-driven IGWs in the vicinity of a synoptic-scale warm front. The wave span predicted through this classical linear theory is consistent with the SAR-observed IGW span. More specifically, the SAR-observed IGW span corresponds to the location along the frontal vertical wind shear zone of a bifurcation from unstable to neutral solutions. This change in wave stability is caused by the decrease in reflection by the rigid lower boundary back to the vertical wind shear zone as a function of horizontal distance perpendicular from the surface warm front.

Similarity formulas are developed for both wave span and the distance from the front of maximum growth rate in terms of static stability, vertical wind shear, frontal slope, and wavenumber. The variation of the SAR-observed IGW amplitude with horizontal distance perpendicular from the surface warm front is explained by the distance from the surface front at which maximum wave growth rate occurs. This distance is nearly as great as that for the bifurcation point which marks the end of
wave growth and hence explains why the SAR-observed IGW amplitude reaches a maximum near the away-from-front end of the IGW signature.

Finally, the angle that the IGWs make with respect to the surface warm front is addressed using a simple wave crest-tracing procedure. An individual wave with its crest initially perpendicular to the surface warm front refracts in the direction of the vertical wind shear vector.

One observation that cannot be explained by the crest-tracing model is that the SAR-observed wave crests are generally straight rather than curved as suggested by the theory presented herein. Two possible explanations are apparent. First, this discrepancy could be the result of young, as yet unbent, IGWs forming in an environment with an along-front temperature gradient and thus non-front parallel thermal wind. While plausible for a single image, this fails to explain the frequency with which straight wave crests were observed by the authors in the JHUAPL SAR image archive. Second, the theory presented herein may fail to capture some aspect of wave dynamics that acts to preserve crest linearity. This latter hypothesis could be tested with a high-resolution nested-grid numerical weather prediction model provided it featured enough levels in the lower troposphere to fully resolve the frontal inversion and vertical wind shear. Such a model would also be appropriate for prediction of IGW in warm front zones.

Acknowledgments. This work was supported in part by Grants N00014-10-1-0569, N00014-07-1-0934, N00014-07-1-0577, and N00014-06-1-0046 from the Office of Naval Research. The authors heartily thank the two anonymous reviewers for insights that greatly improved the presentation of this work. The authors dedicate this work to Donald R. Thompson, a pioneer in the geophysical uses of synthetic aperture radar.

REFERENCES


Helmholtz, H., 1868: Uber Discontinuirliche Flussigkeitsbewegu-


Li, X., 2004: Atmospheric vortex streets and gravity waves. Syn-


Neiburger, M., J. G. Edinger, and W. D. Bonner, 1971: Un-


Sikora, T. D., and S. Ufermann, 2004: Marine atmospheric boundary layer cellular convection and longitudinal roll vorti-


It has come to our attention that Eqs. (5) and (6) of Swales et al. (2012) each contain a typographical error. In Eq. (5), $l_1$ should be replaced with $l_2$. In Eq. (6), $l_2$ should be replaced with $l_2^2$. Thus, the equations should read as follows:

$$\frac{d^2 w_1}{dz^2} + l_2^2 w_1 = 0, \quad z > 0, \quad (5)$$

$$\frac{d^2 w_2}{dz^2} + l_2^2 w_2 = 0, \quad z < 0. \quad (6)$$

The errors admitted herein do not affect the results presented in Swales et al. (2012). We regret any confusion caused by the errors and we thank Prof. Alex DeCaria for bringing the errors to our attention.

REFERENCE