Fourier Finite-Element Method with Linear Basis Functions on a Sphere: Application to Elliptic and Transport Equations

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ABSTRACT
The Fourier finite-element method (FFEM) on the sphere, which performs with an operation count of $O(N^2 \log_2 N)$ for $2N \times N$ grids in spherical coordinates, was developed using linear basis functions. Dependent field variables are expanded with the Fourier series in the longitude, and the Fourier coefficients are represented with a series of first-order finite elements. Different types of pole conditions were incorporated into the Fourier coefficients of the scalar and vector variables in order to avoid discontinuity at the poles. For the Laplacian operator, the linear element was defined as a function of the sine of latitude instead of the latitude. The FFEM was applied to the derivatives of the first- and second-order elliptic equations and the transport equations. The scale-selective high-order Laplacian-type filter was implemented as a hyperviscosity. For the first-order derivative the fourth-order convergence rate of the accuracy, as is expected from the theoretical analysis, was achieved. Elliptic equations were found to be solved accurately without pole discontinuity, and the convergence rate turned out to be second order. The cosine bell advection, time-differenced with a third-order Runge–Kutta method, showed that the squared-norm error convergence rate was slightly above second order. Both the Gaussian bell advection and the deformational flow produced the theoretical convergence rate of fourth order. The high-order filter was found to be effective in maintaining a quasi-uniform resolution over the sphere, and thus allowed a large time step size. Sensitivity experiments of cosine bell advection over the poles revealed that the CFL number, as defined with the maximum grid size on the global domain, can be taken to be as large as unity.

1. Introduction
The spherical harmonics spectral method (SHM) is one of the global spectral-Galerkin methods used to discretize the differential equations describing atmospheric motions on the spherical surface (Bourke 1972, 1974; Hoskins and Simmons 1975; Haltiner and Williams 1980; Krishnamurti et al. 2006). Spherical harmonics, which are the eigenfunctions of the spherical Laplacian operator, are orthogonal to one another and consist of the zonal Fourier function and associated Legendre function. Advantages of the SHM include the accuracy, stability, isotropic resolution, and the simplicity of the semi-implicit time stepping procedure. Considering that the associated Legendre functions are represented with half-ranged sine or cosine functions in the latitudinal direction (e.g., Orszag 1974; Cheong et al. 2012), the spectral Galerkin method with the double Fourier series (DFS) can be implemented in a spherical coordinate system (Orszag 1974; Cheong 2000, 2006). DFS retains many advantageous properties of the SHM, while reducing the operation count of $O(N^3)$ for the SHM to $O(N^2 \log_2 N)$ for $N^2$ complexity.

Recently, the Fourier finite-element method (FFEM), another kind of Galerkin method of $O(N^2 \log_2 N)$ operations for the spherical coordinate system, has been studied (Dubos 2009, 2011), which incorporates the Fourier spectral method in longitude and the finite-element method (FEM) in latitude using the B-spline functions of degree up to three. The FFEM can be used not only for differential equations in the spherical coordinates, but also for those in other coordinates as was demonstrated by previous studies (e.g., Heinrich 1996; Kim and Kweon 2009). Dubos (2009) has applied the FFEM to a variety of problems including simple derivatives of first and second order, the advection equation, and shallow-water models on a sphere.

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(2009), the error (or accuracy) convergence rate of the FFEM with the B splines of degree $d$ turned out to be the $d$th order; the quadratic and cubic splines, for example, provided the discretization errors proportional to $\Delta^2$ and $\Delta^3$, respectively, where $\Delta$ is the grid size. This result is consistent with the theoretical analysis of FEM with high-order elements (Durran 1999). That is, the second-order element produces a rather lower convergence rate than the linear element, which in theory produces the discretization error of $\Delta^4$. While allowing for a better convergence rate compared to the lower order, the high-order elements require an increased operational count that is proportional to the order.

Associated Legendre functions of order $m$, the meridional part of the spherical harmonics of zonal wave-number $m$, is known to pose the pole condition that the derivatives up to order $(m-1)$ vanish at the poles. Basis functions such as DFS functions do not satisfy the pole condition trivially, unlike the associated Legendre functions. Therefore, such basis functions need to be modified for the pole condition; in practice, a modification in such a way that can satisfy the pole condition for the first-order derivative was found sufficient, as was demonstrated for the DFS spectral method (Cheong 2000). These necessary boundary conditions should be satisfied for any Galerkin method on the spherical coordinate system that adopts the Fourier series in the longitude. The fourth-order accuracy, available theoretically for the FEM with linear basis functions, may be not achieved without handling the pole boundary condition in an appropriate manner. In general, the basis functions used for the FFEM cannot fulfill the pole condition, which becomes more serious as the functions become higher order. When not satisfied by the basis functions, the pole condition could be implemented by incorporating a grid point on the opposite side of the pole, as is practiced in the finite-difference method (cf. Haltiner and Williams 1980). In this regard, the linear element is thought to be well suited for the FFEM because only one extra point across the pole is sufficient. With this, the diverse pole conditions, not just one kind of pole condition regardless of the variables, can be imposed with flexibility.

As is well known the latitude–longitude grid system has inherent problems termed as “pole problems,” represented with the singularity associated with the metric term and the narrowing grid-interval toward the poles. The SHM is free from the pole problems due to the isotropic nature of the spherical harmonics series expansion in the model and the Gaussian grids, which are defined off the poles. The pole singularity for the gridpoint-based models is avoidable by adopting the off-pole grids (e.g., the interior grids as is used in DFS spectral model). Dubos (2009, 2011) presented that for the FFEM, which should include the poles, the singularity can be avoided by setting the vanishing values at the poles for the zonal Fourier coefficients other than the zonal-mean components. This kind of simple pole condition, however, does not seem sufficient to avoid the discontinuity at poles, as will be demonstrated in this study.

The narrowing grid intervals near the poles invite severe restriction on the time step size due to the CFL condition. This can be alleviated to a large extent by introducing the quasi-uniform resolution to the zonal Fourier coefficients in terms of either the high-order spectral filtering (Cheong et al. 2004; Cheong 2006) or the reduced number of grid points (or zonal Fourier components) near the poles (Dubos 2009, 2011). In particular, for a given horizontal resolution, the high-order spectral filtering was shown to provide almost the same time step size as that for the spherical harmonics model.

In this study the FFEM on the latitude–longitude coordinate system is investigated using linear basis functions in the meridional direction. One of the major concerns of the present study is the pole condition that was not addressed in detail in the previous studies (Dubos 2009, 2011). Both Dirichlet and Neumann types will be taken into consideration for the zonal-Fourier coefficients, and also are going to be set differently for the scalar and the velocity vector variables. A unique approach, not found in Dubos (2009, 2011), will be taken to the linear elements: they are defined as a function of the latitude or the sine of latitude depending on the differential equations so that the metric terms are represented with polynomials. The paper is organized as follows: in the next section the basic numerical procedure for the FFEM is explained in detail, section 3 is devoted to the numerical experiments for testing the new method, and the summary and conclusions are presented in the final section.

2. Fourier finite-element method

A square integrable variable on the spherical surface can be approximated with finite terms of Fourier series in the zonal direction:

$$h(\lambda, \theta) = \sum_{m=-M}^{M} h_m(\theta) \cos m\lambda, \quad i = \sqrt{-1},$$

where $\lambda$ and $\theta$ are the longitude and the latitude, respectively; $m$ and $M$ are the zonal and the maximum wavenumber, respectively; and $h_m(\theta)$ represents the complex Fourier coefficients of $h(\lambda, \theta)$. The Fourier coefficients are obtained by the Fourier transform.
which, as shown in Cheong (2000), has the following pole condition (see Table 1):

\[ h_m(\theta) = \left\{ \begin{array}{ll} \text{finite}, & m = 0 \\ 0, & m \neq 0 \end{array} \right\}, \quad \theta = \pm \frac{\pi}{2}. \tag{3} \]

To discretize the latitudinal derivatives and the differential operators associated with the elliptic and the transport equations, the meridional domain between the North and South Poles is divided into \( N \) elements with equal distance. The zonal Fourier coefficients, being a function of latitude, are expanded with the linear basis functions \( \phi_j \) as follows:

\[ h_m = \sum_{j=0}^{N} h_{mj} \phi_j, \tag{4} \]

where \( h_{mj} \) means the \( j \)th gridpoint value. As shown in Fig. 1, \( \phi_j \) may be a function of either \( \theta \) or \( x (= \sin \theta) \) on the spherical domain depending on the differential operators of concern. In Dubos (2009), only the latter is used for all variables. One assumption behind this series expansion is that the dependent variables are expressed in polynomials of the dependent variable. Except for the Laplacian operator-related differential equations presented in section 2b, a linear function of \( \theta \) is used for simplicity:

\[ \phi_j = \left\{ \begin{array}{ll} \frac{\theta - \theta_{j-1}}{\Delta \theta}, & \theta_{j-1} \leq \theta \leq \theta_j \\ \frac{\theta - \theta_{j+1}}{\Delta \theta}, & \theta_j \leq \theta \leq \theta_{j+1} \end{array} \right\}, \tag{5} \]

where \( \Delta \theta \) is the grid interval.

### Table 1. Pole conditions for zonal Fourier coefficients of the scalar variable \( (h_m) \) and the velocity vector–coupled variables, where \( m \) means the zonal wavenumber.

<table>
<thead>
<tr>
<th>( m )</th>
<th>( m = 0 )</th>
<th>( m = 1 )</th>
<th>( m = 2, 4, \ldots )</th>
<th>( m = 3, 5, \ldots )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( h_m )</td>
<td>Finite</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( \partial h_m / \partial \theta )</td>
<td>0</td>
<td>Finite</td>
<td>0</td>
<td>Finite</td>
</tr>
<tr>
<td>( u_m, (uh)_m, (vh)_m )</td>
<td>0</td>
<td>Finite</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( \partial u_m / \partial \theta, \partial u_m / \partial \theta )</td>
<td>Finite</td>
<td>0</td>
<td>Finite</td>
<td>0</td>
</tr>
<tr>
<td>( \partial (uh)_m / \partial \theta, \partial (vh)_m / \partial \theta )</td>
<td>Finite</td>
<td>0</td>
<td>Finite</td>
<td>0</td>
</tr>
</tbody>
</table>

with \( \Delta \theta = \pi N^{-1}, j = 0, 1, 2, \ldots, N \), and \( -\pi/2 \leq \theta_j \leq \pi/2 \). The derivatives of the basis functions are given by

\[ \frac{d \phi_j}{d \theta} = \left\{ \begin{array}{ll} \frac{1}{\Delta \theta}, & \theta_{j-1} \leq \theta \leq \theta_j \\ \frac{1}{\Delta \theta}, & \theta_j \leq \theta \leq \theta_{j+1} \end{array} \right\}. \tag{6} \]

**a. Discretization of the gradient and the advection operators**

The spherical-surface gradient operator and its zonal Fourier transform for the unit sphere are given by

\[ \nabla h = \lambda \left( \frac{1}{\cos \theta} \frac{\partial h}{\partial \lambda} + \theta \frac{\partial h}{\partial \theta} \right), \tag{7a} \]

\[ \nabla h_m = \lambda \left( \frac{i m}{\cos \theta} h_m + \theta \frac{\partial h_m}{\partial \theta} \right), \tag{7b} \]

where \( \lambda \) and \( \theta \) are the unit vectors in the longitude and the latitude, respectively, and the variables with the subscript \( m \) denote the zonal Fourier coefficients. Since the first term (i.e., the longitudinal derivative) on the
rhs of (7b) does not include differentiation, it can be evaluated directly without the Galerkin procedure with a special care for the pole singularity as shown below. It is obvious that the meridional gradient component should satisfy the following pole conditions (Cheong 2000):

\[ h'_m = \frac{\partial h_m}{\partial \theta} = \begin{cases} 0, & m = \text{even} \\ \text{finite}, & m = \text{odd} \end{cases}, \quad \theta = \pm \frac{\pi}{2}. \quad (8) \]

Note that the first term on the rhs of (7b) is singular at poles due to the factor \( \cos \theta \). L'Hôpital’s rule then can be used to avoid the singularity at the poles, that is,

\[
\int_{-\pi/2}^{\pi/2} \varphi_k \left( \sum_{j=0}^{N} \varphi_j h'_{m,j} \right) d\theta = \frac{1}{2\Delta \theta} \left\{ h'_{m,j+1} + 4h'_{m,j} + h'_{m,j-1} \right\} \]

where the pole conditions (3) and (8) should be applied. The pole condition requires that \( h_{m,j} \) (and \( h'_{m,j} \) also) for \( j = -1 \) and \( N + 1 \), the grid points across the poles, be replaced by those for \( j = 1 \) and \( N - 1 \) respectively. The discretization accuracy of (10b) can be derived analytically with the use of the finite-difference method (cf. Durrant 1999). Centered finite-difference approximations of the first- and third-order differentiation are written as

\[
\frac{h_{m,j+1} - h_{m,j-1}}{2\Delta \theta} = h''_{m,j} + \frac{1}{6} \left( h'_{m,j+1} + 4h'_{m,j} + h'_{m,j-1} \right) - \frac{1}{3} (\Delta \theta)^2 \left( h'''_{m,j} \right) + O((\Delta \theta)^4), \quad (12a)
\]

which proves that fourth-order accuracy is available for (10b).

When the scalar variable in (7) represents the streamfunction \( \psi \), the gradients are related to the horizontal velocities \( u \) and \( v \) by

\[
u_m = \pm imu_m \quad \text{for} \quad \theta = \pm \frac{\pi}{2}.
\]  

Meridional differentiation of a scalar function as in (7b) by means of the FFEM can be done by projecting the basis functions, which, under the assumption that the residual is orthogonal to each basis function, yields algebraic equations such as

\[
\frac{h_{m,j+1} - h_{m,j-1}}{2\Delta \theta} = h''_{m,j} + \frac{1}{6} \left( h'_{m,j+1} + 4h'_{m,j} + h'_{m,j-1} \right) - \frac{1}{3} (\Delta \theta)^2 \left( h'''_{m,j} \right) + O((\Delta \theta)^4),
\]

where the triple prime denotes the third-order differentiation. If (11b) is substituted into (11a), it follows that

\[
\frac{h_{m,j+1} - h_{m,j-1}}{2\Delta \theta} = h''_{m,j} + \frac{1}{6} \left( h'_{m,j+1} + 4h'_{m,j} + h'_{m,j-1} \right) + O((\Delta \theta)^4),
\]

which can be calculated in the same way as above, implying that the meridional velocity at the poles is identical to the zonal velocity:

\[
u_m = \pm imu_m \quad \text{for} \quad \theta = \pm \frac{\pi}{2},
\]  

From (14) and the boundary condition in (8), the velocities of the zonal-mean component at the poles should be set as vanishing values. It is also obvious from the physical requirement that the velocities vanish at the poles for zonal wavenumbers greater than 1:

\[
u_m(\pm \pi/2) = \nu_m(\pm \pi/2) = 0 \quad \text{for} \quad m = 0 \quad \text{and} \quad |m| > 1.
\]  

This is consistent with Swarztrauber (1993) who showed that the velocities in spherical coordinates are multivalued at the poles and typically are represented by zonal harmonics of wavenumber 1. Such kinds of multivaluedness of velocity at the poles can also be explained by defining the Cartesian coordinate over the poles. For instance, let the
velocity at the North Pole be \( \mathbf{V} = u^C \mathbf{i} + j v^C \), where \( u^C \) and \( v^C \) (\( \mathbf{i} \) and \( \mathbf{j} \)) are single-valued velocity vector components (unit vectors) for the \( x \) and \( y \) directions, respectively. From the identities \( \mathbf{i} = -\lambda \sin \theta \cos \lambda \) and \( \mathbf{j} = +\lambda \cos \theta \sin \lambda \), this can be written as

\[
\mathbf{V} = \dot{\lambda} u + \dot{\theta} v,
\]

(16a)

\[
= \dot{\lambda} (-u^C \sin \lambda + v^C \cos \lambda) + \dot{\theta} (-u^C \cos \lambda - v^C \sin \lambda).
\]

(16b)

This means that the velocities at the poles in spherical coordinates are multiplied and expressed in the wavenumber-1 zonal harmonics.

The vertical component of vorticity is expressed in terms of horizontal velocities as

\[
\zeta = -\frac{1}{\cos \theta} \frac{\partial (u \cos \theta)}{\partial \theta} + \frac{1}{\cos \theta} \frac{\partial v}{\partial \lambda},
\]

(17a)

\[
= -\frac{\partial u}{\partial \theta} + \frac{\sin \theta}{\cos \theta} + \frac{1}{\cos \theta} \frac{\partial v}{\partial \lambda}.
\]

(17b)

Of the two forms of vorticity–velocity relations above, the latter is preferred in this study as it does not include the derivative of a variable that is coupled with the metric term \( \cos \theta \). Equation (17b) can be evaluated for each zonal-Fourier coefficient with special care for the pole singularity as follows:

\[
\zeta_m = \begin{cases}
\frac{-\partial u}{\partial \theta} + \frac{\sin \theta}{\cos \theta} + \frac{1}{\cos \theta} \frac{\partial v}{\partial \lambda}, & \theta \neq \pm \frac{\pi}{2} \\
-2 \frac{\partial u_m}{\partial \theta} - \frac{\partial v_m}{\partial \theta}, & \theta = \pm \frac{\pi}{2}
\end{cases},
\]

(18)

which requires the computation of the derivative \( \partial u_m/\partial \theta \) over the whole domain but only the derivative \( \partial v_m/\partial \theta \) at the poles. As will be seen in the next section, when the advection equation is written in the flux form, the meridional component includes the metric term \( \cos \theta \) as

\[
\frac{1}{\cos \theta} \frac{\partial (uv \cos \theta)}{\partial \theta} = \frac{\partial (uv)}{\partial \theta} \cos \theta - (uv) \frac{\sin \theta}{\cos \theta}.
\]

(19)

The singularity, associated with the second term on the right-hand side, can be avoided in a similar way shown above using L’Hôpital’s rule again:

\[
-(uv) \frac{\sin \theta}{\cos \theta} = \frac{\partial}{\partial \theta} (uv) \cos \theta \text{ for } \theta = \pm \frac{\pi}{2}.
\]

(20)

The pole conditions for the advective fluxes of a scalar variable in (18)–(20) are presented in Table 1, together with those for the scalar variable and the velocities. As will be shown in section 3a, violation of the pole condition causes a nonsmooth behavior of the differentiated variables near the poles.

b. Discretization of the elliptic equations

The elliptic equations, such as Poisson’s equation (e.g., the vorticity streamfunction and the divergence–velocity potential equations) and Helmholtz’s equation, include the horizontal Laplacian operator. Discretization of Poisson’s equation can be readily extended to the Helmholtz equation by simply adding a mass term; hence, only the discretization procedure for Poisson’s equation is explained here. The Poisson’s equation on a spherical surface with unit radius can be written as

\[
\nabla^2_m = \frac{\partial}{\partial \theta} \left( \frac{\cos \theta}{\cos \theta} \frac{\partial \psi}{\partial \theta} - \frac{m^2}{\cos \theta} \right),
\]

(21b)

\[
= -\frac{\partial}{\partial \theta} \left( 1 - x^2 \right) \frac{\partial \psi}{\partial \theta} + \frac{m^2}{1 - x^2},
\]

(21c)

where the variables and operators with subscript \( m \) denote the zonal Fourier transforms. It should be noted that the metric terms in the Laplacian operator in (21c) are expressed as a polynomial when the independent variable is given as \( x \); which is not the case for (21b) where the metric terms are represented with either \( \cos \theta \) or \( \cos^2 \theta \). Taking this into consideration, the linear elements are given as a function of \( x \) rather than \( \theta \) unlike the case of the first-order differential operator:

\[
\phi_j = \begin{cases}
\left. \frac{x - x_{j-1}}{\Delta x_j} \right. & x_{j-1} \leq x \leq x_j \\
\left. \frac{x - x_{j+1}}{\Delta x_{j+1}} \right. & x_j \leq x \leq x_{j+1}; \\
0, & \text{otherwise}
\end{cases}
\]

(22)

where \( x_j = \sin[j(\Delta \theta) - \pi/2] = \sin[jn/N - \pi/2] \), and \( \Delta x_j = x_j - x_{j-1} \).

After multiplying the basis functions of (22) to (21c), the integration over the global domain is performed, yielding the FFEM-discrete form of Poisson’s equation:

\[
\int_{-1}^{1} \phi_k \left( \sum_{j=0}^{N} \xi_{m,j} \phi_j \right) dx = \int_{-1}^{1} \phi_k \left( \nabla^2_m \sum_{j=0}^{N} \psi_{m,j} \phi_j \right) dx \Rightarrow A \xi_m = D \psi_m,
\]

(23)

where \( A \) and \( D \) are tridiagonal matrices with size \( (N+1) \times (N+1) \), and \( \xi_m \) and \( \psi_m \) are column vectors consisting of gridpoint values \( \xi_{m,j} \) and \( \psi_{m,j} \) \( (j = 0, 1, \ldots, N) \), respectively. It turns out that the first-order pole singularity, which is caused by \( (1 - x^2) \) in the denominator, actually does not appear because of the pole condition.
The integrals for the meridional differentiation in (23) include second-order polynomials, so they can be calculated analytically with ease. The integrals for the longitudinal differentiation, which are coupled with \((1 - x^2)^{-1}\), are obtained from the following analytic formula:

\[
\int \frac{-1}{1 - x^2} (x - x_a)(x - x_b) \, dx = (x_u - x_a) + \frac{1 - c}{2} \ln \frac{x_u - 1}{x_d - 1} - \frac{1 + d}{2} \ln \frac{x_u + 1}{x_d + 1} \left[ \begin{array}{l} c = (x_a + x_b - x_a x_b) \\ d = (x_a + x_b + x_a x_b) \end{array} \right],
\]

for \(|x_d| < 1\) and \(|x_u| < 1\), which reduces to a simpler form for the elements at the poles (e.g., for the North Pole):

\[
\int \left[ \frac{-1}{1 - x^2} (x - 1)(x - x_b) \right] \, dx = 1 - x_d + (1 + x_d) \ln \frac{1 + x_d}{2}, \quad (x_d > -1),
\]

and for the South Pole:

\[
\int \left[ \frac{-1}{1 - x^2} (x + 1)(x - x_b) \right] \, dx = 1 + x_u + (1 - x_u) \ln \frac{1 - x_u}{2}, \quad (x_u < 1).
\]

The eigenvectors of the discrete system \(A^{-1}D\) are the discrete forms of the associated Legendre functions (Legendre polynomials for \(m = 0\)). For a given eigenvector, say \(\psi_m\), the function \(\xi_m = \nabla^2 \psi_m\) can be represented with the same function as \(\psi_m\) but with a constant multiple corresponding to the eigenvalue \((\alpha)\). This means that the inversion of \(\xi_m\) to get \(\psi_m\) is obtained by simply dividing \(\xi_m\) by the eigenvalue. Since, in principle, the forward operation of the Laplacian to a constant field yields zero value, the global-mean value should be subtracted from a given field for an accurate inversion without singularity. [It was confirmed that for a constant value, the right-hand side of (23) is a value near the machine round off, e.g., \(O(10^{-15})\) in the case of double-precision computation.]

As is practiced in the SHM model, the zonal Fourier coefficients can be expanded with the eigenfunctions of the Laplacian operator [i.e., the associated Legendre functions, \(P_n^m\) with \(n(\geq m)\) being the degree]. It should be noted that \(P_n^m\) for an odd \(m\) is expressed as a function multiplied by \((1 - x^2)^{m/2}\) (e.g., Haltiner and Williams 1980; Cheong et al. 2012), and, hence, is not a polynomial of \(x\). Therefore, when the basis functions of the FFEM are given as a function of \(x\), a variable transform is necessary to obtain the zonal Fourier coefficients of a global field as polynomials of \(x\). As will be demonstrated below, without such a variable transform, numerical error is unavoidable with its magnitude largest around the poles. This kind of problem is particularly important for \(m = 1\) because the singularity may occur due to the metric term \((1 - x^2)^{-1}\); for example, \(P_6^1\) is proportional to \(\sqrt{1 - x^2}\), which means that a scalar function in this form would invite the singularity due to division by \((1 - x^2)\) as in (21c). [This singularity does not matter for the SHM because the differential operators in the Laplacian operator are not discretized separately.] In the case of even \(m(\neq 0)\), however, the pole singularity does not appear for the Laplacian operator. The Legendre function \(P_n^m\) remains bounded at the poles when divided by \((1 - x^2)\), that is, \((1 - x^2)^{-1} P_n^m\) vanishes for \(m > 2\) whereas it should have a finite value at the poles for \(m = 2\). Therefore, to deal appropriately with these behaviors, variable transform and imposing different pole condition for when \(m(\neq 0)\) is even, are also required.

Variable transforms for Poisson’s equation are performed using the metric term with different order for odd and even zonal wavenumber. After dividing (21) by \(\cos \theta\) for odd \(m\) [multiplying by \(\cos \theta\) is also a viable option as in Dubos (2009)] and by \(\cos^2 \theta\) for even \(m(\neq 0)\), some manipulation yields the Laplacian equation with transformed variables as

\[
\frac{\xi_m}{\cos 2k \theta} = \frac{1}{\cos 2k \theta} \left( \frac{\partial}{\partial \theta} \cos \theta \frac{\partial}{\partial \theta} \cos \theta \right) \frac{m^2}{\cos 2k \theta} \theta - \frac{1}{\cos 2k \theta} \psi_m \cos 2k \theta - \frac{1}{\cos 2k \theta} \psi_m.
\]
in Table 1 and the behavior of the Legendre functions that do not invite singularity because of the pole conditions as shown by dividing the dependent variables by the metric terms, does not invite singularity. Variable transforms in (27), which is worthy to note that the last term in (27c) vanishes for both \( m = 1 \) and \( m = 2 \), which leaves the Laplacian operator without singularity. Variable transforms in (27), which is dividing the dependent variables by the metric terms, does not invite singularity because of the pole conditions as shown in Table 1 and the behavior of the Legendre functions that have \((1 - x^2)^{-\frac{m-1}{2}} P_m^m(x) = 0 \) at \( x = \pm 1 \) for \( m \geq 2 \). When (27c) is inverted from a given \( \zeta_m \), the transformed variable at the poles should be calculated by L’Hôpital’s rule for \( m = 1 \) and \( m = 2 \), while they are given to vanish for \( m \geq 3 \) as

\[
\begin{align*}
Z_m(\pm \pi/2) & \rightarrow \\
\left\{ \begin{array}{ll}
\frac{1}{\sin \theta} \frac{\partial \zeta_m}{\partial \theta} |_{\theta = \pm \pi/2} & \text{for } m = 1 \\
\frac{1}{2 \sin^2 \theta} \frac{\partial^2 \zeta_m}{\partial \theta^2} |_{\theta = \pm \pi/2} & \text{for } m = 2 \\
0 & \text{for } m \geq 3
\end{array} \right.
\end{align*}
\]

The same procedure as above, but this time for \( \psi_m \), is also required in the case of the forward operation. Matrix equations associated with (27c) can be constructed in a similar way to (23) under the pole conditions for transformed variable as

\[
\frac{\partial Z_m}{\partial x} = 0 \quad \text{for } m = 1 \text{ and } 2
\]

\[
Z_m = 0 \quad \text{for } m \geq 3
\]

\[
\Rightarrow (Z_m, Z_{m, N+1}) = \begin{cases} (Z_{m-1}, Z_{m, N-1}) & \text{for } m = 1 \text{ and } 2 \\ (Z_{m-1}, Z_{m, N+1}) = -(Z_{m-1}, Z_{m, N-1}) & \text{for } m \geq 3 \end{cases}
\]

The forward operation of Poisson’s equation in (23) can also be achieved by successively differentiating \( \psi \) twice: first by (13) and second by (17), that is, \( \psi \rightarrow (\psi, -\nu \nabla^2 \psi) \rightarrow \zeta \). The computation time for these procedures would be doubled compared to the direct second-order differentiation. This, however, does not seem to be a serious problem because the matrices are all tridiagonal. More importantly, the increased accuracy for the discretized system of the Laplacian operator is expected because of the inclusion of only the first-order differentiations.

The Helmholtz equation, a combination of mass term and Poisson equations can be easily solved by combining two tridiagonal matrices used in (23):

\[
\zeta_m^* - \epsilon \nabla_m^2 \zeta_m^* = \zeta_m
\]

\[
\Rightarrow (A - \epsilon D)\zeta_m^* = \zeta_m
\]

\[
(1 + (-1)^q \nu \nabla^2) \zeta_m^* = \zeta_m
\]

where \( q \) is the order of the filter, \( \zeta_m^* (\zeta_m) \) is the filtered (to be filtered or forcing function) variable, and \( \nu \) means the positive-valued filter coefficient. This kind of implicit-diffusion-type filter is advantageous over the explicit-type high-order filter in that there is no CFL-condition-like restriction on the magnitude of the filter coefficient (cf. Jablonowski and Williamson 2011). A detailed procedure to solve for the high-order filter equation can be found in Cheong et al. (2004), so it is not repeated here. The key factor for inverting (32) is to split it into multiple Helmholtz equations and invert them successively. If the filter scale is set appropriately, the disturbances smaller than a certain scale can be removed effectively, and at the same time, a quasi-uniform resolution on the spherical surface is accomplished.

3. Numerical experiments and accuracy assessment

The number of grid points for the FFEM on a sphere, including the poles and the equator, is given \( 2N \times (N + 1) \) with equal angular distances with \( N \) being the number of elements from the South to the North Pole. If not stated otherwise, the maximum zonal wavenumber is given as \( M \leq (2N - 1)/3 \), where \( 2N \) is the number of zonal grid
points based on the two-thirds rule to avoid aliasing (e.g., Orszag 1970; Hoskins and Simmons 1975). For the time integrations performed in the present study, the fourth-order filter was applied to the predicted field at every time step except for the simulations for sensitivity tests. The filter coefficient was set to reduce the amplitude of the smallest-scale disturbance by half for one day. The smallest horizontal scale is equivalent to the spherical harmonic function whose degree is the same as the largest zonal wavenumber of the model. While stabilizing the numerical model, the high-order filter at the same time acts to provide an isotropic resolution over the global domain and thus allows a large time step size in spite of the narrowing grid size near the poles.

To assess the accuracy of the FFEM, various test cases are used from simple differentiations to the transport equations. The errors of the new method are measured by the difference between the analytical and numerical solutions using three quantities as follows (e.g., in case of the zonal velocity):

\[
L_1 = \left( \int_S |u - u^R| \, dS \right) \left( \int_S |u^R| \, dS \right)^{-1},
\]

\[
L_2 = \sqrt{\int_S (u - u^R)^2 \, dS} \left[ \sqrt{\int_S (u^R)^2 \, dS} \right]^{-1},
\]

\[
L_\infty = \max_{i,j} |u_{ij} - u^R_{ij}| \left( \max_{i,j} |u^R_{ij}| \right)^{-1}, \tag{33}
\]
where the superscript denotes the reference field; the subscripts $i$ and $j$ mean the grid points in zonal and meridional direction, respectively; and the integration is carried out over the global domain. For the uniform angle grids as in the present study, the global integration can be replaced by summation as

$$L_1 = \left( \sum_j \cos \theta_j \sum_i |u_{ij} - u^R_{ij}| \right) \left( \sum_j \cos \theta_j \sum_i |u^R_{ij}| \right)^{-1},$$

$$L_2 = \sqrt{ \sum_j \cos \theta_j \sum_i (u_{ij} - u^R_{ij})^2 } \sqrt{ \sum_j \cos \theta_j \sum_i (u^R_{ij})^2 }^{-1},$$

(34)

FIG. 3. (top) Differences between the first-order derivatives computed from left (13a) and right (17b) and the reference solution. (bottom) Difference field for the second-order differentiation. Variables $u^R$ and $v^R$ represent the reference velocity fields in Fig. 2.
where the weighting by the cosine of the latitude vanishes at the poles; hence, the errors at the poles is the difference between zonal averages.

a. The first- and second-order differentiation

To measure the accuracy of the FFEM as for the first- and second-order differentiation, the scalar field of the rotated spherical harmonic function is used. Shown in Fig. 2 is the streamfunction given as $P_4^5(x)$ with a poleward rotation angle of $\pi/2 - 0.05$ along with the vorticity, the zonal velocity $u$, and the meridional velocity $v$ associated with the streamfunction. As will be discussed later, the velocities at the North Pole appear multivalued, characterized by a zonal wavenumber-1 structure. Figure 3 shows the first- and second-order differentiation of the reference streamfunction using FFEM with $360 \times 181$ grids, where the difference fields of the zonal velocity and vorticity are presented. To see the effect of the metric term, two types of first-order differentiation, with and without the metric terms each given in (13a) and (18), respectively, were computed. [Meridional velocity is just the streamfunction multiplied by the zonal wavenumber as in (13)–(15), so the difference field is omitted.] The error (or the difference normalized by the maximum gridpoint value) of first-order differentiation in (13a) is as a whole of $O(10^{-8})$, while the error for (18) is of $O(10^{-7})$. For the second-order differentiation [i.e., $\psi \rightarrow (u, v) \rightarrow \nabla^2 \psi$], the error is of $\sim O(10^{-6})$, and it seems to be larger around the North Pole than elsewhere. The results shown in this plots look quite reasonable considering the complexity of the differential equations. The high accuracy available for the first-order differentiation of (18) is particularly important because it is directly applied to the advection terms as in (19) for the time stepping procedure. In Fig. 4, the $L_2$ norm error of the latitudinal differentiation and Poisson’s equation, represented in logarithmic scale, is shown for five different resolutions that vary from 240 grid points in zonal direction to 960. The errors for the latitudinal differentiation decrease from $10^{-5.8}$ on the $240 \times 121$ grid to $10^{-8.3}$ on the $960 \times 481$ grid. As was expected from the theoretical error analysis in section 2a, the convergence rate of the error turns out to be of fourth order $\Delta^4$ with $\Delta$ being the grid size at the equator. Poisson’s equation was solved either by second-order differentiation as in (27) or successive first-order differentiations [i.e., $\psi \rightarrow (u, v) \rightarrow \zeta$], which are shown with rectangles and circles in the right panel of Fig. 4, respectively. The error for the successive differentiations of $\psi \rightarrow (u, v) \rightarrow \zeta$, which corresponds to a second-order differentiation, has slightly increased in comparison with the first-order differentiation (i.e., $\psi \rightarrow u$) shown in the left panel, but still remains almost of the same order. The convergence rate in this case seems to be lying between the fourth and the third order, but is much closer to the fourth order. The error in Poisson’s equation resulting from second-order differentiation, however, has increased by about three orders, compared to first-order differentiation. Furthermore, the error convergence rate remains only in second order.

The accurate and spatially smooth solutions presented in Fig. 3 are believed to have come from incorporation of the proper pole conditions in the discretization procedure. As described in the previous section, the pole condition for the derivatives uses a grid point on the
opposite side of the pole. To elucidate the effectiveness of this strategy, the same calculations as those in Fig. 3 for the zonal velocity were performed under a simple Dirichlet pole condition (i.e., a vanishing value at the poles) without using a grid point across the pole. The results are provided in Fig. 5, where the zonal velocity is plotted for the region around the North Pole. In contrast to the result when using a proper pole condition, which is characterized by the smooth zonal wavenumber-1 structure, the contours of the zonal velocity obtained from wrong pole condition exhibit very nonsmooth distribution near the North Pole. The error associated with the wrong pole condition appears more or less concentrated near the poles, and, hence, may give a rather small impact on the global-domain-averaged error norm. However, the error originated around the poles readily spreads into the global domain during time-integration process by the nature of global treatment of variables through the matrix operations involved in the FFEM.

Variable transforms, which were used in (27), are expected to have contributed to an improvement of the solution of Poisson’s equation by removing the pole singularity. Effects of these transforms may be confirmed by comparing two results each obtained from (21) and (27), respectively. For this purpose, a global field of \( \psi_m = \cos \lambda \sin \theta \cos^m \theta \) is chosen and \( \zeta_m = \nabla^2 \psi_m \) is calculated. Differences from the analytical solution of \( \zeta_m = -[(m + 1)(m + 2)] \psi_m \) are provided in Fig. 6. As a whole, the differences for (27) are significantly reduced compared to those for (21). In case of \( m = 1 \), the difference for (21) is about of \( O(1) \), while the difference for (27) is only of \( O(10^{-10}) \). An improvement rate similar to, but slightly smaller than, the case of \( m = 1 \) is shown for the case of \( m = 2 \). Even for \( m = 3 \) and 4, an accuracy improvement of order 2 and 1, respectively, has been achieved. In contrast to the results from (21), which show a large error near the poles—being indicative of the pole problem (singularity)—the errors at high latitudes for (27) gradually diminish toward the poles. In addition to the increase in accuracy, this can be considered another important feature of the variable transforms used in (27).

Figure 7 shows the \( L_2 \) norm errors associated with the inversion of Poisson’s and Helmholtz equations for the same resolutions as in Fig. 4. The inversion error of Poisson’s equation is of \( \sim O(10^{-3}) \) for the given resolutions, which is comparable to the forward operation. The convergence rate of the error is found to be approximately of second order. In the case of Helmholtz equation, the accuracy appears to improve as the coefficient of the Laplacian operator (\( \varepsilon \)) decreases. This is because the relative contribution of the second-order derivative, which is the only source of numerical error in the Helmholtz equation, becomes smaller. Inversions with different values of \( \varepsilon \), including the two cases in Fig. 7, suggest that the error is approximately equal to \( \varepsilon Q \), where \( Q \) is the error associated with the inversion of Poisson’s equation. It is worth noting, however, that the convergence rate for the Helmholtz equations is not sensitive to the values of \( \varepsilon \), as we can see the same
convergence rate of second order in both cases of the right panel in Fig. 7.

Figure 8 illustrates the errors associated with the high-order filters with a different order and coefficient. The errors are expected to be similar to the case of the Helmholtz equation, because the high-order filtering is performed by successive multiple operations of the Helmholtz equations. The error falls in the range between $O(10^{-4})$ and $O(10^{-7})$, and the error convergence rate is similar to the Helmholtz equations. As in the case of the Helmholtz equation, the errors are found sensitive to the magnitude of the coefficient (i.e., they become smaller at a reduced coefficient). Such a feature appears more conspicuous for the case of $q = 2$ than $q = 4$. This is attributed to the fact that the filter’s scale, which is defined as the horizontal scale whose amplitude is halved by filtering, tends to be affected much more in the case of $q = 2$ than $q = 4$ as the filter coefficient decreases. Remembering that the high-order filter in most cases is used for smoothing out unnecessary, noisy small scales from the global field, this behavior is considered to be a desirable aspect of the high-order filter.

b. Solid-body rotation of the cosine bell and effect of the high-order filter

FFEM is tested with the advection of the cosine bell by super-rotation flows (Williamson et al. 1992; Cheong 2000; Flyer and Wright 2007; Dubos 2009), where the environmental adverting flow is determined to give one rotation for 12 days along the great circle, which has an inclination angle of $\alpha = \pi/2 - 0.05$ from the equator:

$$u = u_0(\cos \theta \cos \alpha + \sin \theta \cos \phi \sin \alpha),$$  
(35a)

$$v = -u_0 \sin \phi \sin \alpha,$$  
(35b)

$$h = \begin{cases} 
(h_0/2)[1 + \cos(r\pi/R)], & r < R \\
0, & r \geq R 
\end{cases}$$  
(35c)

where $R = a/3$ with $a$ the radius of the earth, $h_0 = 1000$ m, $u_0 = 2\pi a/(12$ days) in dimensional units, and $r$
represents the geodesic distance from the bell’s center. The transport equation with nondivergent flow that is scaled by the earth’s radius and rotation rate ($\Omega$) can be written, in either advective or flux form, as follows:

\[
\frac{\partial h}{\partial t} + u \frac{\partial h}{\partial \lambda} + v \frac{\partial h}{\partial \theta} = 0, \quad (36a)
\]

\[
\frac{\partial h}{\partial t} + \frac{1}{\cos^2 \theta} \left[ \frac{\partial (hu \cos \theta)}{\partial \lambda} + \cos^2 \theta \frac{\partial (hv \cos \theta)}{\partial \theta} \right] = 0, \quad (36b)
\]

\[
\frac{1}{\cos \theta} \left[ \frac{\partial u}{\partial \lambda} + \frac{\partial (v \cos \theta)}{\partial \theta} \right] = 0, \quad (36c)
\]

where the last equation represents the nondivergence of the advecting flow. For the advection term in (36b), the quadratic terms should be computed first and then the differentiation is carried out following (19). Unlike the numerical methods capable of providing the quasi-isotropic resolution over the sphere, FFEM is subject to the severe CFL condition for time stepping problems due to the reduced grid size toward the poles. The high-order Laplacian-type filter, as in (32), may be useful for such a problem because it dampens down small-scale disturbances selectively and, hence, gives quasi-isotropic resolution over the global domain. From the viewpoint of the implicit-diffusion operator, the filter coefficient can be determined in such a way to give a halved dampening of the smallest-scale (i.e., the largest total wavenumber $M$) disturbance for a prescribed time scale $\tau$.

**Fig. 7.** As in Fig. 4, but for (left) Poisson’s equation or inverse Laplacian and (right) the Helmholtz equation, where $\varepsilon_1 = 30$ with $\varepsilon$ being the coefficient of the Helmholtz equation in (31). The convergence rate of second order is shown as thin lines for reference.

**Fig. 8.** As in Fig. 4, but for the high-order filter equation with a coefficient of (left) $10^{-2} \times (30)^{-2}$ and (right) $10^{-3} \times (30)^{-4}$, where the order ($q$) of the filter is given as 2 and 4.
\[ \nu = \frac{2\Delta t}{\tau(M(M+1))^{1/2}} \]  
\hspace{1cm} (37)

where \( \Delta t \) is time step size. As the diffusion time scale \( \tau \) becomes small, the damping becomes stronger. Since the high-order filter is applied at every time step, the amplitude of small-scale disturbances is reduced significantly whereas it is not the case for the larger scales. To see how effectively the filter stabilizes the time-integration process, two kinds of filter coefficients are compared for the simulation with a 360 \( \times \) 181 grid resolution: one is \( \tau = 2\Delta t \) and the other is \( \tau = 1 \) day. Even for the former case (i.e., a strong filtering), the filter coefficient is as small as 0.25 \( \times \) 10\(^{-16} \). The CFL condition for the time stepping procedure of the advection equation requires that

\[ \frac{\Delta t}{(u_{\text{max}})^{-1}(\Delta x)_{\text{max}}} = \frac{\Delta t}{(4 \times 10^4 \text{ km})^{-1}(4 \times 10^4 \text{ km})} \leq 1, \]  
\hspace{1cm} (38a)

\[ \Rightarrow \Delta t \leq \frac{1}{30} \text{ day}, \]  
\hspace{1cm} (38b)

where the maximum grid size was taken from the equator. Then, the CFL number can be defined as \( \Delta t \times (30^{-1} \text{ day})^{-1} \) or 30/\( K \) with \( K \) being the number of time steps needed for the 1-day simulation. It is straightforward to calculate the CFL number for a stable time integration over the global domain without high-order filtering. The CFL number is 1/180 or smaller in the case of the 360 \( \times \) 181 grid resolution, if the grid size over the poles is taken into consideration.

The time differentiation is approximated with the third-order Runge–Kutta method (Wicker and Skamarock 2002), which has three times of time differentiating for each time step. The Runge–Kutta method is known to provide a stable time integration with a \( \sqrt{3} \) times larger time step size than the leapfrog method. Table 2 summarizes the results of high-order filtering as an implicit diffusion for the solid-body rotation test case. For a filter coefficient of \( \tau = 1 \) day, the time stepping is unstable until the CFL number is reduced to 0.75, 0.88, and 0.94 for \( q = 3, 4, \) and 5, respectively; this clearly indicates the more effectiveness of the higher-order filter in stabilizing the time stepping procedure. In the case of \( \tau = 2\Delta t \) on the other hand, time stepping remains unstable until the CFL number is reduced to 1.36; this is common to the filters for \( q = 3, 4, \) and 5, implying that the CFL number for stable time integration is not sensitive to the order of the high-order filter. As it should be, the accuracy of time integration becomes better as the time step size and/or the filter coefficient decrease, which is fairly well reproduced in the experiments listed in Table 2. [It should be remembered that for a fixed time scale \( \tau \) in (37) the increase of the order \( q \) means a decrease of the filter coefficient.] The stability achieved in these experiments with a reasonably large CFL number, compared to the CFL number of 1/180 in the case without the filter, is a direct consequence of the quasi-uniform resolution accomplished by high-order filtering.

Figure 9 provides the cosine bell at the initial time and the difference fields from the theoretical field by days 3, 9, and 12 for the 360 \( \times \) 181 grid resolution, simulated with a time step size of 8 min. The maximum absolute difference is as small as 4.64 m by day 12, and the large differences with alternating signs are mainly found on the periphery of the cosine bell. The difference fields show no indication of any problems around the poles (i.e., pole problem), as there is no significant difference among the error patterns by day 3 (over the North Pole), day 9 (over the South Pole), and day 12 (on the equator).
except for the magnitude that increases with time. The norm errors $L_1$, $L_2$, and $L_\infty$ for 12 days are presented in the left panel of Fig. 10. The errors exhibit a steady increase with time, but the increase rate appears to be slightly larger in the earlier stage than at the later stage of the 12-day simulation. The plots of the errors versus resolution indicate that the norm error $L_1$ lies between the second and third order (being 2.57), and $L_2$ is slightly above the second order (2.13), while $L_\infty$ is slightly below the second order (1.87). In terms of the accuracy and convergence rate shown here, the FFEM seems to compare well with other Galerkin methods.

FIG. 9. (top left) The cosine bell at initial time and the difference from the theoretical fields by day: (bottom left) 3, (bottom right) 9, and (top right) 12 for grids with a resolution of $360 \times 181$; CI means the contour interval and the positive (negative) values are in solid (dashed) lines. The minimum and maximum values of difference fields are indicated at the bottom of each panel. Map projections are: orthographic (day 0 and 12), northern polar stereographic (day 3), and southern polar stereographic (day 9).
(Jakob-Chien et al. 1995; Cheong 2000), and is also comparable to the discontinuous Galerkin method (DG; Nair et al. 2005), which has a convergence rate around second order when the resolution is increased in terms of the number of cells. However, the present method gives a slightly lower performance than the DG method (Nair et al. 2005), which has a third-order convergence rate when the resolution is increased through polynomial order, and the transport scheme using radial basis functions (Flyer and Wright 2007), which also has a third order.

c. Gaussian bell advection

The same test as the cosine bell advection was carried out using a smooth initial field of the Gaussian bell (Flyer and Wright 2007), which, unlike the cosine bell whose second-order derivative is discontinuous, maintains a continuous field for the derivatives of any order. The Gaussian bell used as the initial condition is defined as follows:

\[ h = h_0 e^{-\left(\frac{2.5r}{R}\right)^2}, \]  

which gives the horizontal scale comparable to the cosine bell used above. The norm errors of the Gaussian bell advection for 12 days are illustrated in Fig. 11. The norm errors \( L_1, L_2, \) and \( L_\infty \) appear to increase almost linearly with time, and the differences among them are significantly smaller than the cosine bell advection. The error convergence rates for the three norm errors are found to be almost exactly of fourth order as in the case of the first-order derivative in Fig. 4, which is considered a result of the smoothness of the initial field.

d. Deformational flow

As described in Nair et al. (2005) and Flyer and Wright (2007), this test case deals with the passive transport of a scalar field by a deformational flow. Normalized
tangential velocity of the vortical flow on a spherical surface of unit radius is given as follows:

\[ V_t = \frac{3\sqrt{3}}{2} \text{sech}^2(\rho') \tanh(\rho'), \tag{40} \]

where \( \rho' = \rho_0' \cos(\theta') \) represents the radial distance of the vortex with \( \rho_0' = 3 \) being a scale parameter. The exact solution for the scalar field at time \( t \) is

\[ h(\lambda', \theta', t) = 1 - \tanh \left[ \frac{\rho'}{\gamma} \sin(\lambda' - \omega') \right], \tag{41a} \]

\[ \omega' = \begin{cases} 0 & \text{if } \rho' = 0, \\ \frac{V_t}{\rho'} & \text{if } \rho' \neq 0. \end{cases} \tag{41b} \]

where \( \lambda' \) and \( \theta' \) are the rotated spherical coordinates whose poles are at the vortex centers, \( \gamma = 5 \) represents

**FIG. 12.** Distribution of scalar fields in a formational flow with the vortex center at (0,0) simulated by FFEM with a 360 \times 181 grid resolution using an orthographic map projection. (top left) Scalar fields at \( t = 0 \), (top right) the exact solution at \( t = 3 \), (bottom left) the numerical result at \( t = 3 \), and (bottom right) the difference between the exact solution and the simulation. The contour interval is 0.05 \((10^{-2})\) for the color-shaded contour maps (and the difference field), and positive (negative) values are in solid (dashed) lines. The minimum value of the solid lines in the color-shaded maps is 1.
the characteristic width of the scalar fields, and $\omega'$ is the angular velocity. Velocities in spherical coordinates of the rotated deformational flow can be written as

$$
\begin{align*}
    u &= \omega' \left[ \sin \theta_0 \cos \theta - \cos \theta_0 \cos (\lambda - \lambda_0) \sin \theta \right], \\
    v &= \omega' \cos \theta_0 \sin (\lambda - \lambda_0),
\end{align*}
$$

where ($\lambda_0$, $\theta_0$) is the North Pole of the ($\lambda'$, $\theta'$) coordinate system. The vortex center is located either on the equator or near the poles [i.e., ($\lambda_0$, $\theta_0$) = (0, 0) or (0, $\pi/2 - 0.05$)].

Figure 12 shows the results for $t = 3$ for the vortex centered at ($\lambda_0$, $\theta_0$) = (0, 0) in the case of the 360 $\times$ 181 grid with the same time step size as the cosine bell advection. The scalar field at $t = 3$ is very close to the theoretical field, having produced a maximum difference that is as small as $3.6 \times 10^{-5}$. Errors are distributed mainly near the vortex center with alternating signs. Results of the same experiment as in Fig. 12, but with the vortex centered at (0, $\pi/2 - 0.05$), are illustrated in Fig. 13, where the polar stereographic map projection is...
used for the meridional domain poleward of 30°N. Although it is not easy to quantitatively compare because of different map projections, the magnitude and distribution of the errors look almost the same as the case of the vortex located on the equator. As is the case of the cosine bell advection, no significant problems for the case of the vortex located over the poles are apparent in the difference fields. Figure 14 presents the logarithmic norm errors at $t = 3$ of the deformational flow for five resolutions. It should be pointed out that the norm errors share almost the same values for both cases, and again do not show any problem at the poles. In Fig. 14, the error convergence rate for the deformational flow is almost exactly of fourth order regardless of the position of the vortex center. In terms of the $L_2$ norm error, the convergence rate in the present study is better than the result of the discontinuous DG method (Nair et al. 2005), which was found to be slightly lower than of second order. As for the $L_1$ error, however, the FFEM shows a slightly lower performance than the DG method wherein the error lies marginally above fourth order. Though not having the exponential convergence rate as in Flyer and Wright (2007), almost consistent behaviors shown by a straight line across zonal grid resolutions up to 960 that are much higher than those in the literatures, demonstrate a desirable performance of the FFEM.

4. Summary and conclusions

The FFEM with linear elements on a sphere was presented in this study. The basis functions were given as a function of the latitude of the gradient operator, while given as a function for the sine of the latitude for the Laplacian-type operators due to the metric terms that can be expressed as second-order polynomials. Pole singularity that is a result of the metric terms coupled with the derivatives was eliminated by applying L'Hôpital's rule to the differential equations that are indeterminate (i.e., zero divided by zero) at the poles. Boundary conditions at the poles were specified in different ways for the scalar, vector, and flux variables to obtain better accuracy of the differentiation. Variable transforms, which allow the zonal Fourier coefficients to be represented as polynomials of the sine of the latitude, were carried out for the elliptic equations of zonal wavenumbers other than zero. Since the elliptic equations can be discretized into three diagonal matrix equations for each zonal wavenumber, the operation count necessary for the inversion is only of $O(N^2)$ with a small factor, where $N$ is the number of meridional grids. Based on this, it was possible to build a computationally efficient high-order filter, which can be used as a hyperviscosity and a global-domain smoothing operator as well. The overall operation count of the FFEM is of $O(N^2 \log_2 N)$ due to the availability of fast Fourier transforms in the zonal direction and the tridiagonal systems associated with the finite-element discretization in the meridional direction.

For the gradient (or the first-order derivative) of a square-integrable scalar variable, the FFEM produced the theoretical convergence rate accuracy (i.e., the fourth order). Inversion of the elliptic equations as well as the high-order filters, consisting of the Laplacian operator, was shown to provide an almost second-order convergence rate accuracy. Application to the solid-body rotation of the cosine or Gaussian bell and the deformational flow revealed that the FFEM is stable and accurate for time integration. The norm error convergence rate was slightly above second order for the cosine bell, whereas it was shown to be fourth order for the cases of Gaussian bell advection and deformational...
flow. In addition to the high convergence rate, one more important aspect may be the consistency of the convergence rate across the whole range of resolutions used in this study. Without the high-order filtering, the time stepping procedure was found unstable even in the case of the solid-body advection equation. The fact that the high-order filtering in the time stepping procedure did not deteriorate the convergence rate is attributed to the sensitivity of the accuracy of the high-order filter. The accuracy of the elliptic equations significantly increases with decreased filter coefficients, and the filter coefficient is usually given as a very small value only to damp down small-scale numerical noise. The high-order filter has an another important role that of maintaining an isotropic resolution over the global domain, which as a result eliminates the severe CFL condition near the poles. The time step size can be set large enough to keep the CFL number, defined in terms of the largest grid size on the global domain (i.e., on the equator), around unity.

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