Response Functions for Arbitrary Weight Functions and Data Distributions. Part II: Response Function Derivation and Verification

MARK A. ASKELSON
School of Meteorology, University of Oklahoma, and Cooperative Institute for Mesoscale Meteorological Studies, and NOAA/National Severe Storms Laboratory, Norman, Oklahoma

PATRICIA M. PAULEY
Naval Research Laboratory, Monterey, California

JERRY M. STRAKA
School of Meteorology, University of Oklahoma, and Cooperative Institute for Mesoscale Meteorological Studies, Norman, Oklahoma

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ABSTRACT
Distance-dependent weighted averaging (DDWA) is a process that is fundamental to most of the objective analysis schemes that are used in meteorology. Despite its ubiquity, aspects of its effects are still poorly understood. This is especially true for the most typical situation of observations that are discrete, bounded, and irregularly distributed.

To facilitate understanding of the effects of DDWA schemes, a framework that enables the determination of response functions for arbitrary weight functions and data distributions is developed. An essential element of this approach is the equivalent analysis, which is a hypothetical analysis that is produced by using, throughout the analysis domain, the same weight function and data distribution that apply at the point where the response function is desired. This artifice enables the derivation of the response function by way of the convolution theorem. Although this approach requires a bit more effort than an alternative one, the reward is additional insight into the impacts of DDWA analyses.

An important insight gained through this approach is the exact nature of the DDWA response function. For DDWA schemes the response function is the complex conjugate of the normalized Fourier transform of the effective weight function. In facilitating this result, this approach affords a better understanding of which elements (weight functions, data distributions, normalization factors, etc.) affect response functions and how they interact to do so.

Tests of the response function for continuous, bounded data and discrete, irregularly distributed data verify the validity of the response functions obtained herein. They also reinforce previous findings regarding the dependence of response functions on analysis location and the impacts of data boundaries and irregular data spacing.

Interpretation of the response function in terms of amplitude and phase modulations is illustrated using examples. Inclusion of phase shift information is important in the evaluation of DDWA schemes when they are applied to situations that may produce significant phase shifts. These situations include those where data boundaries influence the analysis value and where data are irregularly distributed. By illustrating the attendant movement, or shift, of data, phase shift information also provides an elegant interpretation of extrapolation.

1. Introduction
Distance-dependent weighted averaging (DDWA) can be viewed as a fundamental process in most of the objective analysis techniques that are commonly employed in meteorology (Thiébaux and Pedder 1987, 5–6; Daley 1991, 30–31). In one-pass schemes that use prescribed distance-dependent weight functions (e.g.,
Barnes 1964), analysis values are produced directly through DDWA. Furthermore, multiple-pass schemes using prescribed distance-dependent weight functions (successive-correction schemes) can be rewritten as DDWA (Caracena 1987; Doswell and Caracena 1988). Similarly, schemes that employ least squares function fitting also obtain analysis values through a process that is equivalent to DDWA (Thiébaux and Pedder 1987, 22–23; Daley 1991, p. 49). Statistical objective analysis schemes utilize error correlations to construct a DDWA scheme that minimizes analysis-error variance. Variational schemes, while possibly not generally expressible in terms of DDWA, often utilize techniques that are equivalent to DDWA to facilitate solution (e.g., Testud and Chong 1983).

Given the importance of DDWA to the spatial objective analysis techniques used in meteorology, it is prudent to understand the effects DDWA has upon the data. These effects can be expressed through the response function, which is discussed in detail in Askelson et al. (2005, hereinafter Part I). The response function is a generally complex-valued function that provides information concerning amplitude and phase changes undergone during analysis.

The response function for DDWA analyses of continuous, infinite data has been understood for some time (e.g., Barnes 1964). Observations, however, are rarely continuous or infinite; observations are typically discrete, bounded, and irregularly distributed. Of these three characteristics, the discrete nature of observations is the most straightforward to address in terms of the response function. Numerous texts (e.g., Hamming 1998) and journal articles (e.g., Jones 1972) examine the situation where the observations are both regularly distributed and collocated with the analysis points. A common approach used to determine the response function in this situation is the eigenfunction approach (e.g., Weaver 1983, 259–260). The efficacy of this approach is diminished, however, when either the analysis points are not collocated with the observations or the observations are irregularly distributed.

Pauley and Wu (1990) considered a special case of noncollocated analysis and observation points. They determined the response function for the situation where analysis points are located midway between observations that are discrete, regularly distributed, and infinite in number. The case of irregularly distributed observations was beyond the scope of their investigation.

For irregularly distributed observations, the response function can be viewed from two standpoints. The first standpoint is a domainwide response function, which characterizes the domainwide spectral effects of an analysis. Domainwide response functions for specific data point distributions have been investigated (e.g., Yang and Shapiro 1973; Buzzi et al. 1991), as have domainwide response functions for random data point distributions (Stephens and Polan 1971).

The second standpoint is the local response function, which is the subject of this study. As discussed by Thiébaux and Pedder (1987, p. 105) and Buzzi et al. (1991), the response function for DDWA analyses of discrete, irregularly distributed data depends upon both frequency and location. Others (e.g., Jones 1972; Yang and Shapiro 1973; Schlax and Chelton 1992) have derived the local response function for DDWA analyses of discrete, irregularly distributed data by using what is termed here the back-substitution approach. In this approach a spectral representation of the observation field is first substituted into an expression for a DDWA analysis. Then, the result of this operation is manipulated to obtain the response function. Herein, an alternative approach, labeled the convolution-theorem approach and based upon the approach outlined by Caracena et al. (1984), is utilized to obtain the response function for DDWA analyses of discrete, irregularly distributed data.

The purpose of this paper is to outline a method for determining, using the convolution-theorem approach, the local response function for DDWA analyses of arbitrarily distributed data using arbitrary weight functions. In doing so, the response function for DDWA analyses of discrete, irregularly distributed data is derived. Although this response function is the same as that determined previously (e.g., Jones 1972; Yang and Shapiro 1973; Schlax and Chelton 1992), the method outlined herein is enlightening because of the additional insights it provides. To provide a logical progression to the final result and to clarify issues pertaining to a previous result obtained by Pauley (1990, henceforth P90), the steps taken to obtain the response function for discrete, irregularly distributed data are retraced in the chronological order in which they were discovered. Consequently, the response function for continuous, bounded data is first derived and tested in section 2a and then the response function for discrete, irregularly distributed data is derived and tested in section 2b. The results are subsequently discussed in section 3, with conclusions provided in section 4.

2. Response function derivations and evaluations

a. Continuous, bounded data

Whereas the determination of response functions for DDWA schemes applied to data away from data
boundaries are relatively well understood (e.g., Pauley and Wu 1990), less is known concerning response functions for DDWA schemes applied near data boundaries. Even though this topic has been the subject of past investigations by P90 and Achtemeier (1986), problems with these earlier papers motivate further examination. Herein, the one-dimensional problem is considered for simplicity. Results of this analysis are presumed to generalize to multiple dimensions.

1) THE PROBLEM

It is shown here that the direct application of the convolution-theorem approach to the DDWA analysis equation for continuous, bounded data does not lead to an explicit expression for the response function, where “explicit expression” means here an expression that can be evaluated without applying approximations. Furthermore, the underlying problem that prevents the attainment of an explicit expression for the response function in this manner is illustrated by examining the approach of P90, thus setting up the solution to this problem (the attainment of the response function for continuous, bounded data using the convolution-theorem approach), which is provided in the next subsection.

For one-dimensional, continuous, bounded data with boundaries at \( x_L \) and \( x_R \), the first-pass, DDWA analysis field \( f_A(x) \) is given by

\[
f_A(x) = \frac{\int_{x=x_L}^{x=x_R} f(x) w(x - x) \, dx}{\int_{x=x_L}^{x=x_R} w(x - x) \, dx} = \frac{\int_{x=x_L}^{x=x_R} f(x) p(x) w(x - x) \, dx}{n(x)},
\]

where \( f(x_o) \) denotes the observations, \( w(x_o - x) \) is the weight function, \( n(x) = \int_{x=x_L}^{x=x_R} w(x_o - x) \, dx_o \) is the normalization factor, \( p(x) \) is the pulse function given by

\[
p(x_o) = \begin{cases} 
1 & x_L \leq x_o \leq x_R \\
0 & \text{otherwise} 
\end{cases}
\]

\( x_o \) depicts observation locations, and \( x \) depicts analysis locations. With the substitution \( x' = x_o - x \), (1) becomes

\[
f_A(x) = \frac{\int_{x'=-\infty}^{x'=x_R-x} f(x + x') p(x + x') w(x') \, dx'}{n(x)}.
\]

As indicated in (3), both \( n(x) \) and \( p(x + x') \) are functions of \( x \) (position in the analysis domain).

Examples of the dependence of \( n(x) \) on \( x \) [for the Gaussian weight function given by \( w(x) = \exp(-x^2/\kappa_o) \), with \( \kappa_o = 3 \)] are provided in Fig. 1 for a continuous, bounded case (Fig. 1a); three discrete, regularly distributed cases (Fig. 1b); and a discrete, irregularly distributed case (Fig. 1c). The dependence of \( n(x) \) upon both analysis type (continuous versus discrete) and location \( x \), as illustrated in Fig. 1, requires some explanation.

In the continuous case (Fig. 1a), \( n(x) \) is nearly constant in the center of the observational domain at a value that agrees with the infinite domain value if the boundaries are sufficiently removed from each other, decreases to half the infinite domain value at the boundaries, and shrinks to near zero outside of these boundaries. This pattern arises because, as the analysis point (and weight-function peak) approaches and passes a boundary, the area under the portion of the weight function that resides within the observational domain decreases.

The same pattern is also present in the discrete, regularly distributed cases (Fig. 1b). In these cases, however, \( n(x) \) increases with decreasing data spacing. This can be understood by considering that a discretization of the integrals in (1) is

\[
f_A(x) = \frac{\sum_{i=1}^{N} f(x_i) w(x_i - x) \Delta x_i}{\sum_{i=1}^{N} w(x_i - x) \Delta x_i},
\]

where \( f(x_i) \) denotes the \( i \)th observation, \( N \) is the total number of observations, and \( \Delta x_i \) extends between the midpoints of the \( i - 1 \)st and \( i \)th and the \( i \)th and \( i + 1 \)st observation locations. If observations are regularly distributed, then \( \Delta x_i = \Delta x \), which can be canceled from the numerator and denominator to produce

\[
f_A(x) = \frac{\sum_{i=1}^{N} f(x_i) w(x_i - x)}{\sum_{i=1}^{N} w(x_i - x)}.
\]

This equation holds for all DDWA schemes applied to one-dimensional, discrete data, regardless of whether the data are regularly or irregularly distributed. For regularly distributed data, the denominator of (4) multiplied by \( \Delta x \) provides an approximation of the denominator in (1). Since the denominator of (4) is plotted in Fig. 1b while the denominator of (1) is plotted in Fig.
1a, the values in Fig. 1b are approximately those in Fig. 1a divided by $\Delta x$ and thus the values in Fig. 1b increase with decreasing data spacing.

The final aspect of Fig. 1 that requires explanation is the magnitudes of $n(x)$ for the irregularly distributed case (Fig. 1c). In that case, to get (4) from the discretization of (1), one must pull $\Delta x$ out of both the numerator and denominator. This means that a $\Delta x$ value is substituted for all of the $\Delta x_i$ values, and thus that one obtains an improper discretization of (1). Because the denominator of (4) is plotted in Fig. 1c, values in that figure correspond to pulling $\Delta x = 1$ out of the above
discretization. When the data spacing is less (greater) than one, discretization intervals overlap (are separated) and overestimation (underestimation) results.

In her Eq. (P2a), P90 expresses the first-pass, Barnes, DDWA analysis field for one-dimensional, continuous, bounded data as

\[ g_0(x) = \frac{\int_{x_1}^{x_2} \exp \left( -\frac{x'^2}{\lambda_0^2} \right) f(x + x') \, dx'}{\int_{x_1}^{x_2} \exp \left( -\frac{x'^2}{\lambda_0^2} \right) \, dx'}, \quad (P2a) \]

where \( g_0(x) \) is the analysis field, \( \exp(-x'^2/\lambda_0^2) = w(x', \lambda_0) \) is the (Barnes) weight function, and \( f(x + x') \) denotes the observations. [Equations from P90 are labeled with a P followed by the corresponding equation number in P90]. In order for (P2a) to be consistent with (3), \( x_1 \) must equal \( x_L - x \) and \( x_2 \) must equal \( x_R - x \).

P90 derived the response function for the Barnes scheme applied to one-dimensional, continuous, bounded data by taking the Fourier transform of (P2b). Equation (P2b), which resulted from introducing a pulse function into (P2a), is

\[ g_0(x) = \frac{\int_{-\infty}^{+\infty} \exp \left( -\frac{x'^2}{\lambda_0^2} \right) f(x + x') p(x') \, dx'}{\int_{-\infty}^{+\infty} \exp \left( -\frac{x'^2}{\lambda_0^2} \right) p(x') \, dx'}. \quad (P2b) \]

The pulse function in (P2b), however, was set up incorrectly. Data boundaries should be expressed in terms of observation space—\( x \), as used herein or \( X \) as used in Pauley and Wu (1990). This means that the pulse function \( p(x') \) in (P2b) should be written as \( p(x + x') \), as in (3). The numerator on the rhs of (3) [and (1)], therefore, is the cross correlation of \( w(x) \) and the function \( f(x)p(x') \), denoted as \( w(x) \star f(x)p(x') \). As shown by Papoulis (1962, p. 244), \( w(x) \star f(x)p(x) = w(-x) \star f(x)p(x) \), where \( \star \) denotes convolution. This result can also be obtained directly by recognizing that the numerators of (1) and (3) are the convolution of \( w(x) \) and \( f(x)p(x) \). Thus, a concise expression of (1) and (3) is

\[ f_A(x) = \frac{w(x) \star f(x)p(x)}{n(x)} = \frac{w(-x) \star f(x)p(x)}{n(x)}. \quad (5) \]

Attempting to obtain the response function by taking the Fourier transform of (5), which is the corrected version of (P2b), results in difficulties. Taking the Fourier transform of (5) produces

\[ F_A(v) = \int_{-\infty}^{+\infty} M(\xi) N(v - \xi) \, d\xi, \quad (6) \]

where \( M(v) = \text{FT}[w(-x) \star f(x)p(x)] \), \( N(v) = \text{FT}[1/n(x)] \), \( \xi \) denotes frequency dependence, and the product theorem (Weaver 1983, p. 73) has been applied. Using the convolution (Weaver 1983, p. 72), similarity \( ^3 \) (Bracewell 2000, p. 108), and product theorems, \( M(v) \) can be expressed as \( M(v) = W(-v) \int_{-\infty}^{+\infty} F(\psi)P(v - \psi) \, d\psi \), where \( \psi \) denotes frequency dependence. Substituting this result into (6) results in

\[ F_A(v) = \int_{-\infty}^{+\infty} \left( \int_{-\infty}^{+\infty} F(\psi)P(\xi - \psi) \, d\psi \right) W(-\xi) \]

\[ \times N(v - \xi) \, d\xi. \quad (7) \]

An expression for the response function can be obtained by dividing (7) by \( F(v) \). This does not, however, provide an explicit expression (i.e., one that can be evaluated without applying approximations) for the response function because \( F(v) \) is bound within convolution integrals and thus must be evaluated to compute this response function. Because the data are bounded, however, \( F(v) \) must be approximated and, thus, this approach does not provide an explicit expression for the response function. Moreover, one cannot extract \( F(v) \) from the convolution with \( P(v) \) by rearranging (5) prior to applying the Fourier transform. The numerator on the rhs of (5) is the cross correlation of \( w(x) \) and \( f(x)p(x) \). No matter how (5) is expressed, upon application of the Fourier transform, this cross correlation results in a convolution between \( F(v) \) and \( P(v) \).

P90, however, did obtain an explicit expression for the response function through a process similar to that outlined above, albeit as a consequence of the incorrect specification of the pulse function in (P2b). As stated

\[ ^3 \text{If } f(x) \text{ and } g(x) \text{ have Fourier transforms given by } F(\psi) \text{ and } G(\psi), \text{ the product theorem states that the product of } f(x) \text{ and } g(x) \text{ has a Fourier transform given by the convolution of } F(\psi) \text{ and } G(\psi), \text{ the convolution theorem states that the convolution of } f(x) \text{ and } g(x) \text{ has a Fourier transform given by } F(\psi)G(\psi), \text{ and the similarity theorem states that the Fourier transform of } f(ax) \text{ is } (1/|a|)F(\psi/a). \]
above, the pulse function is used to describe the distribution of the observations and thus is a function of observation location, \( x_o \). P90 incorrectly casted the pulse function as a function of \( x' \), the local coordinate system centered on the analysis point \( x \). Because of this error, the pulse function in the normalization factor of (P2b) is not a function of \( x \) as it should be, and so \( n(x) \) passes through the Fourier transform as a constant in P90. Furthermore, the (incorrect) numerator of (P2b) is a cross correlation of \( w(x') \) and \( f(x) \), rather than a cross correlation of \( w(x) \) and \( f(x)p(x) \) as in (5).

The Fourier transform of the latter binds \( F(u) \) within integrals and so does not lead to an explicit expression for the response function.

Interpretation is facilitated using the concept of a window, which is defined at each analysis point as the effective “view” of the data. For one-dimensional, continuous, bounded data, the window at an analysis point is the product of the pulse function and the normalized weight function, \( p(x + x')[w(x')/n(x)] \), and extends from \( x_L - x \) to \( x_R - x \). Examples of unnormalized [i.e., not divided by \( n(x) \)] windows for this situation are provided in Fig. 2a. [It should be noted that this analysis holds for any \( w(x') \). The \( w(x') \) in Fig. 2 is simply an example, which has been made asymmetric in order to avoid the implication of special characteristics.] As is obvious from Fig. 2a, the analyses at points A and B “see” the data through different windows. In fact, the window is generally different for each analysis point. If, on the other hand, the analysis somehow would view the data through the same window at each analysis point (Fig. 2b), \( n(x) \) would be constant and the pulse function would only depend upon \( x' \), resulting in the numerator of the rhs of (5).

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4 In P90 the normalization factor was erroneously treated as a constant, with a value appropriate for the particular analysis location of interest (as explained further in this, and in the following, sections). P90’s treatment did correct the error in Achtemeier (1986), in which (in effect) the pulse function was not included in the normalization factor and thus \( n(x) \) was constant, regardless of location. Even so, P90’s treatment is inconsistent with (5) because of the previously described error that was made in defining the pulse function.

5 P90 treated the numerator on the rhs of (P2b) as a convolution of \( w_{df}(x) \) and \( f(x) \), where \( w_{df}(x) = w(x, \lambda_d)p(x) \), when it is actually a cross correlation of \( w_{df}(x) \) and \( f(x) \). The two operations would be equal if \( w_{df}(-x) = w_{df}(x) \). This does not hold, however, for (P2b), and P90 incorrectly treated the pulse function as a function of \( x' \), the local coordinate system centered on the analysis point \( x \). Because of this error, the pulse function in the normalization factor of (P2b) is not a function of \( x \) as it should be, and so \( n(x) \) passes through the Fourier transform as a constant in P90. Furthermore, the (incorrect) numerator of (P2b) is a cross correlation of \( w(x') \) and \( f(x) \), rather than a cross correlation of \( w(x) \) and \( f(x)p(x) \) as in (5).

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being a cross correlation of \( w(x)p(x) \) and \( f(x) \). Because of the fixed data boundaries at \( x_L \) and \( x_R \), however, the scenario pictured in Fig. 2b does not hold and the problem investigated by P90 needs to be reexamined.

2) THE SOLUTION

This quandary can be resolved by defining a hypothetical analysis for which the scenario in Fig. 2b holds—i.e., that is, the same window applies across the entire analysis domain. Owing to its equivalent treatment of data across the analysis domain, such an analysis is called herein an “equivalent analysis.” To produce an equivalent analysis for this situation, consider the actual analysis at some point, like point A in Fig. 2. Further, suppose that observations are available not just within the observational domain (i.e., from \( x_L \) to \( x_R \)) but throughout the equivalent analysis domain \(( -\infty, \infty \)). The equivalent analysis field \( f_{EA}(x, x_{ref} = A) \) is then produced by using, throughout the entire equivalent analysis domain, the same weight and pulse functions (i.e., the same window) that are used in the actual analysis at point A. (The symbol \( x_{ref} \) represents the reference location, in this case point A, for which a response function is desired.) In this imaginary analysis, therefore, each point in the infinite domain “sees” the same relative distribution of observations as the actual analysis “sees” at point A. Because of its equal treatment across an infinite domain, this construction enables the determination of an explicit expression for the response function using the convolution-theorem approach.

Mathematically, an equivalent analysis for continuous, bounded data is expressed as

\[
f_{EA}(x, x_{ref}) = \frac{\int_{x_R-x_{ref} \leq x \leq x_L-x_{ref}} f(x_o)w(x_o - x) \, dx_o}{\int_{x_R-x_{ref} \leq x \leq x_L-x_{ref}} w(x_o - x) \, dx_o}.
\]

(8)

Note that the hypothetical equivalent analysis value at each \( x \) is different for each equivalent analysis (i.e., each reference location \( x_{ref} \)). Because within each equivalent analysis \( x_R, x_L, \) and \( x_{ref} \) are constant, the normalization factor in the denominator of (8) is constant. This is more obvious when (8) is transformed using the substitution \( x' = x_{ref} - x \), which results in

\[
f_{EA}(x, x_{ref}) = \frac{\int_{x'_{ref} \leq x' \leq x'_{ref} - x_L} f(x + x')w(x') \, dx'}{\int_{x'_{ref} \leq x' \leq x'_{ref} - x_L} w(x') \, dx'} = \frac{w(x_{ref})}{n_{EA}(x_{ref})},
\]

where

\[
n_{EA}(x_{ref}) = \int_{x_{ref} - x_L \leq x' \leq x_{ref} - x_R} w(x') \, dx' = \int_{x_{ref} - x_R \leq x' \leq x_{ref} - x_R} p_{EA}(x', x_{ref})w(x') \, dx'.
\]

(10)

is the equivalent analysis normalization factor, and the equivalent analysis pulse function \( p_{EA}(x', x_{ref}) \) is given by

\[
p_{EA}(x', x_{ref}) = \begin{cases} 1 & x_L - x_{ref} \leq x' \leq x_R - x_{ref} \\ 0 & \text{otherwise} \end{cases}.
\]

(11)

Because within each equivalent analysis \( x_{ref} \) is constant, \( p_{EA}(x', x_{ref}) \) depends only upon \( x' \). Consequently, for each equivalent analysis, the numerator and denominator on the rhs of (9) are

\[
\int_{x'_{ref} \leq x' \leq x'_{ref} - x_L} f(x + x')p_{EA}(x', x_{ref})w(x') \, dx' = w(x)p_{EA}(x, x_{ref})\text{FT}(f(x))
\]

(12)

and

\[
n_{EA}(x_{ref}) = \int_{x'_{ref} \leq x' \leq x'_{ref} - x_L} w(x') \, dx' = \int_{x'_{ref} \leq x' \leq x'_{ref} - x_L} \text{FT}[w_{eff}(-x, x_{ref})^\ast f(x)]
\]

(13)

respectively. The “effective” weight function \( w_{eff}(x, x_{ref}) = p_{EA}(x, x_{ref}) \), which is the product of the weight function and the pulse function, embodies the actual weights that are applied to the observations during equivalent analyses [cf. (9)].

Using (12), (9) can be expressed as

\[
f_{EA}(x, x_{ref}) = \frac{w(-x)p_{EA}(-x, x_{ref})^\ast f(x)}{n_{EA}(x_{ref})} = \frac{w_{eff}(-x, x_{ref})^\ast f(x)}{n_{EA}(x_{ref})}.
\]

(14)

This expression illustrates an important difference between the actual and equivalent analyses. In the actual analysis (5), the pulse function is associated with the observation field \( f(x) \); in the equivalent analysis (14), the pulse function is associated with weight function \( w(x) \). The equivalent analysis construct results in the pulse function moving across the convolution symbol.

The Fourier transform of (14) and the application of the convolution theorem produce the response function

\[
R(u, x_{ref}) = \frac{F_{EA}(u, x_{ref})}{F(u)} = \text{FT}[w_{eff}(-x, x_{ref})] = \frac{1}{n_{EA}(x_{ref})},
\]

(15)
where the fact that $n_{EA}(x_{\text{ref}})$ is constant for each $x_{\text{ref}}$ has been used. The term $\text{FT}[w_{\text{eff}}(-x, x_{\text{ref}})]$ can be expressed in a more useful form by using the similarity theorem, the definition of the Fourier transform, and the definition of $p_{EA}(x, x_{\text{ref}})$. From the similarity theorem, if $\text{FT}[w_{\text{eff}}(x, x_{\text{ref}})] = W_{\text{eff}}(v, x_{\text{ref}})$, then $\text{FT}[w_{\text{eff}}(-x, x_{\text{ref}})] = W_{\text{eff}}(-v, x_{\text{ref}})$. Using the definitions of the Fourier transform and of $p_{EA}(x, x_{\text{ref}})$, $W_{\text{eff}}(v, x_{\text{ref}})$ can be expressed as

$$W_{\text{eff}}(v, x_{\text{ref}}) = \int_{x = x_{L} - x_{\text{ref}}}^{x = x_{R} - x_{\text{ref}}} w(x) \cos(2\pi vx) \, dx$$

$$+ j \left[ - \int_{x = x_{L} - x_{\text{ref}}}^{x = x_{R} - x_{\text{ref}}} w(x) \sin(2\pi vx) \, dx \right].$$

(16)

Substituting $-v$ for $v$ in (16) to get $\text{FT}[w_{\text{eff}}(-x, x_{\text{ref}})]$, as dictated by the similarity theorem, and inserting the result into (15) produces

$$\mathcal{R}(u, x_{\text{ref}}) = \int_{x = x_{L} - x_{\text{ref}}}^{x = x_{R} - x_{\text{ref}}} \frac{w(x) \cos(2\pi vx)}{n_{EA}(x_{\text{ref}})} \, dx$$

$$- \int_{x = x_{L} - x_{\text{ref}}}^{x = x_{R} - x_{\text{ref}}} \frac{w(x) \sin(2\pi vx)}{n_{EA}(x_{\text{ref}})} \, dx$$

$$- j \int_{x = x_{L} - x_{\text{ref}}}^{x = x_{R} - x_{\text{ref}}} \frac{w(x)}{n_{EA}(x_{\text{ref}})} \exp(j2\pi vx) \, dx.$$  

(17)

This is the response function for DDWA analyses of one-dimensional, continuous, bounded data. The only assumption concerning the weight function is that the integrals in (8)–(10) and (12)–(17) exist. A well-known (e.g., Caracena et al. 1984; Achtemeier 1986; P90) consequence of data boundaries is indicated in (17); when data are bounded, the response function depends upon the weight function $w(x)$, the frequency $v$, and the location $x_{\text{ref}}$.

It is noted that the application of the similarity theorem in the derivation of (17) resulted in a change in the sign of the imaginary term. Consequently, (17) is the complex conjugate of $\text{FT}[w_{\text{eff}}(x, x_{\text{ref}})]/n_{EA}(x_{\text{ref}}) = W_{\text{eff}}(u, x_{\text{ref}})/n_{EA}(x_{\text{ref}}) = W_{\text{eff}}(u, x_{\text{ref}})$ and is denoted here as $W_{\text{eff}}(u, x_{\text{ref}})$. The important result that the local response function is the complex conjugate of the normalized Fourier transform of the effective weight function is succinctly expressed as

$$\mathcal{R}(u, x_{\text{ref}}) = W_{\text{eff}}^{\ast}(u, x_{\text{ref}}).$$

(18)

Before relating this result to that of P90, it is important to resolve issues regarding the equivalent analysis concept. The first issue is whether the response function (17), which was obtained using the equivalent analysis concept and the convolution-theorem approach, applies to the actual analysis (3). It, in fact, does, as illustrated by the following argument. An equivalent analysis applies the same treatment to the data (sees the data through the same window) at every point in its analysis domain as that applied at the analysis location $x_{\text{ref}}$ in an actual analysis. The response function for a given $x_{\text{ref}}$ obtained using the equivalent analysis concept (17) depends upon the data distribution for this $x_{\text{ref}}$ (in this case the pulse function), weight function, and weight normalization factor. This makes sense for both the equivalent analysis and the actual analysis (at $x_{\text{ref}}$) because these dictate the weights applied to (i.e., what is done to) the input field. Thus, they characterize the actual analysis scheme for the analysis location $x_{\text{ref}}$ and in doing so should dictate the response function at that location. Because the equivalent analysis uses the same data distribution, weight function, and weight normalization factor as used in the actual analysis at the analysis location $x_{\text{ref}}$, it gives the response function for that analysis location. Verification of this argument and, thus, that (17) applies to the actual analysis, is provided in the next section.

The second issue is that a complete equivalent analysis field cannot be computed. (Its value at $x_{\text{ref}}$ can be computed and is equal to the actual analysis value at $x_{\text{ref}}$.) This is true for any real situation because the input field is not known everywhere. However, there is no need to compute a complete equivalent analysis field. What is needed is the concept, which enables the attainment of an explicit expression for the response function through the convolution-theorem approach. Thus, the equivalent analysis is an artifice, but a useful one at that.

At this point it is instructive to relate this result to that of P90. P90 applied the Fourier transform to (P2b) and then utilized the convolution theorem and the pulse function definition for (P2b) to obtain an explicit expression for the response function. It has already been shown [see (7)] that when the correct version of (P2b), namely (5), is used, this methodology does not result in an explicit expression for the response function. How, then, did P90 obtain an explicit expression for the response function? The answer is that the incorrect definition of the pulse function in (P2b) emulates equivalent analyses. To examine the local response function for analysis points located at or near a boundary, P90 utilized a pulse function that fixed the boundaries relative to each analysis point. The difficulty is, in DDWA analyses boundaries relate to observation, rather than analysis, locations. The concept of
an equivalent analysis, as in (P2b), allows the boundaries to be specified relative to analysis locations.

With the incorrect definition of the pulse function, the normalization factor in (P2b) is constant and the pulse function depends only upon \( x' \). In fact, with \( x_1 = x_L - x_{\text{ref}} \) and \( x_2 = x_R - x_{\text{ref}} \), P90’s response function (P4), which in slightly modified form is given by

\[
\mathcal{R}(v, x_{\text{ref}}) = \frac{\int_{x_1}^{x_2} w(x) \cos(2\pi v x) \, dx}{\int_{x_1}^{x_2} w(x) \, dx} + j \frac{\int_{x_1}^{x_2} w(x) \sin(2\pi v x) \, dx}{\int_{x_1}^{x_2} w(x) \, dx},
\]

is equivalent to (17). This equivalence is somewhat puzzling, however, since P90 incorrectly regarded (P2b) as a convolution rather than a cross correlation, which should lead to a negative sign in the imaginary term in (P4). In any case, with the pulse function problems in (P2b), if (P2b) were correctly treated as a cross correlation and the same (direct) Fourier transform were used, (P4) would be equivalent to (17). The incorrect treatment of (P2b) as a convolution leads to a sign error in the imaginary term of the response function. The consequence is a difference in the sign of the phase shift undergone during analysis, as illustrated presently.

3) Verification of the Response Function for Continuous, Bounded Data

The veracity of (17) is illustrated herein. First, though, from the definition of the response function (see Part I) and with \( \mathcal{R}(v, x_{\text{ref}}) = W_N^\Psi(v, x_{\text{ref}}) \), the Fourier transform of the equivalent analysis field can be expressed in polar form as

\[
F_{\text{EA}}(v, x_{\text{ref}}) = \left| F(v) \right| W_N^\Psi(v, x_{\text{ref}}) \exp\left( j(\varphi_{\text{FD}} + \varphi_{W_N^\Psi(v_{\text{ref}})}) \right),
\]

(19)

Using (8) from Part I, \( f_{\text{EA}}(v, x_{\text{ref}}) \) can then be written as

\[
f_{\text{EA}}(v, x_{\text{ref}}) = \left| F(v) \right| W_N^\Psi(v, x_{\text{ref}}) \left[ 1 + \delta(v) \right] \cos[2\pi v x_{\text{ref}} + \varphi_{\text{FD}} - \varphi_{W_N^\Psi(v_{\text{ref}})}] \, dv.
\]

(20)

where \( \delta(v) = 0 \) except at \( v = 0 \), where it is 1, and the fact that \( \varphi_{W_N^\Psi(v_{\text{ref}})} = -\varphi_{W_N^\Psi(v_{\text{ref}})} \) has been utilized.

The response function (17) is now verified by using it to predict analysis values and by comparing these predicted analysis values to actual analysis values obtained using (1) or (3). This is accomplished using “observation” fields of the same form as used in Part I,

\[
f(x_v) = A_i \cos(2\pi v x_v + \varphi_i),
\]

(21)

where \( A_i > 0 \) is the amplitude, \( v_i = 0 \) is the frequency, and \( \varphi_i \) is the phase of the input field. In the following test \( v_i > 0 \), and thus from (19) and (20) of Part I \( |F(v)| = (A_i/2) \delta(v - v_i) \) and \( \varphi_{W_N^\Psi} \) is equal to \( \varphi_i \) if \( v = v_i \) and indeterminate otherwise. Insertion of these relations into (20) produces

\[
f_{\text{EA}}(x, x_{\text{ref}}) = A|W_N^\Psi(v, x_{\text{ref}})| \cos[2\pi v x + \varphi_i - \varphi_{W_N^\Psi(v, x_{\text{ref}})}],
\]

(22)

where the sifting property of the Dirac distribution, \( f_v = -\delta(x - a)f(x) \, dx = f(a) \) (Bracewell 2000, p. 79), has been exploited, and the fact that \( \delta(v_i) = 0 \) if \( v_i \neq 0 \) has been utilized. Because the response in (17) and (18) is valid only at \( x = x_{\text{ref}} \), \( f_{\text{EA}}(x, x_{\text{ref}}) \) values from (26) are relevant to the actual analysis values only when \( x = x_{\text{ref}} \) in (22).

Actual and predicted analysis fields for \( x_L = 0, x_R = 10, A_i = 1, v_i = 1/3, \varphi_i = 0, \) and \( w(x) = \exp(-x^2/\kappa_x^2) \), with \( \kappa_x = 3 \), are shown in Fig. 3a. This figure confirms the veracity of (17) since it shows that the predicted analysis values (plus-sign symbols), obtained using (17), match the actual analysis values (solid line).

Predicted analysis values obtained using a response function derived under the incorrect assumption that DDWA entails convolution, instead of cross correlation, are illustrated by the thin-dashed line in Fig. 3a. As this line indicates, this incorrect response function, given by (P4) with a negative sign preceding the imaginary component, does not correctly predict analysis values. The difference between (17) and this incorrect response function can be understood by noting that a repetition of the above analysis for the incorrect response function results in a positive, rather than negative, sign in front of \( \varphi_{W_N^\Psi(v, x_{\text{ref}})} \) in (20) and (22). The response function that results from incorrectly treating DDWA as a convolution correctly specifies the amplitude modulation and the phase shift magnitude but incorrectly specifies the sign of the phase shift. It results in an incorrect response function that is the complex conjugate of the correct response function (18).

The amplitude \( |W_N^\Psi(v, x_{\text{ref}})| \) (solid line, left axis), phase \( \varphi_{W_N^\Psi(v, x_{\text{ref}})} \) (thin-dashed line, right axis), and ideal-amplitude (dotted line, left axis) modulations for the test illustrated in Fig. 3a are shown in Fig. 3b. The ideal amplitude modulation is the response function for infinite, continuous data and is thus also referred to as the
ideal response function. [For the weight function used in these tests, the ideal response function is \( R(v) = \exp[-\kappa_d(v)^2] \) (Barnes 1964).] As Fig. 3b indicates, the response is nearly ideal in the central portions of the observational domain. For \( x \leq -3 \) and \( x \geq 7 \), however, the boundaries affect the analysis and force non-ideal results.

The phase shifts experienced near and outside of the observational domain boundaries are particularly illuminating. Since positive values of \( \varphi_{W_x(v,\kappa_{\text{ref}})} \) result in shifting the input wave to the right while negative values of \( \varphi_{W_x(v,\kappa_{\text{ref}})} \) result in leftward shifts [see (22)], it is apparent that near the boundaries the analysis obtains information from toward the center of the observational domain. This also holds true outside of the observational domain up to a certain distance from the domain boundaries. At that point the phase shift changes from \( \pm 180^\circ \) to \( \mp 180^\circ \), with the phase shift values oscillating with increasing distance from the domain boundaries. These oscillations result from the constraint that \( -180^\circ \leq \varphi_{W_x(v,\kappa_{\text{ref}})} \leq 180^\circ \), which was imposed during the calculations. This constraint does not necessarily produce the correct value of \( \varphi_{W_x(v,\kappa_{\text{ref}})} \). Strictly speaking, any \( \varphi_{W_x(v,\kappa_{\text{ref}})} = \varphi + n(360^\circ) \), where \( -180^\circ \leq \varphi \leq 180^\circ \) and \( n \) is an integer, could be considered valid at any point since \( 360^\circ \)-increment changes in \( \varphi_{W_x(v,\kappa_{\text{ref}})} \) do not alter the results of (22). Within the observational domain it appears as if the restricted \( \varphi_{W_x(v,\kappa_{\text{ref}})} \) values are correct since the analysis does not need to look very far away to obtain information. This is not true, however, for analysis points outside of the observational domain. Consider the situation at \( x = 12 \), where (the restricted) \( \varphi_{W_x(v,\kappa_{\text{ref}})} = -68.69^\circ \). Since the nearest information is two units away, which corresponds to \( 240^\circ \) for this wave (wavelength of \( \lambda = 3 \)), it seems that the correct phase shift value at this point is \( +291.31^\circ \). Note that this value implies a physically plausible rightward migration of information for this point. From reasoning that is supported by the phase shift values near the data boundaries, consequently, it appears that the correct phase shift values for points outside of the observational domain result from incrementing the restricted phase shift values by an appropriate multiple of \( 360^\circ \). To the right (left) of the observational domain, the corrected phase shift values increase (decrease) monotonically with increasing distance from the rightmost (leftmost) data boundary.

The interpretation of analysis consequences in terms of the amplitudes and phases of the basis functions of the Fourier transform, therefore, facilitates understanding. For the analysis illustrated in Fig. 3, for example, extrapolation is achieved by shifting information con-
tained within the observational domain to points outside of the observational domain. This statement, moreover, appears to be fundamental to all extrapolations. A further consequence of the extrapolation illustrated in Fig. 3 is decreasing filtering with increasing extrapolation distance. This is indicated in Fig. 3b by the increasing amplitude modulation with increasing distance from the observational domain. It is not known if this is a fundamental attribute of all extrapolation schemes.

b. Discrete, irregularly distributed data

Herein, the response function for discrete, irregularly distributed data is obtained using the convolution-theorem approach and the result is tested. For simplicity, the problem considered is again one-dimensional. Results are expected to extend to multiple dimensions.

1) Derivation

In this situation, the analysis field \( f_A(x) \) is obtained using (4). As in Caracena et al. (1984) and Pauley and Wu (1990), observation locations can be described using a comb distribution (rather than a pulse function). Comb distributions are typically defined as infinite trains of equally spaced Dirac distributions (e.g., Weaver 1983, p. 131). In this case, however, the irregular data spacing requires an *irregular* comb distribution in the form

\[
\text{iComb}(x_o) = \sum_{i=1}^{N} \delta(x_o - x_{oi}).
\]  

(23)

The sifting property of the Dirac distribution, together with (23), allows (4) to be expressed as

\[
f_A(x) = \frac{\int_{x_{o_1}}^{x_{o_N}} f(x_o) \text{iComb}(x_o) w(x_o - x) \ dx_o}{n(x)} = \frac{\int_{x'=-\infty}^{x'=\infty} f(x + x') \text{iComb}(x + x') w(x') \ dx'}{n(x)}
\]

(24)

where the normalization factor of \( n(x) \) is given by \( n(x) = \Sigma_{i=1}^{N} w(x_{oi} - x) \) and the rightmost form results from the substitution \( x' = x_o - x \). As in the bounded, continuous case, both \( n(x) \) and \( \text{iComb}(x + x') \) depend upon \( x \).

Since the numerator of (24) is \( w(x) \ast f(x) \text{iComb}(x) = w(-x) \ast f(x) \text{iComb}(x) \), (24) can be expressed as

\[
f_A(x) = \frac{w(-x) \ast f(x) \text{iComb}(x)}{n(x)}.
\]

(25)

Following the same steps used to obtain (7), the attempt to determine an explicit expression for the response function by taking the Fourier transform of (25) results in

\[
F_A(v) = \int_{\xi=-\infty}^{\xi=\infty} \left[ \int_{\psi=-\infty}^{\psi=\infty} F(\psi) \text{ICOMB}(\xi - \psi) \ d\psi \right] 
\]

\[
\times W(-\xi)N_f(v - \xi) \ d\xi,
\]

(26)

where \( N_f(v) = \text{FT}[1/n(x)] \) and \( \xi \) and \( \psi \) denote frequency dependence. As with (7), dividing this result by \( F(v) \) does not provide an explicit expression for the response function because \( F(v) \) would have to be estimated owing to the data distribution. Moreover, rearranging (25) prior to applying the Fourier transform does not help produce an explicit expression for the response function.

Again, the artifice of an equivalent analysis allows the determination of the local response function. Consider the hypothetical situation in which the observation field is known everywhere and an equivalent analysis field \( f_{EA}(x, \ x_{ref}) \) is produced using, for all points in the equivalent-analysis domain \((-\infty, \ infty)\), the same relative distribution of observations and weights that is used to produce an actual analysis value at the point \( x_{ref} \). The nonweight component of the window through which the observations are “seen” is, in this case, a sliding irregular comb, as opposed to the sliding pulse function employed earlier in the equivalent analyses of continuous, bounded data. This equivalent analysis field is given by

\[
f_{EA}(x, \ x_{ref})
\]

\[
= \int_{x'=-\infty}^{x'=\infty} f(x + x') \text{iComb}_{EA}(x', x_{ref}) w(x') \ dx'
\]

\[
\frac{n_{EA}(x_{ref})}{w(x_{ref})},
\]

(27)

where

\[
n_{EA}(x_{ref}) = \sum_{i=1}^{N} w(x_{oi} - x_{ref})
\]

\[
= \int_{x'=\infty}^{x'=\infty} \text{iComb}_{EA}(x', x_{ref}) w(x') \ dx'
\]

(28)

is the equivalent analysis normalization factor, and the equivalent analysis irregular comb distribution is

\[
\text{iComb}_{EA}(x', x_{ref}) = \sum_{i=1}^{N} \delta[x' - (x_{oi} - x_{ref})].
\]

(29)

Within each equivalent analysis \( x_{ref} \) and \( n_{EA}(x_{ref}) \) are constant and \( \text{iComb}_{EA}(x', x_{ref}) \) depends only upon \( x' \). Because \( f_{x'=\infty} f(x + x') \text{iComb}_{EA}(x', x_{ref}) w(x') \ dx' = \)

\[
= \int_{x'=\infty}^{x'=\infty} f(x + x') \text{iComb}_{EA}(x', x_{ref}) w(x') \ dx'
\]
\[ w(x) \mid \text{comb}_{n}(x, x_{\text{ref}}) \ast f(x) = w(-x) \mid \text{comb}_{n}(-x, x_{\text{ref}}) \ast f(x), \] (27) can be expressed as

\[ f_{\text{EA}}(x, x_{\text{ref}}) = \frac{w(-x) \mid \text{comb}_{n}(-x, x_{\text{ref}}) \ast f(x)}{n_{\text{EA}}(x_{\text{ref}})}, \]

\[ = \frac{w_{\text{eff}}(-x, x_{\text{ref}}) \ast f(x)}{n_{\text{EA}}(x_{\text{ref}})}, \]

where \( n_{\text{EA}}(x_{\text{ref}}) = \int_{x_{\text{ref}}}^{x_{\text{ref}}} w_{\text{eff}}(x', x_{\text{ref}}) \, dx' \), and the effective weight function \( w_{\text{eff}}(x, x_{\text{ref}}) = w(x) \mid \text{comb}_{n}(x, x_{\text{ref}}) \) embodies not only the structure of the weight function but also the distribution of the observations about the point \( x_{\text{ref}} \).

As in the continuous, bounded case, (30) illustrates an important difference between the actual and equivalent analyses. In the actual analysis (25), the irregular comb distribution is associated with the observation field \( f(x) \); in the equivalent analysis (30), the irregular comb distribution is associated with the weight function \( w(x) \). The equivalent analysis construct, in a manner similar to that in the continuous, bounded case, results in the irregular comb distribution moving across the convolution symbol.

Equation (30) is the discrete, irregularly distributed counterpart to (14), with the irregular comb distribution taking the place of the pulse function. Repeating the steps that follow (14) and utilizing the definition of \( \text{comb}_{n}(x, x_{\text{ref}}) \) results in the response function

\[ \Re(u, x_{\text{ref}}) = \frac{\sum_{i=1}^{N} w(x_{\text{oi}} - x_{\text{ref}}) \cos[2\pi \nu(x_{\text{oi}} - x_{\text{ref}})]}{n_{\text{EA}}(x_{\text{ref}})} \]

\[ - \frac{\sum_{i=1}^{N} w(x_{\text{oi}} - x_{\text{ref}}) \sin[2\pi \nu(x_{\text{oi}} - x_{\text{ref}})]}{n_{\text{EA}}(x_{\text{ref}})} \]

\[ - j \frac{\sum_{i=1}^{N} w(x_{\text{oi}} - x_{\text{ref}})}{n_{\text{EA}}(x_{\text{ref}})} \exp[j2\pi \nu(x_{\text{oi}} - x_{\text{ref}})]. \]

(31)

As in the case for continuous, bounded data, the response function for DDWA analyses of discrete, irregularly distributed data depends upon the weight function \( w(x) \), the frequency \( \nu \) and the location \( x_{\text{ref}} \). Furthermore, as in the case for continuous, bounded data, the response function is \( \Re(u, x_{\text{ref}}) = \text{FT}[w_{\text{eff}}(-x, x_{\text{ref}})]/n_{\text{EA}}(x_{\text{ref}}) = W_{\Re}(u, x_{\text{ref}}) \), and again is thus the complex conjugate of the normalized Fourier transform of the effective weight function.

2) Verification of the response function for discrete, irregularly distributed data

The response function (31) is tested by using it to predict analysis values and by comparing these values to actual analysis values obtained using (4). As in the continuous, bounded case, the input to these tests is prescribed by (21). Furthermore, since in this case \( \Re(u, x_{\text{ref}}) = \text{FT}[w_{\text{eff}}(-x, x_{\text{ref}})]/n_{\text{EA}}(x_{\text{ref}}) = W_{\Re}(u, x_{\text{ref}}) \), as in the continuous, bounded case [(17) and (18)], the analysis from (18) to (22) applies here also. In this case, of course, \( W_{\Re}(u, x_{\text{ref}}) \) is given by (31) rather than by (17).

Actual and predicted analysis fields for \( A_{j} = 1, \nu = 1/5, \varphi_{i} = 0, N = 20, \) and \( w(x) = \exp(-x^{2}/\kappa_{d}) \), with \( \kappa_{d} = 3 \), are shown in Fig. 4a. (Each \( x_{\text{oi}} \) was obtained using a pseudorandom number generator and was restricted such that \( x_{L} \leq x_{\text{oi}} < x_{R} \), with \( x_{L} = 0 \) and \( x_{R} = 10 \).) In this figure the limits of the possible \( x_{\text{oi}} \) values are indicated by thick-dashed lines, the input field is indicated by the dotted line, observations are indicated by diamond symbols, observation locations are indicated by the small arrows at the top of the figure, the analyzed field is indicated by the solid line, and the predicted [using (22)] analysis values are indicated by the plus-sign symbols. As this figure shows, (31) is correct since it predicts actual analysis values.

The amplitude \( |W_{\Re}(u, x_{\text{ref}})| \) and phase \( \phi_{W_{\Re}(u, x_{\text{ref}})} \) modulations for the test illustrated in Fig. 4a are shown in Fig. 4b. Except for the arrows that indicate observation locations along the top of this figure, the elements of Fig. 4b are as in Fig. 3b, with the amplitude modulation field (left axis) indicated by the solid line, the phase modulation field (right axis) indicated by the thin-dashed line, and the ideal amplitude modulation field, or ideal response (left axis), indicated by the dotted line. In this case, amplitude modulations are generally far from ideal, and significant phase shifts are common. (The presence of significant phase shifts in Fig. 4b is consistent with the misalignment in Fig. 4a of the maxima and minima of the input and analysis fields.) Fig. 4 illustrates well the impact an irregular observational distribution can have. When observations are irregularly distributed, the response can be far from ideal both within and outside of the observational domain limits.

The phase shift values that are outside of the observational domain limits in Fig. 4b behave similarly to the phase shift values that are outside of the observational domain limits in Fig. 3b. Because the \( \phi_{W_{\Re}(u, x_{\text{ref}})} \) values plotted in Fig. 4b were restricted as they were in Fig. 3b, that is, such that \(-180^{\circ} \leq \phi_{W_{\Re}(u, x_{\text{ref}})} \leq 180^{\circ}\), the extrapolation phase shift discussion of section 2a(3) ap-
plies here as well. A plot of corrected $\varphi w(x_{v,x_{w}})$ values would thus indicate monotonically increasing (decreasing) phase shifts with increasing distance to the right (left) of the rightmost (leftmost) observational domain limit. As with the continuous, bounded case, extrapolation is achieved by shifting information contained within the observational domain to points outside of the observational domain.

3. Discussion

As indicated in the introduction, the response function can be viewed from either a domainwide or a local perspective. In some situations, like the use of a fixed weight function in the DDWA analyses of continuous, infinite data, these are equivalent. Generally, however, they are not. The local response function concerns spectral effects at a particular location while the domainwide response function denotes some sort of average response. For the purpose of illustrating this very important distinction, the domainwide response function is defined as being composed of the domainwide average amplitude and phase modulations. Given this, consider Fig. 3. In Fig. 3b, the average of the phase modulations is zero. From a phase shift standpoint, this implies a good analysis (no domainwide phase shift). This is a misleading measure of analysis quality, however, since local phase shifts are significant both near and outside of the observational domain boundaries. In fact, the impact of these phase shifts is apparent in the differences in the locations of the extrema of the analysis and input fields near the boundaries of the observational domain (Fig. 3a). The local response function, therefore, appears to be a superior measure of local analysis fidelity.

Differences between domainwide and local response functions are also illuminated by the factors that affect postanalysis Fourier content, which can be examined using (7), (26), (17), and (31). Equation (7) indicates that for continuous, bounded data the postanalysis, domainwide Fourier content results from three steps: 1) the convolution of the input Fourier content with the Fourier transform of the pulse function, 2) the multiplication of the result of 1) with the Fourier transform of $w(x)$, and 3) the convolution of the result of 2) with the Fourier transform of $1/m(x)$. Equation (26) indicates that the situation is much the same for discrete, irregularly distributed data, the only difference being that the input Fourier content is first convolved with the Fourier transform of the irregular comb distribution. Conse-

![Image](https://example.com/image.png)

Fig. 4. (a) The input field (dotted line), observations (diamond symbols), and actual (solid line) and response-function-predicted (plus-sign symbols) analysis fields for a DDWA analysis of one-dimensional, discrete, irregularly distributed data. (b) The amplitude (solid line, left axis) and phase (thin-dashed line, right axis) modulation functions and the ideal response function (dotted line, left axis) for the test shown in (a). In both (a) and (b) the thick-dashed lines indicate the limits of possible observation locations, and the arrows denote actual observation locations. The observational field is given by $f(x) = A_i \cos(2\pi v_i x + \varphi_i)$, with $A_i = 1$, $v_i = 1/5$, $\varphi_i = 0$, and the weight function is given by $w(x) = \exp(-x^2/d)$, with $d = 3$. 

quentiely, the analysis scheme affects the postanalysis, domainwide Fourier content through the Fourier content of both $w(-x)$ and $1/n(x)$. The observation distribution affects the postanalysis, domainwide Fourier content through the Fourier content of the pulse function (continuous, bounded data) or irregular comb distribution (discrete, irregularly distributed data) and the Fourier content of $1/n(x)$ (both the analysis scheme and the observation distribution affect $n(x)$). The postanalysis, local Fourier content, defined here to be the Fourier content under the conditions of an equivalent analysis, is specified by either (17) or (31). From the analyses preceding these equations [e.g., (15)], it is apparent that the postanalysis, local Fourier content results from two steps: 1) the convolution of the Fourier content of $w(-x)$ with the Fourier content of either $p_{\text{EA}}(-x, x_{\text{red}})$ or $\text{icomb}_{\text{EA}}(-x, x_{\text{red}})$ and 2) the multiplication of the result of 1) with $F(v)/n_{\text{EA}}(x_{\text{red}})$. As opposed to the situation for postanalysis, domainwide Fourier content, the normalization factor has a relatively minor affect on the postanalysis, local Fourier content since it only serves to normalize that Fourier content.

Equations (7), (17), (26), and (31) provide a basis for studying the effects observation distributions and analysis schemes have upon response functions. While a rigorous exploration of this topic is beyond the scope of this work, a brief discussion of research progress is appropriate.

The effects rectangular windows have on domainwide spectral content are discussed in both textbooks (e.g., Weaver 1983, 134–137; Hamming 1998, chapter 5) and articles (e.g., Caracena et al. 1984), as are the effects infinite regular comb distributions have on domainwise spectral content (e.g., Weaver 1983, 131–134; Pauley and Wu 1990). The impact of finite regular comb distributions has also been considered (e.g., Caracena et al. 1984). With respect to local spectral content, the impact of pulse functions (Achtemeier 1986; Pauley 1990), infinite regular comb distributions (e.g., Pauley and Wu 1990), finite regular comb distributions (e.g., Jones 1972), and finite irregular comb distributions (e.g., Jones 1972; Schlax and Chelton 2002) have all been considered to varying degrees.

The effects that weight functions have on Fourier content have been considered by numerous investigators (e.g., Barnes 1964; Stephens 1967; Koch et al. 1983). However, the role the normalization factor plays in domainwise spectral content, as indicated by (7) and (26), has not been considered previously.

As discussed in the introduction, the response function (31) has been derived in one form or another by others (Jones 1972; Yang and Shapiro 1973; Thiébaux and Pedder 1987, p. 105; Buzzi et al. 1991; Schlax and Chelton 1992). To obtain the equivalent of (31), these investigators substituted a spectral representation of the observation field into their expressions for DDWA analyses and subsequently manipulated that result. This “back-substitution method” differs from the “convolution-theorem approach” used herein. Since both techniques produce the same response function, one may contend that little has been gained in this exposition. On the contrary, it is argued that the convolution-theorem approach provides insights into DDWA analyses that are not available from the back-substitution method. Specifically, 1) the convolution-theorem approach provides an infrastructure for interpreting both the separate and combined impacts that data distributions and weight functions have upon analyses and 2) the convolution-theorem approach illustrates exactly what the local response function for DDWA analyses is: the local response function is the complex conjugate of the normalized Fourier transform of the effective weight function.

The technique outlined herein has numerous potential applications. The most obvious is the evaluation of the amplitude and phase fidelity of DDWA analysis schemes. In this regard, an interesting use would be the evaluation of the filtering properties of statistical objective analysis (SOA) schemes. It would be particularly interesting to determine whether SOA schemes, in their procurement of a DDWA analysis that minimizes analysis-error variance, also minimize phase shifts.

Another potential use is the evaluation of observation networks. Doswell and Lasher-Trapp (1997) have suggested the use of the gradient of the normalization factor to characterize the degree of irregularity of observation networks. It seems that phase modulation could be used in much the same manner since with even weight functions nonzero phase shifts arise from the inhomogeneity of observation distributions. A potential complication, however, would be the $\pm 2\pi n$ ambiguity in the determination of phase shift values, which is discussed in section 2a(3).

Amplitude modulation could also be quite useful in the evaluation of observation networks. Its principal utility is probably its ability to indicate how well DDWA schemes retain signal while repressing noise (e.g., Jones 1972; Schlax and Chelton 2002). By characterizing filtering potential, the technique outlined herein could be especially useful for networks that are limited in both extent and number of observations, for which the determination of network viability is particularly difficult.

It is noted that the technique outlined in this study is sufficiently general so that it should be applicable to arbitrary weight functions and data distributions as long
as the concomitant integrals and summations are defined. Extension to situations where combinations of discrete and continuous data are available should be possible by combining pulse functions and comb distributions.

Finally, an exciting potential use is in the design of filters that will replicate prescribed amplitude and phase modulations as closely as possible given an observation distribution. This could be a very useful filter design technique, especially considering the difficulties in applying SOA to situations where error covariances are uncertain or background estimates are unavailable. Work on the design of such a filter is currently underway.

Before concluding, it is noted that in the interest of length the multidimensional problem is not considered here. An analysis of the response function for discrete, irregularly distributed, two-dimensional data has been completed. The procedure is the same as that developed herein, with only bookkeeping-type differences arising from the presence of the extra dimension [two-dimensional cross correlations, two-dimensional convolutions, use of the two-dimensional Dirac distribution $\delta(x, y)$ (Bracewell 2000, p. 89), etc.]. The resultant response function is equivalent to that presented by Buzzi et al. (1991) and is given by

$$R(u, v, x_{ref}, y_{ref}) = \frac{\sum_{i=1}^{N} w(x_{oi} - x_{ref}, y_{oi} - y_{ref}) \cos[2\pi(u(x_{oi} - x_{ref}) + v(y_{oi} - y_{ref})]]}{n_{EA}(x_{ref}, y_{ref})} - \frac{j\left[\sum_{i=1}^{N} w(x_{oi} - x_{ref}, y_{oi} - y_{ref}) \sin[2\pi(u(x_{oi} - x_{ref}) + v(y_{oi} - y_{ref})]]\right]}{n_{EA}(x_{ref}, y_{ref})}, \quad (32)$$

where $u$ and $v$ represent frequency in the $x$ and $y$ directions, respectively.

4. Conclusions

The following list summarizes the results of this work:

1) The local response function for DDWA schemes is the complex conjugate of the normalized Fourier transform of the effective weight function. Complex conjugation arises because DDWA is, in general, a cross correlation, not a convolution. Normalization is imposed by the DDWA normalization factor. The effective weight function is the product of the weight function and the function, or generalized function (also called a distribution), that describes the distribution of the observations.

2) To obtain the local response function by way of the convolution theorem the concept of an equivalent analysis is needed. In an equivalent analysis a hypothetical analysis field is produced by using, throughout the entire domain, the same weight function and data distribution that apply to the point of interest. This artifice enables the convolution-theorem-based derivation of the response function by altering the mathematical form that describes the analysis field.

3) The local response function generally depends upon the weight function, frequency, and location.

4) Boundaries significantly affect response functions. In their vicinity they produce significant phase shifts and appreciable alterations of amplitude modulations relative to the ideal response function, which holds for continuous, infinite data.

5) Phase shift information provides a straightforward interpretation for extrapolation. It illustrates the movement, or shift, of information that seems to be fundamental to all extrapolation schemes. During extrapolation analysis values are produced by taking information and moving it to the analysis locations.

6) Irregular data spacing can result in significant phase shifts and significant departures from the ideal amplitude modulation. The degree of filtering at a particular frequency can be either greater or less than that imposed under ideal conditions.

Conclusions 3–4 and 6 reinforce the conclusions of previous investigators (Caracena et al. 1984; Achtemeier 1986; Pauley 1990; Buzzi et al. 1991; Carr et al. 1995; Schlax and Chelton 2002), while conclusions 1–2 and 5 appear to be novel aspects of this work. In closing, it is noted that the technique outlined herein may be extendable to other basis functions [other than $\sin()$ and $\cos()$] for which the convolution theorem applies.

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