A Phase-Plot Method for Diagnosing Vorticity Concentration Mechanisms in Mesoscale Convective Vortices

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ABSTRACT

Mesoscale convective vortex (MCV) analysis results show that these vortices form by way of different evolutionary paths. Rewriting the traditional form of the relative vertical vorticity equation in terms of momentum advection curl produces an alternative form of the equation containing two terms. When the terms are normalized and plotted on orthogonal axes, a phase-plot path depicting MCV evolutionary growth is created. Thermodynamics is included in the phase plot by correlating the path to the heating characteristics of the troposphere. The application of the phase-plot scheme to several cases shows that for MCV formation events, there are two interconnected regions that combine to produce the vortex. The upper-middle- and upper-troposphere vorticity growth is governed primarily by vertical motion, with heating driving the vorticity growth in the upper-middle region. The lower-middle and lower-troposphere vorticity growth is governed primarily by horizontal motion, with the vertical heating gradient driving the vorticity growth in the lower-middle region. Which regime leads the vorticity growth is found to be case dependent. In the middle troposphere, evolutionary paths are governed by the relative strengths of heating and heating gradient. Additional phase-plot and mesoscale analyses clarify the characteristics of two MCV formation modes. In some cases, heating drives the complete formation of the MCV, whereas in cases with lesser heating, tipping is vital to the MCV formation process. In total, these results help synthesize many of the various discoveries regarding the origin and formation of the MCV.

1. Introduction

a. General discussion

In the introduction to Raymond and Jiang’s (1990) theory for long-lived mesoscale convective systems, the circumstances leading to a positive potential vorticity anomaly in the middle troposphere are described in general terms. Prior to and subsequent to this theory, discoveries regarding the details of formation of the mesoscale convective vortex (MCV) have been presented. Consider various discoveries regarding the origin and intensification of mesoscale convective vortices:

• Low-to-midlevel convergence spreading upward (Zhang and Fritsch 1988; Olsson and Cotton 1997a);
• midlevel convergence spreading downward (Chen and Frank 1993);
• midlevel horizontal advection, convergence, and tipping (Verlinde and Cotton 1990);
• tipping at lower levels than midlevel convergence (Brandes 1990; Zhang 1992; Weisman and Davis 1998);
• convergence acting on planetary vorticity (Bartels and Maddox 1991);
• strong upper-level tipping, midlevel convergence, and weak low-level tipping (Knievel and Johnson 2003);
• convergence acting on planetary vorticity, an upper-level warm anomaly (and associated mesolow), and midlevel tipping (Skamarock et al. 1994); and
• tipping early in time followed by convergence (Cram et al. 2002).

While there is some consensus among the results, there is enough variety to suggest that perhaps there are different evolutionary paths to the same typical MCV final structure. Kirk (2003) showed evidence for such a possibility. Such a result leads to several questions that are addressed in this paper. Is there a way to view processes in the troposphere that would efficiently expose the...
similarities and differences between MCV formation cases? What are the defining characteristics for different modes of formation? Can the results of others be unified by identifying regions of the troposphere that exhibit similar behavior? This paper addresses these questions by providing a framework for cases to be compared and contrasted. The ultimate objective is to gain a better perspective on the backdrop within which MCV studies are occurring.

b. Brief background

Fritsch and Forbes (2001) describe the dynamical similarities between mesoscale convective complexes (MCCs) and larger long-lived squall line mesoscale convective systems (MCSs). MCC cases will be examined in this paper to illustrate the new analysis framework. MCCs were identified and defined as a meso-γ-scale weather phenomenon in the very early 1980s by Maddox (1980) and Fritsch and Maddox (1981). The discovery prompted new studies in an attempt to understand the gross features of such storms, large-scale setting, frequency of severe weather associated with them, precipitation structure, and diurnal cycle. Individual case studies were carried out by Bosart and Sanders (1981), Wetzel et al. (1983), Leary and Rappaport (1987), Zhang and Fritsch (1987), Menard and Fritsch (1989), Verlinde and Cotton (1990), and Brandes (1990). Papers were also written documenting the frequency of occurrence during certain years (Augustine and Howard 1988, 1991), and the global population was studied by Laing and Fritsch (1997). Composite studies were carried out by Maddox (1983) and Cotton et al. (1989). Conceptual models of formation for MCVs associated with MCCs were proposed by Cotton et al. (1989), Olsson and Cotton (1997b), and Chen and Frank (1993). Fritsch and Forbes (2001) provide a review of MCCs.

An intriguing internal feature of MCCs and some long-lived squall lines, the MCV, was identified as early as 1981 by Johnston (1981). Bartels and Maddox (1991) reported MCV cases from a variety of convective systems from 1981 to 1988. Bosart and Sanders (1981) and Menard and Fritsch (1989) documented the convective redevelopment aspects of MCSs and their associated MCVs. Davis and Trier (2002) more recently showed that MCVs were essential for the redevelopment of convection in a squall line system simulation. Raymond and Jiang (1990) and Fritsch et al. (1994) proposed a mechanism for the redevelopment of convection resulting from the existence of the MCV. Preconditioning issues as defined by Johnson and Mapes (2001) were found to be important in the Fritsch et al. (1994) model by Trier and Davis (2002). Several more papers (Bosart and Sanders 1981; Velasco and Fritsch 1987; Miller and Fritsch 1991; Laing and Fritsch 1993a,b) developed the connection of MCVs to tropical cyclogenesis. Simulations by Skamarock et al. (1994), Davis and Weisman (1994), Wicker (1996), and Weisman and Davis (1998) explored the role of shear in the development of squall line MCVs, and studies by Raymond (1992), Davis and Weisman (1994), Jiang and Raymond (1995), and Olsson and Cotton (1997b) attempted to find a balance condition that would describe the MCV. A recent balance study was performed by Knievel et al. (2004).

c. Motivation and objectives

Scott and Rutledge’s (1995) observational study obtained different MCV formation results compared to that of Brandes and Ziegler (1993), and they attributed the differences to the nature of data and their processing. Because of the complexity of MCSs and the difficulty in acquiring data, it is likely that many presumed that literary contrariety regarding the origin and intensification of MCVs could be attributed to the varied nature of data sources and the subsequent studies they are based upon. Kirk (2003) was the first to compare two cases in precisely the same manner for the express purpose of identifying different organizational modes of formation for MCVs. The results of Kirk (2003) suggest a need to identify and classify the different modes of MCV formation in a way similar to the Parker and Johnson (2000) taxonomy. In the case of MCVs, however, the classification would be of an important internal feature of MCSs as opposed to classifying the MCSs themselves [there has already been broad classification of MCCs by Fritsch and Forbes (2001) and Davis et al. (2002)]. One of the major impediments to doing this is the difficulty of finding a straightforward method of comparison. The method of Kirk (2003) is effective, but the myriad of plots and graphs is cumbersome. Also, a thermodynamic analysis in that paper was not performed. It would be best to compare two or more cases using a single diagram that depicts the path of the troposphere to a defined final state and includes information on the effects of condensational and depositional warming [Bohren and Albrecht (1998) explain why the term “latent heat” is problematic]. Such a diagram is proposed in this paper and will be called a phase plot.

To begin the construction of the phase plot, a different form of the vertical vorticity equation than the traditional one found in Holton (1992, p. 105) is used. The different form of the vertical vorticity equation is arguably more fundamental since it identifies the process of the curl of the advection of horizontal momentum (from here forward called “momentum advection curl”) as the single process responsible for vorticity...
growth within the usual approximations. In this paper, the physical implications of this view of vorticity are compared and contrasted with the more traditional view of vorticity changes. The equations that form the basis of the phase-plot depiction are then developed, and the phase plot is introduced. The nature of the phase-plot scheme is such that a pathway in the form of a curve in two-dimensional phase space from an initial to final state will be defined. The color and linestyle of the curve are governed by the thermodynamic features of the system. The result is a phase-plot depiction containing an evolutionary history of both the dynamics and thermodynamics of the MCV. The phase plot further provides a concise method of comparison for different cases so that formation modes for MCVs can be defined. Phase plots will be presented for the Kirk (2003) cases, the study of Chen and Frank (1993), and the MCV analyzed by Rogers and Fritsch (2001). Features common to the four cases and dissimilarities between them are identified and discussed. The results will illustrate the features of the phase plot and demonstrate how the broader objective of creating a computationally efficient and insightful means to compare MCV cases for the purpose of identifying various organizational modes has been achieved.

Summarily, the objectives of this paper are to
1) construct the phase plot with the interpretation of its basis and the processes it can depict,
2) use the phase plot to interpret the differences and similarities for a variety of cases,
3) further analyze the cases explored by Kirk (2003), and
4) synthesize results of the phase-plot analysis with the discoveries of other scientists.

2. Methodology and data sources

The four cases examined by using the phase plot in this study are the two presented by Kirk (2003), the simulation of Chen and Frank (1993, hereafter CF93), and the case of Rogers and Fritsch (2001, hereafter RF01). The two cases studied by Kirk (2003) occurred on 6 May 2000 (M6 case) and 21 June 1997 (J21 case).

In all cases, numerical values for the plots to be produced are obtained from the fifth-generation Pennsylvania State University–National Center for Atmospheric Research (PSU–NCAR) Mesoscale Model (MM5). Readers seeking details of the model runs performed should refer to the individual papers. To perform the phase-plot analysis, results of the model output were processed as described by Kirk (2003) for the M6 and J21 cases. Some of the details on the processing of the model output are summarized in section 5a where the thermodynamic analysis is added. To generate the phase plots, the terms for the alternative form of the vorticity equation were computed from the original terms used by Kirk (2003).

For the CF93 and RF01 cases, the numerical values used to generate the phase plots were obtained by digitizing the appropriate figures in those papers. The numerical values for the alternative vorticity equation terms and thermodynamic terms were computed from numbers extracted from the figures. The CF93 case had much stronger heating than both M6 and J21 because the averaging technique used in this case included a smaller region. For the RF01 case, the averaging technique was not specified. Other relevant facts are that the CF93 case was a simulation with initial conditions specifically chosen to produce an MCC, and the RF01 simulation was performed to replicate a documented serial MCV (Fritsch et al. 1994). The RF01 and CF93 cases are included here to help assess the general features of MCV growth. Beyond this there will be little further discussion for these cases. The M6 and J21 cases, however, will be further analyzed to illustrate phase-plot features and enhance the work initiated by Kirk (2003). In all cases, the earliest time for the growth period was when the vorticity at some level in the troposphere was between 10% and 20% of the maximum that developed in the midtroposphere.

After the numerical values were obtained for all cases, the data were entered into MATLAB for the phase-plot depiction. The MATLAB program is written in version 7 code that runs on a Linux computer system. It carries out all data manipulation for producing the phase plots presented herein.

It is noted here that while the resolution of the model runs being used is not consistent with the current practice for modeling case studies, it is sufficient to assess the dominant MCV organizational features. The reason for the coarse simulations is that the cases presented are from an earlier era.

3. Vorticity equation in terms of momentum advection curl: A different basis

a. The vorticity basis

Holton (1992, p. 105) derives the vertical vorticity equation in pressure coordinates for an inviscid atmosphere in a rotating frame of reference. The equation is

\[ \frac{\partial \zeta}{\partial t} = -\mathbf{V} \cdot \nabla (\zeta + f) - (\zeta + f) \nabla \cdot \mathbf{V} - \omega \frac{\partial \zeta}{\partial p} + k \cdot \left( \frac{\partial \mathbf{V}}{\partial p} \times \nabla \omega \right). \]  (1)

This is the same as the advection curl definition of vorticity where the vorticity equation is written as a vertical component of the curl of the advection. The result is a phase-plot depiction of the evolutionary history of both the dynamics and thermodynamics of the MCV. The phase plot further provides a concise method of comparison for different cases so that formation modes for MCVs can be defined. Phase plots will be presented for the Kirk (2003) cases, the study of Chen and Frank (1993), and the MCV analyzed by Rogers and Fritsch (2001). Features common to the four cases and dissimilarities between them are identified and discussed. The results will illustrate the features of the phase plot and demonstrate how the broader objective of creating a computationally efficient and insightful means to compare MCV cases for the purpose of identifying various organizational modes has been achieved.

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3. Vorticity equation in terms of momentum advection curl: A different basis

a. The vorticity basis

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Here, \( \mathbf{V} = u \mathbf{i} + v \mathbf{j} \) is the horizontal momentum per unit mass vector. Vertical motion is denoted by \( \omega \) in units of pressure per unit time. The direction of vertical motion is parallel to the unit vector \( \mathbf{k} \), which points parallel to the local vertical. The \( \nabla \) operator is the horizontal gradient operator on a surface of constant pressure, \( \zeta \) is the relative vertical vorticity, and \( f \) is the Coriolis parameter.

Equation (1) is a conventional vorticity equation, and each of the terms on the rhs has an interpretation in terms of vorticity processes. These are, moving from left to right, respectively, horizontal advection of vorticity, concentration or dilution of vorticity by a convergent or divergent velocity field (called convergence here), vertical advection of vorticity, and tipping representing the rotation of horizontal vorticity into the vertical. Notwithstanding the problems with some of these interpretations [see, e.g., Adlerman et al. (1999, p. 2053) and Haynes and McIntyre (1987)], this set of processes and their associated terms in the vorticity equation reflect a choice of perspective on how vorticity in the atmosphere changes. Because the interpretation emphasizes processes in terms of vorticity, it is referred to here as the “vorticity basis.” Introducing this terminology is only useful if other bases exist, which is examined next.

b. The momentum basis

The beginning step to changing basis is to rewrite Eq. (1) by combining the first and second terms and the third and fourth terms, respectively, on the rhs of Eq. (1). This results in the following:

\[
\frac{\partial \zeta}{\partial t} = -\nabla \cdot [\mathbf{V}(\zeta + f)] - \mathbf{k} \cdot \left( \nabla \times \frac{\partial \mathbf{V}}{\partial p} \right). \tag{2}
\]

The first term on the rhs has a reasonably clear interpretation in terms of the vorticity basis as the divergence of vorticity flux due to horizontal motion. The second term’s interpretation in the vorticity basis is not so straightforward, however. Haynes and McIntyre (1987) discussed, and Davis and Weisman (1994), Weisman and Davis (1998), and Davis and Trier (2002) used, the flux form of the vorticity equation, which is closely related to Eq. (2), but none gave a clear interpretation of the term involving the vertical motion field.

The most direct way to interpret the second term on the rhs of Eq. (2) is to think in terms of momentum. First, imagine a single square cell of area \( \Delta A = \Delta x \Delta y \) that is oriented perpendicular to the local vertical direction. The second term on the rhs of Eq. (2) can be expanded in terms of momentum, yielding

\[
\mathbf{k} \cdot \left( \nabla \times \left( \frac{\partial \mathbf{V}}{\partial p} \right) \right) = -\frac{\partial}{\partial x} \left( \frac{\partial u}{\partial p} \right) + \frac{\partial}{\partial y} \left( \frac{\partial v}{\partial p} \right). \tag{3}
\]

Each term on the rhs of Eq. (3) has an interpretation for how vorticity can change as a result of the vertical motion field. Figure 1 illustrates the interpretation of the terms. Figures 1a,b represent differential momentum advection caused by vertical motion. The vorticity of the depicted cell is changed because the momentum advection at “side a” is different than the momentum advection at “side b.” Such a difference in momentum advection between these two sides can be caused by (i)
a horizontal variation in vertical velocity combined with vertical shear or (ii) vertical velocity combined with a horizontal variation in the vertical shear. Thus, the explanation of how vertical motion changes vorticity is in terms of momentum advection processes. This choice of interpretation will be referred to as the “momentum basis.” Switching to the vorticity basis, it is recognized that if the difference in momentum advection between sides a and b of the cell is due to (i), then it is interpreted as tipping. If the difference in momentum advection is due to (ii), then it is interpreted as vertical advection. Thus, in the momentum basis, the vorticity basis processes of tipping and vertical advection appear together in a single momentum basis term.

Returning to Eq. (2), the second term on the rhs is momentum advection curl due to vertical motion and will be called vertical momentum advection curl.

c. Vorticity equation in terms of momentum advection curl

The introduction of the momentum advection curl interpretation and the momentum basis concept raises the question of how the first term on the rhs of Eq. (2) can be interpreted in the momentum basis. To answer this question, Eq. (2) must be transformed to the earth’s center frame of reference (an inertial frame).

Switching basis processes of tipping and vertical advection appear together in a single momentum basis term.

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4. A phase-plot depiction of MCV growth

The concentration of vorticity in an MCV (or any vortex) occurs over a period of time that can be split into N finite time steps of length \( \Delta t \). Writing Eq. (2) in finite difference form yields

\[
\zeta_{t+1} = \zeta_t + (\text{HMAC}_t)\Delta t + (\text{VMAC}_t)\Delta t,
\]

where \( \text{HMAC}_t = -\nabla \cdot [\nabla (\zeta + f)] \) and \( \text{VMAC}_t = -\mathbf{k} \cdot \nabla \omega \). The bar across the top represents the spatial average value of the quantity in a portion of the troposphere.

The total effect of earlier time steps producing the nth time step value of vorticity is
and the final value of vorticity for a given region is

\[ \zeta_N = \zeta_0 + \sum_{i=1}^{N} (\text{HMAC})_{i-1} \Delta t + \sum_{i=1}^{N} (\text{VMAC})_{i-1} \Delta t, \]

or as

\[ 1 = \text{HMAC}_f + \text{VMAC}_f, \]  

(10)

with

\[ \text{HMAC}_f = \sum_{i=1}^{N} (\text{HMAC})_{i-1} \Delta t \]

\[ \text{VMAC}_f = \sum_{i=1}^{N} (\text{VMAC})_{i-1} \Delta t \]

(11)

This normalization, by dividing by total vorticity growth, generates an equation that contains two terms
representing the fractional growth at the end of the MCV growth period due to horizontal momentum advection curl (HMACf) and vertical momentum advection curl (VMACf). If the two terms (HMACf and VMACf) are plotted on orthogonal axes, the single point representing the final atmospheric state must lie somewhere along a line with a slope of $-1$ and an intercept of $+1$. Thus, the manipulations leading from Eq. (6) to Eq. (10) have created a graphical depiction of the final atmospheric state for any MCV. To create the path leading to such a final state, the following quantities are further defined:

\[
HF = \sum_{i=1}^{n} \frac{(HMAC)_{i-1} \Delta t}{\zeta_N - \zeta_0} \quad \text{and} \quad VF = \sum_{i=1}^{n} \frac{(VMAC)_{i-1} \Delta t}{\zeta_N - \zeta_0},
\]

where HF and VF are now the fractional contributions to vorticity growth at time $t_i$ earlier than the final time $t_N$. Where vorticity is steadily being concentrated, these two quantities, when summed, yield a number less than $+1$. Thus, if all points between the initial time $t = 0$ and the final time $t = t_N$ are computed and plotted on the same set of orthogonal axes, the result is a curve representing the path of vorticity growth toward the final state along the line defined by Eq. (10).

The basic idea of the phase plot and its representation is shown in Fig. 3. It shows plotting HF on the ordinate and VF on the abscissa. If the plot is rotated $45^\circ$, the line representing the final atmospheric state is horizontal.

Many studies (e.g., Chen and Frank 1993; Rogers and Fritsch 2001; Kirk 2003; Knievel and Johnson 2003) have shown that horizontal advection is minimal for MCV formation. Thus, the contribution from HF can reasonably be assumed to be primarily from convergence. Vertical advection and tipping combine to yield the VF term, and it is not clear which one is dominant. It is reasonable to assume that for early times tipping would dominate and for later times vertical advection may dominate. However, as shown in the previous section, both of these processes are different versions of the single process of vertical momentum advection curl in the momentum basis. What is important is that the growth was caused by the combination of vertical motion and vertical shear, the essential ingredients for all vertical momentum advection curl.

5. Thermodynamics

a. Review

To present a thermodynamic analysis for the M6 and J21 cases, it is first necessary to briefly review some important details from Kirk (2003). For convenience, the large-scale setting (Fig. 4) and satellite views of the two cases (Fig. 5) are duplicated here. The troposphere was divided into layers, and the terms on the rhs of Eq. (1) were numerically averaged (both horizontally and vertically) over a $72 \text{ km} \times 72 \text{ km}$ horizontal region centered on the MCV. This size was chosen in an attempt to find a compromise between capturing the essential processes responsible for vorticity concentration and diluting the computations with peripheral details (note that for M6, because the vortex exhibited some tilt, an expanded domain of $144 \text{ km} \times 144 \text{ km}$ was tested and showed no significant change in the vorticity growth pattern). The averaging was performed at various times and the terms were plotted as functions of
Fig. 4. The (a) observed surface features of the J21 case prior to MCV formation. Winds are in m s$^{-1}$. Sea level pressure is in hPa and temperatures and dewpoints are in °C. The shaded region shows dewpoints above 20 °C and the box shows the MCC genesis region. (b) The observed 500-hPa chart for J21. (c) The M6 case surface that precedes MCV formation. Everything is as in (a) except that the shaded region shows dewpoints in excess of 18 °C. (d) The observed 500-hPa chart for M6. Both (b) and (d) are at 0000 UTC with heights in m and temperatures (actual and dewpoint) in °C.
time. Vorticity was also computed. The troposphere was divided into layers in the following way:

- lower troposphere—centered on 850 hPa and 200 hPa thick;
- lower-middle troposphere—centered on 650 hPa and 200 hPa thick;
- upper-middle troposphere—centered on 450 hPa and 200 hPa thick; and
- upper troposphere—centered on 250 hPa and 200 hPa thick.

The results of the analysis were the following.

1) Vorticity growth in the J21 case began low and moved upward, whereas in the M6 case, there was initial growth in both the lower- and upper-middle troposphere that merged into a single vortex maximum (i.e. $\partial \zeta / \partial p = 0; \partial^2 \zeta / \partial p^2 < 0$) at maturity.

2) Convergence was responsible for the low-level vorticity growth of both storms.

3) The upper-midlevel growth of the M6 case was primarily due to tipping early followed by vertical advection. The J21 case upper-midlevel growth was initiated by vertical advection that eventually combined with convergence.

b. Thermodynamic computations

To perform the thermodynamic energy budget analysis, the terms in the thermodynamic energy equation were computed in the same fashion as the vorticity terms by Kirk (2003). The model thermodynamic energy equation in pressure coordinates can be written as

$$
\frac{\partial T}{\partial t} = -\mathbf{V} \cdot \nabla T - \left< \omega \left[ \frac{\partial T}{\partial p} - \frac{C_p(1 + 0.8q_v)}{T} \right] \right> + Q_t + (F_h + F_v)T,
$$

where $T$ is temperature, $C_p$ is the specific heat capacity for dry air, $\rho$ is density, $q_v$ is the water vapor mixing ratio, $Q_t$ is the total heating rate, and $F_h$ and $F_v$ are the horizontal and vertical diffusion operators, respectively. The terms on the rhs of Eq. (13) are (moving from left to right) horizontal advection, change due to vertical motion (which includes vertical advection and adiabatic cooling), total heating, and model diffusion. Chen and Frank (1993) showed that diffusion can be ignored for the purpose of thermal energy budget computations related to MCVs.

Adjusting for the motion of the MCV, as in Kirk (2003), the thermodynamic energy equation is

$$
\frac{D T}{D t} = -(\mathbf{V} - \mathbf{V}_{vor}) \cdot \nabla T - \left< \omega \left[ \frac{\partial T}{\partial p} - \frac{C_p(1 + 0.8q_v)}{T} \right] \right> + Q_t + (F_h + F_v)T,
$$

where $\mathbf{V}_{vor}$ is the horizontal velocity of the MCV.

The lhs of Eq. (14) is a derivative in time no longer with fixed $x$, $y$, and $p$ but with fixed horizontal location of the vortex. All terms in the equation were computed with the exception of the diffusion term. Because the heating term is not part of the MM5 output, it was computed as a residual. Support for the accuracy of the residual value of the heating term comes from the combination of Chen and Frank (1993, their Fig. 18), Rogers and Fritsch (2001, their Fig. 17), and the computed M6 and J21 thermodynamic terms. In CF93 and RF01, the total heating computed from MM5 is nearly equal to the temperature tendency from vertical motion, the
other terms being negligibly small. Because of this, it can be readily deduced (by visual inspection of the plots) that the total heating is the residual of the other terms for their computations. In the computations for M6 and J21, the temperature tendency from vertical motion is much larger than the other terms computed (these results are not shown here), as in CF93 and RF01. This consistency with CF93 and RF01 indicates that for the M6 and J21 cases, the computed residual heating term is also consistent with their results and is accurately computed. For the terms computed in Eq. (14), 8 numerical averages for boxes 100-hPa thick over the 72 km × 72 km locations centered on the MCV (see Kirk 2003, p. 868 and 871) were smoothed and plotted. The results of the thermodynamic computations are shown in Fig. 6, which depicts the vertical profile of heating at various times for each MCV.

Basic dynamics shows that where heating increases with height, there is convergence. The approximate balance of vertical motion in response to heating that induces convergence is described by Fritsch (1986). The results of the thermodynamic computations for the M6 and J21 cases (not shown here) show that in Eq. (14), to a good approximation, we have

\[
Q_t = \sum_{i} a_i \frac{\partial T}{\partial p} - \left[ C_{p} (1 + 0.8 q_{w} p)^{-1} \right].
\]  

This result is consistent with the CF93 and RF01 cases as well. Equation (15) shows that upward velocity is proportional to heating. Taking the derivative of Eq. (15) with respect to pressure [assuming that the coefficient of \( \omega \) does not vary significantly with height—a reasonable assumption (see Fritsch 1986)] yields

\[
\frac{\partial Q_t}{\partial p} = \rho \frac{\partial \omega}{\partial p} \frac{\partial T}{\partial p} - \left[ C_{p} (1 + 0.8 q_{w} p)^{-1} \right].
\]

However, the continuity equation is

\[
\frac{\partial \rho}{\partial p} = -\mathbf{\nabla} \cdot \mathbf{V}.
\]

Thus,

\[
\frac{\partial Q_t}{\partial p} = \rho \frac{\partial \omega}{\partial p} \frac{\partial T}{\partial p} - \left[ C_{p} (1 + 0.8 q_{w} p)^{-1} \right].
\]

Equation (18) shows that where heating increases with height there is a converging wind, and where heating decreases with height there is a diverging wind. The computations performed confirm this mesoscale response to heating with Fig. 6, combined with the results of Kirk (2003) aiding in this confirmation. A particularly nice illustration of this result appears in Hertenstein and Schubert (1991), and such behavior is also apparent in Chen and Frank (1993). Heating increasing with height will be referred to as a “positive” heating gradient since its effect is to increase vorticity in regions where positive absolute vorticity already exists. Heating decreasing with height will be referred to as a “negative” heating gradient. The differences in the J21 and M6 heating profiles can be summarized as follows: (i) generally, maximum heating occurs at lower elevations in the M6 case compared to the J21 case and (ii) the heating in the J21 case is nearly double in strength compared to the M6 case over much of the troposphere.

c. Adding the thermodynamics to the phase plots

Completing the design of the phase-plot method of distinction for MCV evolutionary paths requires the inclusion of thermodynamic computational results. To

Fig. 6. Vertical profiles of heating in units of \( 10^{-5} \) K s\(^{-1} \) for the M6 and J21 cases. The numbers beside the curves represent UTC.
accomplish this, the locus of points used to define the evolutionary path will be colored according to the heating at the given time. The resulting curve will then change color according to the heating. In the presentation here, blue was used for weak heating and red was used for strong heating (the complete color map used is shown in Figs. 7, 8, and 11).

To include the effect of the vertical gradient of heating on MCV growth, the phase-plot linestyle has been adjusted to reflect the heating gradient. Where heating increases with height, the points are indicated by an open circle. Where there is a strong increase with height, the plus symbol is used, and where heating decreases with height, the asterisk is used. The cutoff for the plus symbol was chosen to be a nominal figure of $1.00 \times 10^{-5}$ K s$^{-1}$ hPa$^{-1}$, and heating gradients exceeding this threshold will be referred to as “extreme.”

6. Results

a. Four-case comparison

Figure 7 shows the phase plots for the four cases. Linestyle threshold is the same for each case, but heating is specific to each case. In the lower troposphere, the systems have developed a cold pool (CF93, p. 2413), or as the green coloration in the early growth period shows, are continuing to develop a cold pool (M6, J21, and RF01). The interaction of the cold pool with the low-level advection of moisture (Fritsch and Forbes 2001; type 2 system) combined with frontal conver-
gence (Fritsch and Forbes 2001; type 1 system) promotes both lifting and convergence. Heating gradients are moderate to extreme, which tends to enhance the convergent flow. In all cases, HF in the form of convergence is responsible for vorticity growth. Because the lower troposphere is dominated by HF regardless of the heating characteristics, this portion of the troposphere will be referred to as the convergence-dominated region.

The lower-troposphere growth is nearly identical in nature for the M6 and J21 cases (this can be seen in both Fig. 7 and Fig. 11). The negative effects on vorticity growth from VF are apparent in all cases, but in the M6 and J21 cases they follow HF, showing that convergence led in time the tendency for vertical motion to reduce vorticity. The large-scale setting shown in Fig. 4 shows the circumstances favoring convergence in the lower troposphere for the M6 and J21 cases. Knievel and Johnson (2003) showed that mesoscale convergence was primarily responsible for vorticity concentration in their case study, but it is unclear whether or not this result applies to the M6 and J21 cases.

The CF93 and RF01 cases show extreme gradients for all times in the lower-middle troposphere, and this gradient accompanying the dominant HF growth (convergence) is apparent. Heating is moderate, thus there is a tendency for the growth to have a significant VF component in cases where the heating gradient is not extreme. This is shown in the early growth of the J21 and M6 cases. However, as shown in the inset, when the M6 gradient turns from negative to positive, and when the J21 gradient turns from positive to extreme, convergence immediately follows and persists until the end of the growth period. Figure 8 shows a better view of the 450- and 650-hPa phase-plot levels for the M6 and J21 MCVs. The plots are as in Fig. 7, but the view angle and zoom is different and several key points labeled P1–P6 are marked. The different view helps illustrate

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**Fig. 8.** Magnified view of the middle atmospheric layer phase plots for the J21 and M6 cases. The layer and case are identified at the end of the curves and some of the interesting features creating differing evolutionary paths are identified. Six key points labeled P1–P6 are enlarged and identified in the plot and on the colormap to illustrate evolutionary features (see text).
the fundamental differences between the cases. For the M6 MCV at 650 hPa, there are equal contributions from HF and VF early, but HF begins to dominate and VF remains unchanged for the remainder of the vortex growth period after the heating gradient turns from negative to positive and the relative heating remains weak (see P5 in Fig. 8). In the J21 case, the equal contributions persist even though the heating gradient is positive. This is due to the stronger J21 heating driving vertical motion and VF as indicated by P6. Eventually HF dominates, overtaking the growth after a period of time where the heating gradient was extreme and the heating was decreasing.

The evidence presented here shows that the lower-middle troposphere vorticity growth is dominated mainly by the characteristics of the heating gradient and thus will be referred to as the heating gradient portion of the troposphere. This result is consistent with Hertenstein and Schubert’s (1991) result, showing that MCVs originate in regions where the stratiform heating at midlevels and evaporative cooling at low levels produce large vertical heating gradients. The phaseplot analysis adds to their conclusion by showing that 1) this behavior may be a general result that goes beyond their single case study and 2) the degree to which the heating gradient is correlated to the vorticity growth is related to the magnitude of the heating itself. The heating gradient is less significant where there is more heating (J21 compared to M6). The details of Chen and Frank (1993), Rogers and Fritsch (2001), and Kirk (2003; M6 case only) show that from a vorticity basis viewpoint, there is a tendency for moderate tipping and vertical advection to cancel in the lower-middle troposphere. This pattern, combined with moderate heating and a strong heating gradient, creates a circumstance where the heating gradient is dominant.

In contrast to this, the upper-middle troposphere vorticity growth is dominated by heating itself. All cases begin with VF producing the growth, as either the heating is very strong (the J21 case) or the moderate heating is accompanied by weak or negative heating gradients. In some cases the VF is tipping (M6), and in others it is vertical advection (J21, CF93 case, RF01 case). Figure 8, for the M6 and J21 cases, shows the path beginning to curve toward a line parallel to the HF axis as the moderate heating gradient drives convergence. Points P3 and P4 along the J21 path show that the heating level was decreasing as vorticity grew. Thus, the extreme heating gradient extending into this region combined with the decreased heating results in the evolutionary path sharply curving toward significant HF. This contrasts with the M6 case where the heating became stronger in time as vorticity grew, as can be seen from points P1 and P2 in Fig. 8. The VF dominates the growth subsequent to the sharp increase in heating beginning at P2. At the same time, the heating gradient was weakening, eventually turning negative, as the maximum level of heating decreased. Thus, the vorticity growth in the region turned out to be exclusively from VF, first as tipping and later from vertical advection (shown by Kirk 2003). This layer of the troposphere marks the principal differences between the J21 and M6 cases. In the CF93 case, the gradient was negative, indicating that the heating peaked lower in the troposphere and the vorticity growth followed a nearly linear path to the final state along the VF isoline of HF equaling zero. In the RF01 case, HF was negative, showing divergence at this lower than usual level. Eventually, heating did spread upward and the developing extreme gradient led to curvature, very late in time, toward a positive contribution from HF (convergence). While the differences in the cases are significant and are nicely correlated to the heating profiles of the cases, overall, this region of the troposphere is dominated by heating and its associated VF growth and thus will be referred to as the heating-dominated portion of the troposphere.

Finally, the upper-troposphere characteristics are that of weak heating, negative heating gradient with accompanying divergence, and VF-induced growth in all cases. The growth in this portion of the troposphere is quite weak as expected, since in many cases, the regions of the troposphere at or slightly above this level develop anticyclonic rotation. The region is dominated by VF in the form of vertical advection and thus will be called the vertical advection region of the troposphere.

b. More detail in the M6 and J21 cases

The M6 MCC was on the threshold of the requirements to be an MCC, and tipping played a major role in the upper-middle troposphere vorticity growth for the developing MCV. Without this upper-level contribution to the vortex growth, it is conceivable that the MCV may have never fully formed. The J21 case on the other hand was a large MCC, and the MCV formation was dominated by the strong heating. Figure 9 shows the vertical shear in the MCV genesis region for both the M6 and J21 cases 2 h into the growth period. The M6 case shows more vertical shear, helping to explain the role of VF domination. The shear is strong enough to aid in the vorticity growth, but weak enough to not shear apart the developing MCV. Figure 9 also shows that the M6 shear is partly storm generated since the shear 2 h into the growth period is greater than that shown at the time the vortex began to form. The very weak J21 shear is consistent with the Fig. 8 (see P3).
result showing maximum heating early that has produced little vorticity growth. While there was strong heating, there was not enough vertical shear to create vertical momentum advection curl.

In Fig. 10, several horizontal cross sections 1 h into the vorticity growth period for both the M6 and J21 cases are shown. Winds show that in both cases, the swath of larger water content was being transported into the genesis region. These plots might help explain why the J21 heating was stronger than the M6 case. More moisture was available and concentrated by stronger convergence at lower levels. The higher dewpoints in the middle troposphere suggest that the greater moisture supply was more likely to produce the stronger J21 heating.

The phase-plot and supporting analysis show how the overall effect of heating dominated the J21 case. They also show that while heating was weaker in the M6 case, tipping at higher levels compensated for the weaker heating to still create the MCV. These two cases provide evidence for two formation modes: one where the strength and vertical extent of heating completely drive the system and the second where lower-level moderate heating combined with upper-level dynamics produces the MCV.

c. Timing

Figure 11 shows the four cases for the first 50% of the growth period and the full-growth period. The J21 case growth originates in the low-level convergence portion of the troposphere during the same period of time that the upper-middle troposphere showed very strong heating (see Fig. 8) with little corresponding vorticity growth. Whether or not this low-level growth was induced by the strong heating remains unclear. With the very weak shear in the upper-middle troposphere, it is
FIG. 10. Mean values of mixing ratio (g kg$^{-1}$), divergence ($\times 10^{-5}$ s$^{-1}$), and relative humidity (%) 1 h into the growth period for the M6 and J21 cases. Winds are at 850 hPa (half barb is 5 m s$^{-1}$). The boldface box shows the MCV genesis region.
Fig. 11. Plan views of the phase plots for each case: the (left) growth of vorticity in the labeled atmospheric layers for 50% of the growth period and (right) full-growth phase-plot paths. (middle) Because the RF01 case may have undergone a transition, the growth at 20% is shown.
conceivable that the heating preceded the low-level vorticity growth and that the VF in the form of vertical advection of vorticity did not develop until the lower levels concentrated vorticity. In contrast, for the M6 case, vorticity growth was initiated in the heating portion of the troposphere (even though the heating was not strong early in time) and this was followed by lower-level convergence. The M6 pattern is also apparent in the CF93 case. While it is not shown in Fig. 11, the 50% growth pattern remained the same throughout the growth period for the J21, M6, and CF93 cases. However, the RF01 case seems to have undergone a transition. The growth began in the lowest level (see the 20% diagram in Fig. 11) and was followed by the upper-middle level leading. This suggests a possible transition in the MCV growth pattern, as alluded to by Knievel and Johnson (2002, p. 1767), and may be a feature exhibited by other MCVs. Such a transition may also be related to the serial nature of this MCV. These timing conclusions represent another aspect of the different evolutionary path hypothesis that can be studied by the phase-plot method of analysis.

d. Comparison with other cases

In the introduction, the discoveries of others were presented. These can now be examined in light of the results from the phase-plot analysis. The cases of Knievel and Johnson (2003) and Verlinde and Cotton (1990) fit nicely into the suggested scheme of four different regions of the troposphere and their corresponding dominant processes. In both of these cases, low-level HF in the convergence and heating gradient portions of the troposphere is consistent with their results. Additionally, the upper-level tipping scenario is consistent with the heating-dominated portion of the troposphere. Thus, these two cases with somewhat different claims are drawn together by the phase-plot analysis results.

The cases of Olsson and Cotton (1997a), Zhang and Fritsch (1988), and the study of Bartels and Maddox (1991) fit into the scenario of low-level convergence-dominated and lower-middle heating gradient-dominated regions of the troposphere being a common feature of MCV formation. The results discussed by Chen and Frank (1993) fit into the suggested scheme with the exception that mid- to upper-level convergence is a major factor in the growth. This is not the case as shown here and is related to the fact that their Fig. 16 shows upper-level convergence that is strong early but weak in comparison to the later growth terms (see their p. 2418). In addition, the case study result of Skamarock et al. (1994) is consistent with the four regimes suggested in this paper.

Cases that do not appear to fit into the suggested scheme include those of Brandes (1990), Zhang (1992), and Weisman and Davis (1998). According to their results, it might be expected that the lower-middle troposphere exhibits mostly VF and the upper-middle troposphere exhibits mainly HF. It might also be true, however, that the paths taken are similar to the J21 case. In addition, the cases of Brandes (1990), Weisman and Davis (1998), and Cram et al. (2002) show that low-level tipping is important early in the vorticity growth period. The simulation results presented herein may have underrepresented the processes at the lowest levels of the atmosphere, or alternatively, these cases may represent another mode of formation not represented in the four cases analyzed herein.

Regarding the two modes of MCV formation proposed, the results of Zhang and Fritsch (1988) and Bartels and Maddox (1991) are consistent with the strong heating with large vertical extent mode. Although it is somewhat difficult to compare results because of the style of analysis and data sources, it seems that the findings of Verlinde and Cotton (1990), Skamarock et al. (1994), and Knievel and Johnson (2003) are consistent with the weaker heating and upper-level dynamics mode. The case of Olsson and Cotton (1997a,b) fits the strong heating with large vertical extent scenario with the exception that tipping is also important at the lower levels. Scott and Rutledge (1995) does not seem to fit into either scenario since the heating does not appear to be strong, yet tipping was not a major factor in the MCV evolution. Thus prior literature, in some cases, is consistent with the two modes of formation proposed.

7. Conclusions

The phase-plot method of comparison when applied to MCVs depicts the evolutionary path of both dynamical and thermodynamical fields. The direct comparison it allows simplifies pinpointing the similarities and differences between cases and reveals important details regarding different modes of MCV formation.

The new method of comparing MCV growth has led to the hypothesis that the fundamental nature of MCV formation is contained within the existence of four atmospheric regimes that are the four layers of the troposphere defined herein. These are the convergence-dominated, heating gradient-dominated, heating-dominated, and vertical advection-dominated regions working from the bottom to the top of the troposphere. The similarities in the vorticity growth forcing support the hypothesis that these four regimes exist for MCV formation cases. What creates a difference between cases is the relative strengths and location of the heat-

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ing and the heating gradient and how these quantities change in time, the existence of either storm generated or ambient vertical shear and its location, the large-scale forcing that is present, and the timing of events. Each of the cases examined here show how the MCV can form via different evolutionary paths that have both the similarities and differences mentioned. Two modes of formation are proposed: one that requires strong heating with large vertical extent and the other that involves weaker heating and upper-level dynamics. Prior literature, in some cases, is consistent with the proposed regime structure and two modes of formation. However, there is still more work to be carried out to determine additional modes of formation should they exist and whether or not the proposed regime structure of the atmosphere must be modified if it is to include all MCV cases.

While the phase-plot technique has been used exclusively on model output in the four cases presented herein, it is possible that the technique could be used with observational data. There are several possibilities, but perhaps the most straightforward would be to use the National Oceanic and Atmospheric Administration (NOAA) Profiler Network (NPN) data processed to produce gridded wind data, as in Knievel and Johnson (2002). Thermodynamic data could partly come from National Weather Service soundings, but supplemental data would be needed to improve temporal resolution. Should there be no additional soundings available, an acceptable model run may be used in tandem with observations. Either way, after the MCV location is identified, acceptable thermodynamic data would have to be added to complete the phase plot. Another possibility would be to use data from the Bow Echo and MCV Field Experiment (BAMEX) to construct phase plots for MCV cases where the vortex growth stages were captured.

Should only kinematic data be available, the phase plot could be constructed without color and linestyle detail in order to assess the dynamics alone. If data are filtered as in Knievel and Johnson (2002), the intriguing possibility of creating two phase plots for each case exists, one for mesoscale wind and the other for synoptic wind. Even without thermodynamic data, this could yield interesting results by showing differing mesoscale and synoptic-scale behavior.

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APPENDIX

Vorticity Equation in the Inertial Frame

Transforming Eq. (3) to the earth’s center frame of reference can be accomplished by using the following set of equations:

\[
\omega = \omega_e, \\
\mathbf{v} = \mathbf{v}_e, \\
u = u_e - u_R, \\
\zeta_e = \zeta + f, \quad \text{and} \quad u_R = \Omega a \cos \phi,
\]

where the subscript \(e\) denotes the quantity from a reference frame fixed with respect to the earth’s center. The symbol \(\Omega\) is the angular rate of rotation of the earth, \(a\) is the radius of earth, \(\phi\) is the latitude, and \(u_R\) is the movement of the earth’s surface relative to its center. Because the time rate of change of \(f\) is zero, it is true that

\[
\frac{\partial \zeta_e}{\partial t} = \frac{\partial \zeta_e}{\partial t}.
\]

To change to the nonrotating frame, a horizontal advection of vorticity term must be added to the rate of change of absolute vorticity:

\[
\left( \frac{\partial \zeta_e}{\partial t} \right)_e = \frac{\partial \zeta_e}{\partial t} - u \nabla \cdot \nabla \zeta_e,
\]

where the time derivative on the left is now the local rate of change of absolute vorticity in the earth’s frame.
of reference. The same must added to the rhs of Eq. (2) as well, and when doing so, this yields
\[-\nabla \cdot (V(\zeta + f)) - u_{p}\nabla \cdot \nabla \zeta = -\nabla \cdot (V_{e} \zeta),\]
(A4)
where \((\nabla \cdot u_{p})\zeta = 0\) has been used to simplify the equation.

Finally, the second term on the rhs of Eq. (2) is transformed by first expanding the term
\[k \cdot (\nabla \times \omega \cdot \frac{\partial V}{\partial p}) = k \cdot (\nabla \times \omega \cdot \frac{\partial V_{e}}{\partial p})\]
\[-k \cdot (\nabla \times \omega \cdot \frac{\partial u_{p}}{\partial p}),\]
(A5)
and recognizing that scaling arguments allow for the approximation
\[k \cdot (\nabla \times \omega \cdot \frac{\partial u_{p}}{\partial p}) \approx 0,\]
(A6)
so that
\[k \cdot (\nabla \times \omega \cdot \frac{\partial V}{\partial p}) \approx k \cdot (\nabla \times \omega \cdot \frac{\partial V_{e}}{\partial p}).\]
(A7)
Using Eqs. (A2), (A3), (A4), and (A7), Eq. (2) can be transformed to the earth’s frame, yielding
\[
\left(\frac{\partial \zeta_{e}}{\partial t}\right) = -\nabla \cdot (V_{e} \zeta_{e}) - k \cdot (\nabla \times \omega \cdot \frac{\partial V_{e}}{\partial p}).\]
(A8)
The first term on the rhs of Eq. (A8) can be rewritten as
\[-\nabla \cdot (V_{e} \zeta_{e}) = -k \cdot \nabla \times [(V_{e} \cdot \nabla)V_{e}].\]
(A9)
Substituting Eq. (A9) into Eq. (A8) produces Eq. (4).

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