

## Theoretical Considerations for the Analysis of Population Coding in Motor Cortex

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Recent evidence of population coding in motor cortex has led some researchers to claim that certain variables such as hand direction or force may be coded within a Cartesian coordinate system with respect to extrapersonal space. These claims are based on the ability to predict the rectangular coordinates of hand movement direction using a "population vector" computed from multiple cells' firing rates. I show here that such a population vector can always be found given a very general set of assumptions. Therefore the existence of a population vector constitutes only weak support for the explicit use of a particular coordinate representation by motor cortex.

### 1 Introduction

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Recent results suggest that the representation of arm movement in motor cortex involves the simultaneous activity of many cells, and that the pattern of activation over the group of cells specifies the motion that occurs (Caminiti *et al.* 1990; Kalaska and Crammond 1992, for review). These results have led many researchers to ask whether movement variables are coded internally in terms of a particular coordinate system such as a Cartesian or polar representation of extrapersonal space, a representation of muscle lengths around relevant joints, or some other set of coordinates. In the following, I distinguish between a coded variable (such as hand position) and the coordinates used to represent that variable (such as Cartesian coordinates). I will summarize certain experiments that demonstrate that hand movement direction is represented within motor cortex, but I claim that these experiments cannot be used to determine the coordinate system in which movements are coded.

I discuss a set of experiments that investigated the relationship between cell firing rates during free arm movements in awake monkeys, and the direction in which the hand was moved to a target in space (Georgopoulos *et al.* 1988; Kettner *et al.* 1988; Schwartz *et al.* 1988). These experiments led to the following results:

- R1: The firing rate of 89.1% (486/568) of the cells tested in motor cortex varied consistently with the direction of hand motion within a

limited region of space, and many cells would be simultaneously active for any given direction.

- R2: A statistically significant component of the variance of the firing rate of 83.6% of the cells could be accounted for by a broadly tuned function of the form

$$d_i(M) \approx b_i + k_i \cos(\theta_i - \theta_M) \quad (1.1)$$

where  $d_i(M)$  is the firing rate of cell  $i$  for hand motion in the direction of a unit vector  $M$ ,  $\theta_i$  is the direction of motion in which the cell has maximal response,  $\theta_i - \theta_M$  is the angle<sup>1</sup> between the direction of hand motion  $\theta_M$  and the cell's preferred direction, and  $b_i$  and  $k_i$  determine the average firing rate and modulation depth, respectively. (Here and in the following, capital letters indicate vector quantities.)

- R3: The preferred directions  $\theta_i$  are approximately uniformly distributed with respect to directions in the workspace.
- R4: The hand direction vector  $M$  can be approximated in Cartesian coordinates by a population vector  $P$  computed from a linear combination of the cell firing rates.
- R5: The coefficients of this linear combination are given by unit vectors  $C_i$  along the preferred direction  $\theta_i$  for each cell, so that

$$M \approx P = \sum_{i=1}^N C_i d_i \quad (1.2)$$

where the  $d_i$  have been normalized to account for resting firing rate and response amplitude, and both the movement direction  $M$  and the preferred direction vectors  $C_i$  are given in Cartesian coordinates with respect to the external workspace.

Together, these results might suggest that a Cartesian representation of the direction of hand motion is coded in motor cortex (Schwartz *et al.* 1988).

I will show that results R2, R4, and R5 are direct consequences of results R1, R3, and the experimental design. This in no way reduces the importance of these experiments, but rather emphasizes the fact that results R1 and R3 contain the most significant information. Although their importance was recognized in Georgopoulos *et al.* (1988), the fact that they imply the other results was not. Previous investigations have studied the conditions under which results R4 and R5 hold, and it has been shown that the population vector predicts the direction of hand motion if

<sup>1</sup>A difference of 3D angles is defined by  $\theta_i - \theta_M = \cos^{-1}(C_i \cdot M)$ , where  $C_i$  and  $M$  are both unit vectors.

the tuning curve is symmetric and the distribution of preferred directions is uniform (Georgopoulos *et al.* 1988) or has no second harmonic components (Mussa-Ivaldi 1988). I derive a necessary and sufficient condition that is even broader, since it requires only that the three components of the preferred directions be uncorrelated with each other over the population. Before doing this, I will first show that the cosine tuning curves found in (Schwartz *et al.* 1988) may be an artifact of the analytic techniques used.

## 2 Single Unit Tuning Curves

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In Schwartz *et al.* (1988), the firing rate of each tuned cell  $d_i(M)$  is approximated by a linear combination of the normalized Cartesian coordinates of the target toward which the monkey is reaching. These coordinates are relative to the initial hand position  $X_0$  and are given by a unit vector in the direction of motion  $M = (m_x, m_y, m_z)$ . The linear approximation is

$$d_i(M) \approx b_i + b_{ix}m_x + b_{iy}m_y + b_{iz}m_z \quad (2.1)$$

and an  $F$  test showed that the variance of 83.6% of all cells was at least partly accounted for by this linear regression. The preferred direction vector  $C_i$  is calculated from

$$k_i = (b_{ix}^2 + b_{iy}^2 + b_{iz}^2)^{1/2} \quad (2.2)$$

$$C_i = \langle b_{ix}/k_i, b_{iy}/k_i, b_{iz}/k_i \rangle \quad (2.3)$$

and we can now write

$$d_i(M) \approx b_i + k_i C_i \cdot M \quad (2.4)$$

which is equivalent to equation 1.1. Note that cells with  $k_i = 0$  are not sensitive to the direction of movement and were not analyzed further, so  $k_i \neq 0$ .

To understand results R1 and R2, I perform a simplified analysis of movement in two dimensions (the extension to three dimensions is straightforward but complicates the notation significantly). For a fixed initial hand position and with all other variables held constant, consider any arbitrary firing rate function  $d(\theta_M)$  that depends on the direction of hand movement  $\theta_M$ .  $\theta_M$  is a periodic variable, so the output of  $d(\theta_M)$  will be periodic, and if eight uniformly spaced directions are tested then the complete behavior can be described by a discrete Fourier series with periods up to  $\pi/2$

$$d(\theta_M) = \sum_{k=0}^4 \alpha_k \cos(k\theta_M + \phi_k) \quad (2.5)$$

where  $\phi_k$  is the phase for each angular frequency component  $k$ . Note that for  $k > 1$  the terms have no directional component, since they consist of

either two, three, or four “lobes” symmetrically placed around the circle. Thus a linear regression on the Cartesian coordinates  $x = \cos(\theta_M)$ ,  $y = \sin(\theta_M)$  will be unaffected by the values of  $\alpha_2, \alpha_3$ , and  $\alpha_4$  and will depend only on  $\alpha_0$  and  $\alpha_1$ . To see this, note that linear regression computes the three expected values:

$$\begin{aligned} E[d(\theta_M)] &= E\left[\sum_{k=0}^4 \alpha_k \cos(k\theta_M + \phi_k)\right] \\ &= \alpha_0 \\ E[d(\theta_M)x] &= E\left[\sum_{k=0}^4 \alpha_k \cos(k\theta_M + \phi_k) \cos(\theta_M)\right] \\ &= E[\alpha_1 \cos(\theta_M + \phi_1) \cos(\theta_M)] \\ &= \left(\frac{\alpha_1}{2}\right) \cos(\phi_1) \\ E[d(\theta_M)y] &= E\left[\sum_{k=0}^4 \alpha_k \cos(k\theta_M + \phi_k) \sin(\theta_M)\right] \\ &= E[\alpha_1 \cos(\theta_M + \phi_1) \sin(\theta_M)] \\ &= \left(\frac{\alpha_1}{2}\right) \sin(\phi_1) \end{aligned}$$

where the expectation operator  $E[\ ]$  is taken over all tested directions  $\theta_M$ . The preferred direction is therefore equal to  $\phi_1$  and is independent of  $\phi_2, \phi_3$ , or  $\phi_4$ . Even if more than eight directions are tested, the linear regression will respond only to the  $\phi_1$  component.

The “goodness of fit” to the linear regression is the extent to which the  $k = 0$  and  $k = 1$  terms capture the behavior of  $d(\theta_M)$ . However, it is important to realize that a statistically significant  $F$  test does not indicate a good fit to a linear model in the sense of having small prediction error variance. Fit is determined by mean squared error, which distributes according to a  $\chi^2$  statistic. The  $F$  test estimates only the probability that the linear model accounts for some portion of the total variance. This is equivalent to testing if  $\alpha_1$  is significantly different from 0. A significant  $F$  test does not imply that  $\alpha_1$  describes the dominant response behavior, and  $\alpha_2, \alpha_3$ , or  $\alpha_4$  might well be larger. If a set of tuning curves were generated randomly by selecting the coefficients  $\alpha_k$  independently from a normal distribution, then one would expect 95% of the tuning curves to have statistically significant values of  $\alpha_1$ . Thus the observed value of 83.6% [93% in Caminiti *et al.* (1990)] does not support statistical arguments that the population has been “engineered” to have directional tuning.

Since this method of analysis ignores terms for  $k > 1$ , it in effect low-pass filters the tuning curves. So the cosine tuning results from the method of analysis and may not be justified by the original data. These considerations show that result R2 does not provide any information beyond result R1, since R2 would be true for a randomly chosen set of tuning curves satisfying R1 that were analyzed in this way.

True cosine tuning could be verified by fitting equation 1.1 to data samples from many different directions and measuring the average mean-squared approximation error over the population using a  $\chi^2$  statistic. A similar test was done in the two-dimensional case, where it was found that 75% of 241 cells had a normalized mean-squared approximation error less than 30% of the total variance (Georgopoulos *et al.* 1982). Although this is not a statistically good fit to the population, there may have been individual cells whose response was well predicted by cosine tuning.

What is the significance of the cells that were well fit by a cosine tuning curve? As shown in equation 2.4, these cells have a response  $d$  that is approximately linearly related to the hand movement vector  $M$ . We can thus claim either that these cells are in fact linear in the movement direction, or else that they are linear in the testing region but may be nonlinear if tested in other regions of space. So if we write the response as  $d(X_0, X)$  where  $X_0$  is the initial hand position and  $X$  is the target, then we know that  $d(X_0, X)$  must be sufficiently smooth that it appears locally linear for the positions  $X$  that were tested.

Over larger distances,  $d$  may not be well approximated linearly, but it can still be written as

$$d(M) \approx b + kC(X_0) \cdot M \quad (2.6)$$

where  $C(X_0)$  emphasizes that the preferred direction may become dependent on the initial position, as was indeed found in Caminiti *et al.* (1990). But equation 2.6 is a general representation for arbitrary smooth functions, so even an accurate fit to a locally linear function does not allow one to claim much beyond the fact that the preferred direction remains approximately constant over the tested region.

### 3 Population Vectors

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Result R4 that there exists a linear combination of the firing rates that can predict the Cartesian coordinates of hand motion follows as a direct consequence of well-known results on coarse coding and the theory of radial basis functions (Poggio and Girosi 1990, for example), since a raised cosine function of angle can be thought of as a local basis function centered on the preferred direction. An alternate way to prove this fact follows. Define an  $N \times 3$  matrix  $Q$  whose rows are the preferred direction vectors  $C_i$ . Let  $D$  be an  $N$ -dimensional column vector formed from the firing rates of all the cells by the formula  $[D]_i = (d_i - b_i)/k_i$  as in Georgopoulos *et al.* (1988). Then from equation 2.4 we can write

$$D \approx QM \quad (3.1)$$

We seek a  $3 \times N$  weighting matrix  $H$  such that a population vector of the form  $HD$  predicts hand direction  $M$  according to

$$M \approx HD \approx HQM \quad (3.2)$$

There are many matrices  $H$  that will satisfy this equation. One possibility is to use linear least-squares regression, giving

$$H = (Q^T Q)^{-1} Q^T \tag{3.3}$$

where the inverse  $(Q^T Q)^{-1}$  will always exist so long as there are three linearly independent preferred direction vectors. We now have

$$M \approx HD \approx HQM = (Q^T Q)^{-1} Q^T Q M = M \tag{3.4}$$

as desired. This equation means that so long as there exist three linearly independent direction vectors, the hand direction will be approximately linearly related to the cell firing rates  $d_i$  in any coordinate system for  $M$  that satisfies equation 3.1. So far I have shown that result R1 implies both results R2 and R4, given the method of analysis.

In Georgopoulos *et al.* (1988) the columns of  $H$  were not found by performing a regression of the cell firing rates against the hand direction according to equation 3.3, but instead were assumed a priori to be equal to the preferred direction  $C_i$  for each cell, so that  $H = Q^T$ . I now discuss under what conditions result R5 holds, so that this particular linear combination will give the right answer. The population vector is given by equation 1.2, which we can rewrite in vector notation as

$$M \approx P = Q^T D \approx Q^T Q M \tag{3.5}$$

and if this holds for all directions  $M$  then we must have  $Q^T Q = I$ . This is a necessary condition for the existence of a population vector. In Georgopoulos *et al.* (1988) a more restrictive sufficient condition satisfying equation 1.2 is that the distribution of preferred directions is uniform over the sphere. Another necessary and sufficient condition based on Fourier analysis of the distribution of preferred directions for the planar case is given in Mussa-Ivaldi (1988).

To understand the meaning of equation 3.5, we can write each component of  $Q^T Q$  as

$$[Q^T Q]_{jk} = \sum_{i=1}^N [C_i]_j [C_i]_k \tag{3.6}$$

and  $I = Q^T Q$  implies that  $\sum_{i=1}^N [C_i]_j [C_i]_k = 0$  whenever  $j \neq k$ . This expression is the correlation of the  $j$ th and  $k$ th components of the preferred direction vectors  $C_i$ , so a necessary and sufficient condition for equation 1.2 to work is that the  $x$ ,  $y$ , and  $z$  components of these vectors are uncorrelated and have equal variance. The result that equation 1.2 is satisfied is thus implied by the approximately uniform distribution of cell preferred directions in result R3. Note that for other coordinate systems, even if the components of the  $C_i$ s are correlated there will still exist a linear combination  $H \neq Q^T$  of the firing rates that will predict the desired values, although the matrix  $H$  may need to be found by regression using

equation 3.3. But if both results R1 and R3 hold, then result R5 must hold.

Suppose that rather than using the predicted value  $QM$  we use the true measured value  $D$  and this includes significant noncosine (nonlinear) terms. Then we have

$$D = QM + E$$

where  $E$  is a vector with components

$$e_i(\theta_M) = \sum_{k=2}^4 \alpha_{ik} \cos(k\theta_M + \phi_{ik})$$

If the terms  $\alpha_{ik}$  and  $\phi_{ik}$  are distributed independently of the components of  $C_i$ , then  $Q^T E = 0$  and these terms will not affect the value of the population vector. So even if the individual cells do not have cosine tuning, the population vector will correctly predict hand direction if the terms for  $k > 1$  do not correlate with the terms for  $k = 1$  in the expansion given in equation 2.5.

If the experiments are repeated with differing initial positions as in Caminiti *et al.* (1990), then the preferred directions  $C_i$  may change. This will lead to a new matrix  $Q'$  so that  $D' = Q'M$ . Population vector analysis under the new conditions will give  $P' = Q'^T Q'M$ , so again the requirement for success is that the components of the new preferred directions are uncorrelated. The fact that population vectors "proved to be good predictors of movement direction regardless of where in space the movements were performed" (Caminiti *et al.* 1990, p. 2039) provides no information beyond the knowledge that the components of the preferred directions remain uncorrelated as the initial hand position changes.

#### 4 Coordinate-Free Representations

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One might ask if the experiments described above could be modified to determine the "true" coordinate system used by motor cortex to describe hand movement direction. However, I claim that for certain classes of distributed representation this is not a well-defined question. Distributed representations of measured variables can be coordinate-free in the sense that they do not imply any particular coordinate system. To see this, let  $X$  be any variable represented in cortex (such as hand movement direction), and let  $D(X)$  be a vector-valued function representing the outputs of a large set of basis functions  $d_i(X)$  that describe the behavior of (motor) cortical cells.  $D(X)$  is then a distributed representation of the variable  $X$ . Now, consider a vector function  $T(X)$  that measures  $X$  in a particular coordinate system [ $T(X)$  might give the three Cartesian components of hand movement direction, for example]. If there exists a matrix  $H$  such that  $HD(X) \approx T(X)$ , then one can say that the distributed representation  $D$  codes the coordinate system  $T$ . Yet this will hold for any  $T(X)$  that is

close to the linear span of the basis functions  $d_i(X)$ , so we cannot claim that  $D$  encodes any single coordinate system for  $X$  within this span better than another.

## 5 Conclusion

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In this letter, I have extended the generality of previous results (Georgopoulos *et al.* 1988; Mussa-Ivaldi 1988) to show that cosine tuning curves will be found for large classes of arbitrary response functions if they are analyzed according to the statistical techniques in Schwartz *et al.* (1988), and that the existence of a population vector as found in Georgopoulos *et al.* (1988) is determined by very general necessary and sufficient conditions that depend only on the distribution of preferred directions rather than on any intrinsically coded coordinate system. The concept that a distributed representation codes a particular coordinate system may not be well-defined, since certain types of representations can be considered "coordinate-free." These considerations imply that experiments of the type described may yield population vectors which predict many different three-dimensional coordinates (such as Cartesian, polar, muscle lengths, or joint angles).

It is important to understand that the considerations presented here in no way reduce the importance of the results reported in Schwartz *et al.* (1988), Georgopoulos *et al.* (1988), Kettner *et al.* (1988), Caminiti *et al.* (1990), and elsewhere. The fact that results R2, R4, and R5 are direct consequences of R1 and R3 serves only to underscore the significance of these two results. They show that large populations of motor cortical cells respond to hand motion in a predictable way, and that the preferred directions are approximately uniformly distributed with respect to a Cartesian representation of extrapersonal space. No additional conclusions can be drawn from the population vector, since its existence is a mathematical consequence of these two facts. However, if the distribution of preferred directions is nonuniform with respect to other coordinate systems or if the distribution can be modified through experience, then this would provide significant information about cortical representations. In addition, if cosine tuning can be verified by explicitly fitting cell tuning curves to a linear regression model, then further studies may discover constraints that explain why more than 486 linear cells are needed to code for only 3 linearly independent components of hand direction.

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