Sampled-Data State Feedback Stabilization of Boolean Control Networks

Yang Liu
liuyang@zjnu.edu.cn
Department of Mathematics, Southeast University, Nanjing 210096, China, and College of Mathematics, Physics and Information Engineering, Zhejiang Normal University, Jinhua 321004, China

Jinde Cao∗
jdcao@seu.edu.cn
Research Center for Complex Systems and Network Sciences, Department of Mathematics, Southeast University, Nanjing 210096, China

Liangjie Sun
1546258649@qq.com
Department of Mathematics, Southeast University, Nanjing 210096, China, and College of Mathematics, Physics and Information Engineering, Zhejiang Normal University, Jinhua 321004, China

Jianquan Lu
jqluma@seu.edu.cn
Department of Mathematics, Southeast University, Nanjing 210096, China

In this letter, we investigate the sampled-data state feedback control (SDSFC) problem of Boolean control networks (BCNs). Some necessary and sufficient conditions are obtained for the global stabilization of BCNs by SDSFC. Different from conventional state feedback controls, new phenomena observed the study of SDSFC. Based on the controllability matrix, we derive some necessary and sufficient conditions under which the trajectories of BCNs can be stabilized to a fixed point by piecewise constant control (PCC). It is proved that the global stabilization of BCNs under SDSFC is equivalent to that by PCC. Moreover, algorithms are given to construct the sampled-data state feedback controllers. Numerical examples are given to illustrate the efficiency of the obtained results.

∗Corresponding author.
1 Introduction

Boolean networks (BNs), the simplest logical dynamic systems, were first proposed by Kauffman (1969) for modeling complex and nonlinear biological systems. In BNs, each state variable attains two possible values (0 or 1, true or false), and its update is governed by logical functions at each discrete time. It has been proved to be a powerful tool for describing, analyzing, and simulating the cellular networks and gene regulatory networks (Davidson et al., 2002). Despite the fact of their conceptual simplicity, BNs can capture basic dynamic behavior and provide useful information on many real systems. Boolean control networks (BCNs) are BNs with Boolean inputs. In fact, a BCN can be regarded as a switched system with switching among different BNs (Fornasini & Valcher, 2013).

The study of BNs and BCNs has received great attention since the semi-tensor product (STP), a new matrix product, was proposed by Cheng, Qi, and Li (2011) and has been successfully applied to express and analyze BNs. Using STP, BCNs and BNs can be easily converted into algebraic forms as discrete systems. Then the analysis on BCNs and BNs turns to be much easier. Many fundamental and interesting results have been obtained for BCNs and BNs, such as fixed points, cycles, basin of attractors (Cheng et al., 2011; Fornasini & Valcher, 2013), as well as the systematic analysis of BCNs, for example, controllability (Cheng & Qi, 2009; Laschov & Margaliot, 2012; Liu, Lu, & Wu, 2014; Liu, Chen, Lu, & Wu, 2015; Lu, Zhong, Huang, & Cao, 2015), and optimal control (Laschov & Margaliot, 2011, 2013). Complete synchronization and partial synchronization for BNs have been studied in Li & Lu (2013a, 2013b) and Li (2014). The synchronization of BNs with impulsive effects and time delays has been investigated by Lu et al. (2014) and Zhong, Lu, Liu, and Cao (2014), and the synchronization of master-slave probabilistic Boolean networks also been considered in Lu, Zhong, Li, Ho, and Cao (2015). The stabilization of BCNs has been investigated in Li, Yang, and Chu (2013) and Zhao and Cheng (2014), and the construction of the feedback controller was introduced in Li, Yang, and Chu (2013, 2014), Li and Wang (2013), Liu, Sun, Lu, and Liang (2015), and Liu, Li, and Lou (2016).

The problem of stabilization for many dynamical systems sometimes cannot be well solved by continuous state feedback controllers. Therefore, it is necessary to design time-dependent or discontinuous feedback controllers to stabilize systems. The idea of using discontinuous stabilizers instead of continuous stabilizers has been broadly discussed. For example, Coron (1992) considered the use of time-dependent continuous feedback laws. And Emel’yanov, Korovin, and Nikitin (1989) discussed discontinuous feedback controls.

Moreover, it is natural to consider discontinuous controllers because the sampled data, obtained from sensors, transmitters, and controllers, for example, are often discontinuous. A vast framework also focuses on sampled...
data techniques concerning system and control. For wide application in the real world, mainly due to the flexibility induced by the adoption of digital controllers (Chen & Francis, 2015) and communication networks (Wang & Liu, 2008) in the architecture of the closed-loop system, the controller design problem using sampled data has attracted considerable attention (Naghshtabrizi, Hespanha, & Teel, 2006). In the control community, the problems of sampled-data control have been well investigated, and important results have been proposed for linear systems (Fridman, Seuret, & Richard, 2004; Fridman, 2010), nonlinear systems (Lam, 2011), chaotic systems (Wu, Shi, Su, & Chu, 2014a, 2014b; Cao, Sivasamy, & Rakkaiyappan, 2015), and complex dynamical networks (Wu, Shi, Su, & Chu, 2013; Rakkiyappan, Sakthivel, & Cao, 2015). Rodrigues (2005) and Geromel and Gabriel (2015) studied a kind of sampled-data control called sampled-data state feedback control (SDSFC). For discrete-time systems, if the sampling period is one, then the SDSFC can be regarded as a normal state feedback control. Therefore, SDSFC is also a generalization of the state feedback control to some extent. For BCNs, the problem of state feedback control has been well investigated (Fornasini & Valcher, 2013; Li & Wang, 2013; R. Li et al., 2013, 2014; Zhao & Cheng, 2014). However, the SDSFC problem for BCNs is still open and challenging, and to the best of our knowledge, there is no result on the construction of sampled-data controllers for BCNs as well.

Piecewise constant control (PCC) (Nikitin, 1999; Schaller, 2014) is another kind of discontinuous control, which is defined on each sampling interval to stabilize some dynamical systems or networks. When a control parameter is switched instantaneously from one value to another, it can take the solution as the new initial state and evolve it further until the next switching event occurs. The PCC allows one to account for higher-order terms normally neglected while using a discretizing sampled data controller or a continuous feedback controller (Voytsekhovsky & Hirschorn, 2007). There are many interesting results on PCC of systems. For example, constructing a PCC for a continuous Roesser system has been studied in Majewski (2005). An algorithm for PCC design in minimal time has been proposed by Lefebvre (2011).

In this letter, we study the SDSFC problem for BCNs with constant sampling period. We first give some necessary and sufficient condition, under which a BCN can be globally stabilized by SDSFC. Some differences between SDSFC and state feedback control for BCNs are noted. We then discuss the PCC stabilization problem. We show that a BCN can be globally stabilized by SDSFC only if it can be globally stabilized under PCC. As a special case with the sampling period being one, the results reduce to the state feedback control case as obtained in Fornasini and Valcher (2013). Two kinds of algorithms are presented to construct SDSFCs: a direct method from the analysis of SDSFCs and the other based on PCCs and transition
matrices of BCNs. The latter one is observed to be much easier to use for the construction.

2 Preliminaries

2.1 Semitensor Product of Matrices. We first give some basic notation. Let \( \Omega_k = \{1, 2, \ldots, 2^k\} \). Define a delta set as \( \Delta_n := \{\delta_i^n | i = 1, 2, \ldots, n\} \), where \( \delta_i^n \) is the ith column of identity matrix \( I_n \). An \( n \times m \) matrix \( L \) is called a logical matrix if for all \( 1 \leq j \leq m \), the jth column of \( L \), denoted by \( Col_j(L) \), is in \( \Delta_n \). Let \( Col(L) \) be the collection of all the columns of \( L \). Denote the set of \( n \times m \) logical matrices by \( L_{m \times n} \). If \( L \in L_{m \times n} \), it can be expressed as \( L = [\delta_m^1, \delta_m^2, \ldots, \delta_m^n] \) or simplified by \( L = \delta_m^1[n, \delta_m^2, \ldots, \delta_m^n] \).

Definition 1 (Cheng et al., 2001). A swap matrix \( W_{[m, n]} \) is an \( mn \times mn \) matrix, defined as

\[
W_{[m, n]} = \delta_{mn}[1, m + 1, 2m + 1, \ldots, (n - 1)m + 1, 2, m + 2, 2m + 2, \ldots, (n - 1)m + 2, \ldots, m, 2m, 3m, \ldots, nm].
\]

When \( m = n \), we briefly denote \( W_{[n]} := W_{[m, n]} \).

Lemma 1 (Cheng et al., 2011). Let \( X \in R_m \) and \( Y \in R_n \) be two column vectors. Then \( W_{[m, n]} XY = YX \). For a given \( A \in M_{m \times n} \), and \( Z \in R_t \), one has \( ZA = W_{[m, t]}AW_{[t, n]}Z = (I_t \otimes A)Z \).

Definition 2 (Cheng et al., 2011). The semitensor product of two matrices \( A \in M_{m \times n} \) and \( B \in M_{p \times q} \) is defined by

\[
A \times B = (A \otimes I_{\alpha/n})(B \otimes I_{\alpha/p}),
\]

where \( \alpha = \text{lcm}(n, p) \) is the least common multiple of \( n \) and \( p \) and \( \otimes \) is the tensor (or Kronecker) product.

When \( n = p \), \( A \times B = (A \otimes I_1)(B \otimes I_1) = AB \). So STP is a generalization of the conventional matrix. We simply call it product and omit the symbol \( \times \) if no confusion arises.

Lemma 2 (Cheng et al., 2011). Let \( x = x_1, x_2, \ldots, x_n \) with \( x_i \in \Delta_2 \), \( (i = 1, 2, \ldots, n) \); then \( x^2 = \Phi_n x \), where \( \Phi_n = \delta_{2^n}[1, 2^n + 2, 2 \cdot 2^n + 3, \ldots, (2^n - 2) \cdot 2^n + 2^n - 1, 2^{2n}] \).
2.2 Algebraic Representations of Boolean Networks. Using matrix expression, we denote a logical domain by $\mathcal{D}$ in which 1 and 0 are represented by $\delta_1$ and $\delta_2$, respectively. Therefore, $\mathcal{D}$ equals $\Delta_2$. Then a logical function with $n$ arguments $f : \mathcal{D}^n \rightarrow \mathcal{D}$ can be expressed in the algebraic form by using STP of matrices.

**Lemma 3** (Cheng et al., 2011). Let $f(x_1, x_2, \ldots, x_n) : \mathcal{D}^n \rightarrow \mathcal{D}$ be a logical function. Then there exists a unique matrix $M_f \in L_{2 \times 2^n}$, called the structural matrix of $f$, such that

$$f(x_1, x_2, \ldots, x_n) = M_f \star_{i=1}^n x_i, \; x_i \in \Delta_2,$$

(2.1)

where $\star_{i=1}^n x_i = x_1 \star \cdots \star x_n \in \Delta_2^n$.

In this letter, we consider the Boolean control network as

$$\begin{aligned}
\begin{cases}
  x_1(t+1) = f_1(x_1(t), \ldots, x_n(t), u_1(t), \ldots, u_m(t)), \\
  \vdots \\
  x_n(t+1) = f_n(x_1(t), \ldots, x_n(t), u_1(t), \ldots, u_m(t))
\end{cases},
\end{aligned}$$

(2.2)

for $t \geq 0$, where $f_i : \mathcal{D}^n \rightarrow \mathcal{D}$, $1 \leq i \leq n$ are logical functions, $x_i \in \mathcal{D}$, $1 \leq i \leq n$, and $u_j \in \mathcal{D}$, $1 \leq j \leq m$ are states and control inputs, respectively. Using lemma 3, for each logical function $f_i$, $i = 1, 2, \ldots, n$, we can find its unique structure matrix $M_i$. Let $x(t) = \star_{i=1}^n x_i(t)$, $u(t) = \star_{i=1}^m u_i(t)$. Then system 2.2 can be converted into an algebraic form as

$$x_i(t+1) = M_i u(t) x(t), \; i = 1, \ldots, n.$$  

(2.3)

Multiplying the equations in equation 2.3 together yields

$$x(t+1) = Lu(t) x(t), \quad t \geq 0,$$

(2.4)

where $L = M_1 \star_{i=2}^n [(l_{2i-1} \otimes M_i) \Phi_{m+n}] \in L_{2^n \times 2^{n+m}}$.

The feedback law to be determined for system 2.2 is in the following form,

$$\begin{aligned}
\begin{cases}
  u_1(t) = e_1(x_1(t_1), \ldots, x_n(t_1)), \\
  \vdots \\
  u_m(t) = e_m(x_1(t_1), \ldots, x_n(t_1)),
\end{cases} \quad t_1 \leq t < t_{1+1},
\end{aligned}$$

(2.5)

where $e_j : \mathcal{D}^n \rightarrow \mathcal{D}$, $1 \leq j \leq m$ are Boolean functions, $t_l = l \tau \geq 0$ for $l = 0, 1, \ldots$ are sampling instants, and $t_{l+1} - t_l = \tau$ denotes the constant
sampling period. Similar to the analysis above, equation 2.5 is equivalent to the following algebraic form,

$$u(t) = Ex(t_l), \quad t_l \leq t < t_{l+1},$$  \hspace{1cm} (2.6)

where $E = \tilde{M}_1 \bigotimes_{j=2}^m [(I_{2^n} \otimes \tilde{M}_j) \Phi_n] \in \mathcal{L}_{2^n \times 2^n}$, and $\tilde{M}_j$ is the structure matrix of $e_j$.

If $\tau = 1$, then equation 2.6 can be regarded as a normal state feedback control as in Fornasini and Valcher (2013), Li and Wang (2013), R. Li et al. (2013, 2014), Zhao and Cheng (2014), and others.

**Definition 3.** For a given state $X_e = (x^e_1, \ldots, x^e_n) \in D^n$, system 2.4 is said to be globally stabilizable to $X_e$ if there exist a logical control sequence $U = \{u(t), t = 0, 1, 2, \cdots\}$ and an integer $T > 0$, such that $X(t; X_0; U) = X_e$ for $\forall X_0 \in D^n$ and $\forall t \geq T$.

### 3 Main Results

#### 3.1 Sampled-Data State Feedback Control for BCNs.

Considering system 2.4 and the SDSFC, equation 2.6, for $0 \leq t \leq \tau$, we have

$$x(1) = Lex(0)x(0) = LEW_{[2^n]}x(0)x(0) = LEW_{[2^n]} \Phi_n x(0),$$

$$x(2) = Lex(0)x(1) = LEW_{[2^n]}x(1)x(0) = LEW_{[2^n]} \Phi_n x(0),$$

$$= LEW_{[2^n]} (LEW_{[2^n]} \Phi_n x(0)) x(0) = (LEW_{[2^n]})^2 \Phi_n^2 x(0),$$

$$\vdots$$

$$x(t) = (LEW_{[2^n]})^t \Phi_n^t x(0).$$  \hspace{1cm} (3.1)

Similarly, regarding $x(\tau)$ as an initial state for $\tau < t \leq 2\tau$, one gets from equation 3.1 that

$$x(t) = (LEW_{[2^n]})^{t-\tau} \Phi_n^{t-\tau} x(\tau) = (LEW_{[2^n]})^{t-\tau} \Phi_n^{t-\tau} ((LEW_{[2^n]})^\tau \Phi_n^\tau) x(0).$$  \hspace{1cm} (3.2)

Consequently, for $l\tau < t \leq (l+1)\tau$, we have by the induction that

$$x(t) = (LEW_{[2^n]})^{t-l\tau} \Phi_n^{t-l\tau} ((LEW_{[2^n]})^\tau \Phi_n^\tau)^l x(0).$$  \hspace{1cm} (3.3)

**Theorem 1.** System 2.4 can be globally stabilized to $\delta^e_{2^n}$ by SDSFC in the form of equation 2.6, if and only if there exists $k > 0$, such that
\[
\begin{align*}
\left\{ \begin{array}{l}
((L E W_{2^n} \Phi_n)^r)^k = \delta_{2^n}[r \: r \: \cdots \: r], \\
(L E W_{2^n} \Phi_n)_{rr} = 1,
\end{array} \right.
\end{align*}
\]  \tag{3.4}

where \((L E W_{2^n} \Phi_n)_{rr}\) is the \((r, r)\)th element of the matrix \(L E W_{2^n} \Phi_n\).

**Proof.** (Sufficiency) Suppose that equation 3.4 holds. It then follows from equation 3.3 that

\[x(k\tau) = ((L E W_{2^n} \Phi_n)^r)^k x(0) = \delta_{2^n}^r\]

for all \(x(0) \in \Delta_{2^n}\). Then \(x(k\tau + 1) = L E W_{2^n} \Phi_n x(k\tau) = \delta_{2^n}^r\) from equation 3.4, and so does \(x(k\tau + 2)\). Similarly, we have \(x(t) = \delta_{2^n}^r\) for \(k\tau \leq t \leq (k + 1)\tau\).

By induction, one can see that when \(t \geq T \triangleq k\tau\), \(x(t) \equiv \delta_{2^n}^r\) for all \(x(0) \in \Delta_{2^n}\). Sufficiency is proved.

(Necessity) Assume that system 2.4 is globally stabilizable to \(\delta_{2^n}^r\) by SDSFC in the form of equation 2.6. Then there exists \(T > 0\) such that \(\forall t \geq T\), \(x(t) = \delta_{2^n}^r\). Without loss of generality, supposing that \(T = k\tau\) for some \(k > 0\), we have \(x(k\tau) = ((L E W_{2^n} \Phi_n)^r)^k x(0) \equiv \delta_{2^n}^r\) for \(\forall x(0) = \delta_{2^n}^1, 1 \leq i \leq 2^n\). Therefore, the first equation in equation 3.4 holds. Furthermore, \(x(k\tau + 1) = L E W_{2^n} \Phi_n x(k\tau) = L E W_{2^n} \Phi_n \delta_{2^n}^r \equiv \delta_{2^n}^r\). Hence the second equation in equation 3.4 holds as well. Necessity is now proved.

**Remark 1.** When \(\tau = 1\), the SDSFC reduces to state feedback control. Then \(((L E W_{2^n} \Phi_n)^r)^k = \delta_{2^n}[r \: r \: \cdots \: r]\) implies \((L E W_{2^n} \Phi_n)_{rr} = 1\). In fact,

\[
\begin{align*}
(L E W_{2^n} \Phi_n)^{k+1} x(0) &= (L E W_{2^n} \Phi_n) (L E W_{2^n} \Phi_n)^k x(0) \\
&= (L E W_{2^n} \Phi_n) \delta_{2^n}[r \: r \: \cdots \: r] x(0) \\
&= (L E W_{2^n} \Phi_n) \delta_{2^n}^r.
\end{align*}
\]  \tag{3.5}

On the other hand,

\[
\begin{align*}
(L E W_{2^n} \Phi_n)^{k+1} x(0) &= (L E W_{2^n} \Phi_n)^k (L E W_{2^n} \Phi_n) x(0) \\
&= \delta_{2^n}[r \: r \: \cdots \: r] (L E W_{2^n} \Phi_n) x(0) \\
&= \delta_{2^n}^r.
\end{align*}
\]  \tag{3.6}

Combining equations 3.5 and 3.6 gives that \((L E W_{2^n} \Phi_n) \delta_{2^n}^r = \delta_{2^n}^r\), that is, \((L E W_{2^n} \Phi_n)_{rr} = 1\).

When \(r > 1\), this argument does not work, which means that \((L E W_{2^n} \Phi_n)_{rr} = 1\) cannot be guaranteed by \(((L E W_{2^n})^r \Phi_n)^k = \delta_{2^n}[r \: r \: \cdots \: r]\).

This is different from the case \(\tau = 1\). Therefore, both conditions in
equation 3.4 are necessary for $\tau > 1$ in theorem 1. The following example illustrates this point.

Example 1. Let $\tau = 2$, $n = 1$, $r = 1$ and

$$LEW_{[2]} = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad \Phi_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$  

Then

$$(LEW_{[2]})^\tau = (LEW_{[2]})^2 = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

and

$$\Phi_1^\tau = \Phi_1^2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}^T.$$  

As a result,

$$((LEW_{[2]}^2)^2 \Phi_1^2)^1 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}.$$  

However, it is noticed that

$$LEW_{[2]} \Phi_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

and $(LEW_{[2]} \Phi_1)_{11} \neq 1$. Therefore, $(LEW_{[2]} \Phi_1)_{rr} = 1$ is not necessarily guaranteed by $((LEW_{[2]}^r \Phi_1^r)^k = \delta_{2^r}[r r \cdots r]$ for $\tau > 1$.

It should be noticed that only when $t$ is sufficiently large ($t \geq k\tau$, where $k$ is given by equation 3.4) is the state $\delta_{2^n}$ fixed. Otherwise, when $t < k\tau$, although the state of the system reaches $\delta_{2^n}$ by the SDSFC (i.e., $x(t) = \delta_{2^n}$), we cannot guarantee $x(t + 1) = \delta_{2^n}$. It is a very different fact from the normal state feedback controlled system with $\tau = 1$. We use the following example to illustrate the statement.

Example 2. Let $\tau = 3$, $n = 1$, $r = 1$ and

$$LEW_{[2]} = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}, \quad \Phi_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^T.$$
Then

\[(LEW_{2n})^3 \Phi_1^3 \eta = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix},\]

and \((LEW_{[2]} \Phi_1)_{11} = 1\). So the system can be globally stabilized to \(\delta^1\) for \(t \geq 3\) from theorem 1. When \(x(0) = \delta^2, x(1) = LEW_{[2]} \Phi_1 x(0) = \delta^1,\) while \(x(2) = (LEW_{[2]} \Phi_1)^2 x(0) = \delta^2 \neq \delta^1\). It means that \(\delta^1\) is not fixed for \(l = 2 < 3\).

**Lemma 4.** Let \(L = \delta_{2n}[\alpha_1 \alpha_2 \cdots \alpha_{2^n}]\) and assume \(E = \delta_{2n}[p_1 p_2 \cdots p_{2^n}]\). Define sets by

\[S_i(i) = \{x_0 \in \Delta_{2n} : ((LEW_{[2^n]} \Phi_1)^i \eta x_0 = \delta^i_{2^n})\}, 1 \geq 1, 1 \leq i \leq 2^n. \quad (3.7)\]

Then we have the following results.

I. \(S_i(i)\) can be rewritten by a set \(\delta^i_{2^n} : 1 \leq i \leq 2^n, \beta_i^1 = r, \) for some \(1 \leq p_i \leq 2^n\), where

\[\begin{align*}
\beta_i^1 &= \alpha_{(p_i - 1)2^n + i}, \\
\beta_i^{l+1} &= \alpha_{(p_i - 1)2^n + \beta_i}, l \geq 1.
\end{align*} \quad (3.8)\]

II. \(S_{i+1}(r) = \bigcup \{S_i(i) : 1 \leq i \leq 2^n, \delta^i_{2^n} \in S_i(r)\}\) for all \(l \geq 1\).

**Proof.** I. Assuming \(\delta^i_{2^n} \in S_i(r),\) we have from equation 3.7 that

\[
\delta^r_{2^n} = (LEW_{[2^n]})^r \Phi_1^r \delta^i_{2^n}
= (LEW_{[2^n]})^r (\delta^i_{2^n})^r + 1
= (LEW_{[2^n]})^{-1} \left( (LEW_{[2^n]} (\delta^i_{2^n})^2 \right) (\delta^i_{2^n})^{-1}
= (LEW_{[2^n]})^{-1} \left( \alpha_{(p_i - 1)2^n + i} \right) (\delta^i_{2^n})^{-1}
= (LEW_{[2^n]})^{-1} \left( \delta^i_{2^n} \right) (\delta^i_{2^n})^{-1}.
\]

Moreover, for \(1 \leq k \leq r - 1,\)

\[
(LEW_{[2^n]})^{r-k} \left[ \delta^i_{2^n} \right] (\delta^i_{2^n})^{r-k}
= (LEW_{[2^n]})^{r-(k+1)} \left[ (LEW_{[2^n]} \delta^i_{2^n}) (\delta^i_{2^n})^{r-(k+1)}
= (LEW_{[2^n]})^{r-(k+1)} \left[ \delta^i_{2^n} \right] (\delta^i_{2^n})^{r-(k+1)}
= (LEW_{[2^n]})^{r-(k+1)} \left[ \delta^i_{2^n}^{k+1} \right] (\delta^i_{2^n})^{r-(k+1)}.
\]


By induction, we have $\delta^r_{2^i} = \delta^r_{2^i}$ for $k = \tau - 1$. Therefore, $\beta^r_i = r$, and we proved the first part of the lemma.

II. From the definition of $S_l(r)$ in equation 3.7, $S_{l+1}(r) = \{x_0 \in \Delta_{2^i} : ((\text{LEW}_{[2]}^r)^\tau \Phi^r_{n})^l x_0 = \delta^r_{2^i}\}$. We notice that $((\text{LEW}_{[2]}^r)^\tau \Phi^r_{n})^l x_0 = ((\text{LEW}_{[2]}^r)^\tau \Phi^r_{n})^l x_0$. Therefore, $S_{l+1}(r) = \{x_0 \in \Delta_{2^i} : ((\text{LEW}_{[2]}^r)^\tau \Phi^r_{n}) x_0 = \delta^r_{2^i}, \delta^r_{2^i} \in S_l(r)\} = \bigcup \{S_l(i) : 1 \leq i \leq 2^n, \delta^r_{2^i} \in S_l(r)\}$.

**Theorem 2.** System 2.4 is globally stabilizable to $\delta^r_{2^i}$ by SDSFC, equation 2.6, Then

I. $\delta^r_{2^i} \in S_l(r)$.

II. There exists $1 \leq N \leq 2^n$ such that $\Delta_{2^i} = \bigcup_{1 \leq l \leq N} S_l(r) \setminus S_{l-1}(r)$ with $S_0(r)$ denoted by $\emptyset$.

**Proof.** I. If system 2.4 is globally stabilizable to $\delta^r_{2^i}$ by equation 2.6, then there exists $k > 0$ such that $x(t) \equiv \delta^r_{2^i}$ for all $t \geq k\tau$ and $x(0) \in \Delta_{2^i}$. Therefore, $\delta^r_{2^i} = x((k + 1)\tau) = (\text{LEW}_{[2]}^r)^\tau \Phi^r_{n} x(k\tau) = (\text{LEW}_{[2]}^r)^\tau \Phi^r_{n} \delta^r_{2^i}$. From equation 3.7, we conclude that $\delta^r_{2^i} \in S_l(r)$.

II. If $S_l(r) = \{\delta^r_{2^i}\}$, since $S_{l+1}(r) = \bigcup \{S_l(i) : 1 \leq i \leq 2^n, \delta^r_{2^i} \in S_l(r)\}$ for all $l \geq 1$, it follows from lemma 4 that $S_l(r) = \{\delta^r_{2^i}\}$ for any $l > 1$. It is a contradiction to the condition that system 2.4 can be globally stabilizable to $\delta^r_{2^i}$. Therefore, there must be $\delta^r_{2^i} \in S_l(r)$ besides $\delta^r_{2^i}$.

Due to $S_2(r) = \bigcup \{S_l(i) : 1 \leq i \leq 2^n, \delta^r_{2^i} \in S_l(r)\}$ and $\delta^r_{2^i} \in S_l(r)$, then $S_1(r) \subset S_2(r)$. If $S_2(r) \setminus S_1(r) = \emptyset$, then $S_1(r) = S_2(r)$ for any $l > 1$. If $S_1(r) = \Delta_{2^i}$; then $N = 1$. Otherwise it contradicts the fact that system 2.4 can be globally stabilizable to $\delta^r_{2^i}$. Therefore, $S_2(r) \setminus S_1(r) \neq \emptyset$.

Similarly, we can conclude that $S_l(r) \setminus S_{l-1}(r) \neq \emptyset$ for $l = 1, 2, \ldots$. The upper boundary of $l$, denoted by $N$, will be less than or equal to $2^n$, since the entire set is $\Delta_{2^i}$ containing $2^n$ elements. Therefore, $\Delta_{2^i} = \bigcup_{1 \leq l \leq N} S_l(r) \setminus S_{l-1}(r)$.

Now we are ready to solve equation 3.4 to get the SDSFC.

**Theorem 3.** If the SDSFC in the form of equation 2.6 exists such that system 2.4 can be globally stabilized to $\delta^r_{2^i}$, then for every $1 \leq i \leq 2^n$, there is a unique integral $1 \leq l_i \leq N$ such that $\delta^r_{2^i} \in S_{l_i}(r) \setminus S_{l_i-1}(r)$ with $S_0(r) = \emptyset$. Let $p_i$ be the solution such that

$$
\begin{align*}
\alpha^{(p_i-1)2^n+r} = r, \\
\beta^r_i = r \text{ for } l_i = 1 \text{ with } i \neq r, \\
\delta^r_{2^i} \in S_{l_i-1}(r) \setminus S_{l_i-2}(r) \text{ for } l_i \geq 2.
\end{align*}
$$

(3.9)
Then the SDSFC can be determined by \( E = \delta_{2^n}[p_1 \ p_2 \ \cdots \ p_{2^n}] \).

**Proof.** From theorem 2, \( \Delta_{2^n} = \bigcup_{1 \leq l \leq N} S_l(r) \setminus S_{l-1}(r) \). Then for any \( 1 \leq i \leq 2^n \), it is easy to see that there exists a unique integral \( 1 \leq l_i \leq N \) such that \( \delta_{2^n}^r \in S_{l_i}(r) \setminus S_{l_i-1}(r) \).

Since \((LEW_{[2^n]} \Phi_n)_r = 1\) from theorem 1,

\[
\delta_{2^n}^r = LEW_{[2^n]} \Phi_n \delta_{2^n}^r = LE(\delta_{2^n}^r)^2 = \alpha_{(p_{r-1})2^n + r}.
\]

Therefore, \( \alpha_{(p_{r-1})2^n + r} = r \).

For \( l_i = 1 \) and \( i \neq r \), from lemma 4, we need to find a possible \( p_i \) such that \( \beta_{i}^r = r \). By the definition of \( \beta_{i}^l \), \( l \geq 1 \) in equation 3.8, if \( \beta_{i}^r = r \), then \( \alpha_{(p_{r-1})2^n + \beta_{i}^l - 1} = r \). Define \( D(i) = \{ j : \alpha_j = i, 1 \leq j \leq 2^{n+1} \} \) for \( 1 \leq i \leq 2^n \).

**Step 1.** Since \( 1 \leq \beta_{i}^{l-1} \leq 2^n \), then for any \( j_1 \in D(r) \), there exists unique \( 1 \leq p_{i} \leq 2^n \) such that \( (p_{i} - 1)2^n + \beta_{i}^{l-1} = j_1 \).

**Step 2.** Consider \( \beta_{i}^{l-1} = j_1 - (p_{i} - 1)2^n \) which is between 1 and \( 2^n \). Then \( \alpha_{(p_{r-1})2^n + \beta_{i}^{l-2}} = j_1 - (p_{1} - 1)2^n \) from equation 3.8. To determine \( \beta_{i}^{l-2} \), we consider an element \( j_2 \in D(j_1 - (p_{i} - 1)2^n) \) such that \( 1 \leq j_2 - (p_{i} - 1)2^n \leq 2^n \).

Then \( j_2 = (p_{i} - 1)2^n + \beta_{i}^{l-2} \), and therefore, \( \beta_{i}^{l-2} = j_2 - (p_{i} - 1)2^n \).

**Step 3.** Since \( S_{r}(r) \setminus \{ \delta_{2^n}^r \} \neq \emptyset \), there exists at least \( j_{r-1} \in D(j_{r-2} - (p_{r} - 1)2^n) \) such that \( 1 \leq j_{r-1} - (p_{r} - 1)2^n \leq 2^n \). Then \( j_{r-1} = (p_{r} - 1)2^n + \beta_{i}^{l-1} \), that is, \( \beta_{i}^{l-1} = \alpha_{(p_{r-1})2^n + i} = j_{r-1} - (p_{r} - 1)2^n \).

**Step 4.** Solving the equation \( \alpha_{(p_{r-1})2^n + i} = j_{r-1} - (p_{r} - 1)2^n \) gives at least one possible \( i \) such that \( p_i \) satisfies \( \beta_{i}^{l} = r \).

For \( l_i \geq 2 \), we only need to regard \( \delta_{2^n}^l \in S_{l-1}(r) \setminus S_{l-2}(r) \) to be \( \delta_{2^n}^l \), for \( l_i = 1 \). Now consider all \( \delta_{2^n}^l \in S_{l-1}(r) \setminus S_{l-2}(r) \). Then \( p_i \) can be obtained with a similar argument as the one above. Consequently, the matrix \( E \) is constructed.

Based on theorem 3, we give algorithm 1 to get \( E \).

**Algorithm 1.**

Step 1. Solving \( \alpha_{(p_{r-1})2^n + r} = r \) to get \( p_r \). If there is no solution, \( E \) does not exist.

Step 2. Solving \( \beta_{i}^{l} = r \) for \( l_i = 1 \) with \( i \neq r \) to get one solution of \( p_i \) for \( \delta_{2^n}^l \in S_{l}(r) \setminus \{ \delta_{2^n}^r \} \). If there is no solution of such \( p_r \), \( E \) does not exist.

Step 3. Regard \( \delta_{2^n}^l \in S_{l-1}(r) \setminus S_{l-2}(r) \) to be \( \delta_{2^n}^r \), for \( l_i = 1 \), and solve \( \beta_{i}^{l} \) for \( \delta_{2^n}^l \in S_{l-1}(r) \setminus S_{l-2}(r) \) for \( l_i \geq 2 \). Get one solution of each \( p_i \). If there is no solution of \( p_r \), \( E \) does not exist.

Step 4. Get \( u \) with \( E = \delta_{2^n}[p_1 \ p_2 \ \cdots \ p_{2^n}] \).
Theorem 3 presents an approach to construct SDSFC such that system 2.4 can be globally stabilized. If the SDSFC exists, it can be constructed by equations 3.9. On the other hand, if there is no solution of \( p_i \) to equations 3.9, then the SDSFC does not exist. In other words, we do not know if the SDSFC exists unless we solve all equations in equations 3.9.

**Example 3.** Let us consider the BCN model presented in H. Li, Wang, and Liu (2013). It is a reduced Boolean model for the lac operon in the bacterium *Escherichia coli*, which shows that lac mRNA and lactose form the core of the lac operon. The model is given as follows:

\[
\begin{align*}
    x_1(t+1) &= \neg u_1(t) \land (x_2(t) \lor x_3(t)), \\
    x_2(t+1) &= \neg u_1(t) \land u_2(t) \land x_1(t), \\
    x_3(t+1) &= \neg u_1(t) \land (u_2(t) \lor (u_3(t) \land x_1(t))).
\end{align*}
\]  

(3.10)

where \( x_1, x_2, \) and \( x_3 \) are state variables denoting the lac mRNA, the lactose in high concentrations, and the lactose in medium concentrations, respectively; \( u_1, u_2, \) and \( u_3 \) are control inputs that represent the extracellular glucose, the high extracellular lactose, and the medium extracellular lactose, respectively.

Using the vector form of logical variables and setting \( x(t) = \bigoplus_{i=1}^{3} x_i(t) \) and \( u(t) = \bigoplus_{i=1}^{3} u_i(t) \), by the STP, we can express system 3.10 in its algebraic form as

\[
x(t+1) = Lu(t)x(t),
\]

where

\[
L = \delta_8[8, 8, 8, 8, 8, 8, 8, 8, \]
\[
8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8],
\]

The feedback law to be determined for system 3.10 is in the form of equation 2.6 as

\[
u(t) = Ex(t_l), \quad t_l \leq t < t_{l+1},
\]

where \( E = \tilde{M}_1 \otimes_{j=2}^{3} [(I_{2^3} \otimes \tilde{M}_j) \Phi_3] \in \mathcal{L}_{2^3 \times 2^3}, \) and \( \tilde{M}_j \) is the structure matrix of \( e_j \). Assume that the state feedback matrix \( E \) is given by

\[
E = \delta_8[p_1, p_2, \ldots, p_8].
\]  

(3.11)
Let $\tau = 2$. We now construct the SDSFC such that system 3.10 is globally stabilizable to $\delta^1_8$.

**Step 1.** Solving $\alpha(p_1 - 1)^2 + 1 = 1$, we get $p_1 = 5$ or $6$.

**Step 2.** Since $\beta_2 = 1$, $\alpha_j = 1$, then $j_1 \in D(1) = \{33, 34, 35, 41, 42, 43\}$. Solve $\alpha(p_1 - 1)^2 + 1 = j_1 - (p_1 - 1)^2$. If $j_1 = 35, p_5 = p_6 = p_7 = 5$; if $j_1 = 41, p_2 = p_3 = 6$; if $j_1 = 43, p_5 = p_6 = p_7 = 6$. We have $\delta^2_8, \delta^3_8, \delta^4_8 \in S_1(1) \setminus \{\delta^1_8\}$.

**Step 3.** Regard $\delta^4_8, \delta^5_8 \in S_2(1) \setminus S_1(1)$. Assume $\beta_2 = 3$, $\alpha_j = 3$; then $j_1 \in D(3) = \{37, 38, 39, 45, 46, 49, 50, 51\}$. Solve $\alpha(p_1 - 1)^2 + 1 = j_1 - (p_1 - 1)^2$. If $j_1 = 37, p_4 = 5$; if $j_1 = 39, p_8 = 5$.

Consequently, one of the possible $E$ is given by

$$E = \delta^1_8[5, 6, 6, 5, 6, 5, 6, 5],$$

and the corresponding SDSFC is in the form of

$$\begin{cases} u_1(t) = 0, \\
 u_2(t) = 1, \quad 2l \leq t < 2(l + 1), \\
 u_3(t) = x_2(2l) \leftrightarrow x_3(2l), \end{cases}$$

for $l = 0, 1, 2, \ldots$.

Figure 1 shows the trajectories of system 3.10 with initial states $x_1(0) = \delta^1_2, x_2(0) = \delta^2_2, x_3(0) = \delta^2_2$ under the constructed SDSFC as shown in Figure 2.

### 3.2 Piecewise Constant Control of BCNs

Considering system 2.4 and the fact of $Lu(t) \in \mathcal{L}^n_{2n \times 2n}$, we split $L$ into $2^m$ equal blocks as

$$L = [^1L \quad ^2L \quad \ldots \quad ^{2^m}L],$$

where $^iL \in \mathcal{L}^n_{2n \times 2n}, i \in \Omega_m$. Denote the controllability matrix

$$\hat{L}^l = \bigvee_{j=1}^{2^m} (^iL)^l, l \geq 1.$$  \hspace{1cm} (3.13)

Assume that $u(t) = \delta^1_{2m}$. Then $Lu(t) = ^iL$. Therefore, system 2.4 can be regarded as a Boolean switched system (Fornasini & Valcher, 2013),

$$x(t + 1) = \sigma^{(i)}Lx(t), t \geq 0.$$  \hspace{1cm} (3.14)
Figure 1: The trajectories of system 3.10 with initial states $x_1(0) = \delta_1^1$, $x_2(0) = \delta_2^2$, $x_3(0) = \delta_2^2$ are globally stabilized to $\delta^1$ under the SDSFC as shown in Figure 2.

Figure 2: An illustration of the designed $u(t)$ to achieve stabilization of system 3.10.
where $1 \leq \sigma(t) \leq 2^n$ is a switching sequence and $x(t + 1) = iLx(t)$ is the subsystem of Boolean switched system 3.14.

**Definition 4.** Consider system 2.4. Let $X_0 \in \mathcal{D}^n$. $X_d \in \mathcal{D}^n$ is said to be reachable from $X_0$ if there exist $l > 0$ and a control sequence $U = \{u(0), \ldots, u(l - 1)\}$ such that steers system 2.4 from $X_0$ to $X_d$. $X_d \in \mathcal{D}^n$ is said to be globally reachable if $X_d$ is reachable from any $X_0 \in \mathcal{D}^n$. System 2.4 is said to be controllable if any $X_d \in \mathcal{D}^n$ is globally reachable.

We learn from Laschov and Margaliot (2012) that $X_l = \delta^j_{2^n}$ is reachable from $X_0 = \delta^i_{2^n}$ at the $l$th step if and only if $(\hat{L}^1)^{ji}_{\tau} > 0$, where $\hat{L}^1$ is given by equation 3.13. $X = \delta^j_{2^n}$ is reachable from $x(0) = \delta^i_{2^n}$ if and only if there exists a positive integer $l$ such that $(\hat{L}^1)^{ji}_l > 0$. $X = \delta^j_{2^n}$ is globally reachable if and only if there exists a positive integer $l$ such that $\text{Row}_r(\bigvee_{j=1}^N (\hat{L}^1)^j) > 0$.

We now consider the PCC for system 2.4 as follows,

$$u(t) = u(t_l) \in \Delta^m_{2^n}, \quad t_l \leq t < t_{l+1},$$  \hspace{1cm} \text{(3.15)}

where $t_l = l\tau \geq 0$ for $\tau > 0$ and $l \geq 0$. Define

$$S^0_i(l) = \{x_0 \in \Delta^m_{2^n} : \exists u(0), \ldots, u((l - 1)\tau) \in \Delta^m_{2^n} \text{ such that } \text{(Lu)((l - 1)\tau)})^T \cdots (Lu(0))^T x_0 = \delta^i_{2^n}, 1 \leq i \leq 2^n, l \geq 1. \text{ (3.16)}$$

**Theorem 4.** There exists a sequence of PCCs in the form of equation 3.15 such that system 2.4 can be globally stabilized to $\delta^r_{2^n}$ if and only if there exists $1 \leq N \leq 2^n$ such that

$$\text{Row}_r(\bigvee_{j=1}^N (\hat{L}^1)^j) > 0, \quad \text{and} \quad (\hat{L}^1)^{rr}_{\tau} = 1. \text{ (3.17)}$$

**Proof.** (Sufficiency) From equation 3.13, $(\hat{L}^1)^{rr}_{\tau} = 1$ implies that there exists at least one $1 \leq j \leq 2^n$ such that $(iL)^{rr}_{\tau} = 1$,

$$\delta^r_{2^n} = L\delta^j_{2^n}\delta^r_{2^n}. \text{ (3.18)}$$

It is easy to see from 3.18 that $\delta^r_{2^n} \in S^0_i(\tau)$.

On the other hand, $\text{Row}_r(\bigvee_{j=1}^N (\hat{L}^1)^j) > 0$ in equation 3.17 means that

$$1 = (\delta^r_{2^n})^T \bigvee_{j=1}^N (\hat{L}^1)^j \delta^r_{2^n}, \quad \text{for any } 1 \leq i \leq 2^n.$$
Step 1. Since \( \delta^i_{2n} \in S^0_1(r) \), \( S^0_1(r) \neq \emptyset \). There exist at least one \( x(0) = \delta^i_{2n} \) and a corresponding \( u(0) \), denoted by \( u^*(0, \delta^i_{2n}, \delta^o_{2n}) \), such that \( \delta^i_{2n} = (Lu^*(0, \delta^i_{2n}, \delta^o_{2n}))^{\tau} \delta^i_{2n} \).

\[
1 = (\delta^i_{2n})^T (Lu^*(0, \delta^i_{2n}, \delta^o_{2n}))^{\tau} \delta^i_{2n}.
\]

It also implies that \( 1 = (\delta^i_{2n})^T (\hat{L}^\tau) \delta^i_{2n} > 0 \). From the controllability criteria obtained by Laschov and Margaliot (2012) and the definition of \( S^0_1(r) \) in equation 3.16, we conclude that all such \( \delta^i_{2n} \) consists \( S^0_1(r) \).

If \( S^0_1(r) = \Delta_{2n} \), that is, \( \delta^i_{2n} = \delta^i_{2n} \), holds for all \( 1 \leq i \leq 2^n \), then \( \text{Row}_i(\hat{L}^\tau) > 0 \) and \( N = 1 \) here. Considering equation 3.18, \( \delta^i_{2n} \) can be fixed for any \( t \geq \tau \). Then system 2.4 can be globally stabilized to \( \delta^i_{2n} \). The proof is finished. Otherwise, \( \Delta_{2n} \setminus S^0_1(r) \neq \emptyset \), and \( \text{Row}_i(\hat{L}^\tau) > 0 \) does not hold any more.

Step 2. It is easy to see from \( x(2\tau) = (Lu(\tau))^{\tau} (Lu(0))^{\tau} x(0) \) that \( S^0_2(r) = \bigcup \{S^0_1(i) : 1 \leq i \leq 2^n, \delta^i_{2n} \in S^0_1(r) \} \). Therefore, \( S^0_2(r) \subset S^0_1(r) \) due to \( \delta^i_{2n} \in S^0_1(r) \). There exist \( u(0), u(\tau) \) and \( x(0) = \delta^i_{2n} \) such that \( \delta^i_{2n} \) is reachable. Denote such controls by \( u^*(0, x(0), x(\tau)), u^*(\tau, x(\tau), x(2\tau)) \). Similar to step 1, we have

\[
1 = (\delta^i_{2n})^T (Lu^*(\tau, x(\tau), x(2\tau)))^{\tau} (Lu^*(0, x(0), x(\tau)))^{\tau} \delta^i_{2n}.
\]

It implies that \( 1 = (\delta^i_{2n})^T (\hat{L}^\tau)^2 \delta^i_{2n} > 0 \), and as in step 1, all such \( \delta^i_{2n} \) consists of \( S^0_2(r) \).

If \( S^0_2(r) = \Delta_{2n} \), that is, \( \text{Row}_i(\sqrt^N_{j=1}(\hat{L}^\tau))^i > 0 \), then \( \delta^i_{2n} \) is globally reachable with \( N = 2 \). From equation 3.18 again, \( \delta^i_{2n} \) is fixed for any \( t \geq 2\tau \), and system 2.4 is globally stabilized to \( \delta^i_{2n} \). We finish the proof. Otherwise, \( \Delta_{2n} \setminus S^0_2(r) \neq \emptyset \).

Moreover, \( S^0_2(r) \setminus S^0_1(r) \neq \emptyset \). Otherwise, \( S^0_2(r) = S^0_1(r) \). In this way, one has \( S^0_N(r) = S^0_{N-1}(r) = \ldots = S^0_1(r) \), which means that \( \text{Row}_i(\hat{L}^\tau)^N = \text{Row}_i(\hat{L}^\tau)^{N-1} = \ldots = \text{Row}_i(\hat{L}^\tau) \). As a result, \( \text{Row}_i(\sqrt^N_{j=1}(\hat{L}^\tau))^i = \text{Row}_i(\hat{L}^\tau) \). Since \( \Delta_{2n} \setminus S^0_1(r) \neq \emptyset \), which means that \( \text{Row}_i(\hat{L}^\tau) > 0 \) is not satisfied. It is a contradiction to \( \text{Row}_i(\sqrt^N_{j=1}(\hat{L}^\tau))^i > 0 \).

Step 3. Similarly, one can also prove that \( S^0_{j+1}(r) \setminus S^0_j(r) \neq \emptyset \) with \( S^0_0(r) \) denoted by \( \emptyset \). Then one can conclude that there exists \( N \leq 2^n \) such that \( \Delta_{2n} = \bigcup \{S^0_{j+1}(r) \setminus S^0_j(r) \} \). From the definition of \( S^0_i(r) \), we obtain that \( \delta^i_{2n} \) is globally reachable. Moreover, for \( t \geq N\tau \), \( \delta^i_{2n} \) is fixed from equation 3.18. Therefore, system 2.4 is globally stabilizable to \( \delta^i_{2n} \).

The necessary part is easy to get from the proof of the sufficiency, and we omit it here.

When \( \tau = 1 \), the PCC can be regarded as a normal control sequence (see Cheng et al., 2011; Laschov & Margaliot, 2012; Fornasini & Valcher, 2013;
Liu et al., 2014; Liu, Chen et al., 2015). Then we get the following corollary from theorem 4.

**Corollary 1.** System 2.4 can be globally stabilized to $\delta_{2^n}$ if and only if there exists $1 \leq N \leq 2^n$, such that

$$\text{Row}_r \left( \sum_{j=1}^{N} (\hat{L}^1)^j \right) > 0, \text{ and } (\hat{L}^1)_{rr} = 1.$$  

(3.19)

**Remark 2.** It is easy to see that condition 3.19 in corollary 1 is actually equivalent to conditions 1 and 2 in proposition 3 of Fornasini and Valcher (2013), when the cycle considered there is an equilibrium point. Motivated by Fornasini and Valcher (2013), it is also possible to consider the stabilization of system 2.4 to a limit cycle by the PCC.

The following result presents the relationship between SDSFCs and PCCs for system 2.4:

**Theorem 5.** System 2.4 can be globally stabilized to $\delta_{2^n}$ by PCC, equation 3.15 if and only if it is globally stabilizable by means of SDSFC, equation 2.6.

**Proof.** The sufficiency is obvious by letting $u(t_k) = Ex(t_k)$. For the necessary part, from condition 3.17 of theorem 4, $(\hat{L}^1)_{rr} = 1$, there is $1 \leq j \leq 2^n$ such that $(\hat{L}^1)^j = 1$. Then $\delta_{2^n}^j = iL\delta_{2^n}^j = L\delta_{2^n}^j \delta_{2^n}^j$, which means that $u(t_k) = \delta_{2^n}^j = E\delta_{2^n}^j = Ex(t_k)$ for some $k$, and it implies $Col_j(E) = \delta_{2^n}^j$.

From step 3 in the proof of theorem 4, there exists $N \leq 2^n$ such that $\Delta_{2^n} = \bigcup_{0 \leq j \leq N-1} (S_{j+1}^0(r) \setminus S_j^0(r))$. For any $x(t_j) = \delta_{2^n}^j \in S_{j+1}^0(r) \setminus S_j^0(r)$, there exists $u(t_j) = \delta_{2^n}^j$ such that $(L\delta_{2^n}^j)^\tau \delta_{2^n}^j \in S_j^0(r)$, which means that $u(t_k) = \delta_{2^n}^j = E\delta_{2^n}^j = Ex(t_j)$ and further implies $Col_j(E) = \delta_{2^n}^j$. Considering every $\delta_{2^n}^j \in \bigcup_{0 \leq j \leq N-1} (S_{j+1}^0(r) \setminus S_j^0(r))$, all the columns of $E$ can be determined.

Consequently, the global stabilization by PCC, equation 3.15, is equivalent to the global stabilization by means of SDSFC in the form of equation 2.6.

When $\tau = 1$, we have the following corollary:

**Corollary 2.** System 2.4 can be globally stabilized to $\delta_{2^n}$ if and only if it is globally stabilizable by state feedback control.

Based on theorem 5, algorithm 2 gives another approach to construct the SDSFC:

**Algorithm 2.**

Step 1. Judge whether $(\hat{L}^1)_{rr} = 1$. If yes, find $j$ such that $(\hat{L}^1)^j = 1$; then $Col_j(E) = \delta_{2^n}^j$. Otherwise, $E$ does not exist.
Step 2. Find $1 \leq N \leq 2^n$, such that $\text{Row}_r \left( \bigvee_{j=1}^N (\hat{L}^r)^{1/2} \right) > 0$. If such $N$ exists, then $E$ can be constructed. Otherwise, there is no $E$.

Step 3. Find all $i$, the set of which is denoted by $s_1(r)$ such that the $(\delta^i_{2^n})^T (\hat{L}^r) \delta^i_{2^n} > 0$. For each $i \in s_1(r) \setminus \{r\}$, find $j_i$, such that $(\delta^i_{2^n})^T (\hat{L}^r) \delta^j_{2^n} > 0$. Then $\delta^i_{2^n} = E \delta^j_{2^n}$, and as a result, $p_i = \delta^j_{2^n}$ for all $i \in s_1(r)$.

Step 4. For each $j \in s_1(r)$, we get a set $s_1(j)$ as in step 3. Then we define $s_2(r) = \bigcup_{j \in s_1(r)} s_1(j)$. Repeat step 3 for any $i \in s_2(r) \setminus s_1(r)$. Then $p_i$ for all $i \in s_2(r) \setminus s_1(r)$ can be determined.

Step $N+2$. For each $j \in s_{N-1}(r)$, we get a set $s_1(j)$. Then we define $s_N(r) = \bigcup_{j \in s_{N-1}(r)} s_1(j)$. Repeat step 3 for any $i \in s_N(r) \setminus s_{N-1}(r)$. Then $p_i$ for all $i \in s_N(r) \setminus s_{N-1}(r)$ can be determined.

Step $N+3$. Get the SDSFC $u$ with $E = \delta^i_{2^n}[p_1 \ p_2 \ \cdots \ p_{2^n}]$.

**Remark 3.** Compared with algorithm 1, algorithm 2 is much more effective. First, we can judge the existence of the SDSFC by checking the existence of $N$ such that equation 3.17 is satisfied. However, in algorithm 1, we need to compute each solution of $p_i$ to learn the existence of $E$ until all of them are found. Second, we notice that in algorithm 1, one should solve the equation $\beta_i^T = r$ to get $p_i$. Although the equation looks very simple, it actually includes $\tau$ equations in the induction (see the proof of theorem 2). When $\tau$ is a big number, the computation will be complex. For algorithm 2, to get the solution of $p_i$, one needs only to solve the equation $\delta^i_{2^n} = E \delta^j_{2^n}$, and it is very easy to get $p_i = \delta^j_{2^n}$.

**Example 4.** Consider example 3 by algorithm 2. We can get

\[
\hat{L}^1 = \begin{bmatrix}
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 \\
1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{bmatrix}.
\]

Step 1. It is found that $(\hat{L}^1)_{11} = 1$. For $j = 5$ or $6$, $(\hat{L})_{11} = 1$, then $\text{Col}_1(E) = p_1 = \delta^5_8$ or $p_1 = \delta^6_8$. 

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Step 2. We have

\[
\hat{L}^2 = \begin{bmatrix}
1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\end{bmatrix},\quad (\hat{L}^2)^2 = \begin{bmatrix}
2 & 2 & 2 & 1 & 2 & 2 & 2 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & 3 & 3 & 3 & 2 & 2 & 2 & 2 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
3 & 3 & 3 & 3 & 3 & 3 & 3 & 2 \\
\end{bmatrix}.
\]

So when \( N = 2 \), then Row\(_i\)\((\sqrt{\hat{L}^2})^j\) > 0 is satisfied. Then \( E \) can be constructed.

Step 3. Since \((\delta_8^1)^T(\hat{L}^2)\delta_8^i > 0\), we can find \( i = 1, 2, 3, 5, 6, 7 \). For \( i = 2 \), we can find \( j_2 = 5 \) or \( j_2 = 6 \), such that \((\delta_8^1)^T(5L^2)\delta_8^5 > 0\) and \((\delta_8^1)^T(6L^2)\delta_8^6 > 0\). Then \( \delta_5^5 = E\delta_8^5 \) or \( \delta_6^6 = E\delta_8^6 \) and hence \( p_2 = 5 \) or 6. Repeating the same method, finally, we can get \( p_3 = 5 \) or 6, \( p_5 = 5 \) or 6, \( p_6 = 5 \) or 6, and \( p_7 = 5 \) or 6.

Step 4. Since \( 3 \in s_1(1) \), we get a set \( s_1(3) \) as in step 3. Repeating step 3, \((\delta_8^3)^T(\hat{L}^2)\delta_8^i > 0\), then \( i = 4, 8 \). For \( i = 4 \), we can find \( j_4 = 5 \) or \( j_4 = 6 \), such that \((\delta_8^3)^T(5L^2)\delta_8^5 > 0\) and \((\delta_8^3)^T(6L^2)\delta_8^6 > 0\). Then \( \delta_5^5 = E\delta_8^4 \) or \( \delta_6^6 = E\delta_8^4 \) and hence \( p_4 = 5 \) or 6. We can also get \( p_8 = 5 \) or 6.

Consequently, we can construct the same SDSFC as in example 3 with \( E \) given by \( E = \delta_8[5, 6, 6, 5, 6, 6, 6, 5] \).

In example 4, algorithm 2 is used to get the solution of \( p_i \) for system 3.10. It is simple to solve the equation \( \delta_{2m}^i = E\delta_{2n}^i \), and obtain \( p_i = \delta_{2n}^i \).

4 Conclusion

In this letter, we have studied the sampled-date control stabilization problem of BCNs. Necessary and sufficient conditions for the global stabilization by SDSFC have been derived. Different from the normal state feedback control, some new observations have been presented. In terms of the controllability matrix, PCC of BCNs has been discussed as well. We have proved the equivalence of SDSFC and PCC for the control of BCNs. In detail, a BCN can be globally stabilizable by PCC if and only if it is globally stabilizable by SDSFC. Two algorithms have been presented to construct SDSFC. Examples have been given to show the efficiency of the proposed results.
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