The linkage between box-counting and geomorphic fractal dimensions in the fractal structure of river networks: the junction angle

Xianmeng Meng, Pengju Zhang, Jing Li, Chuanming Ma and Dengfeng Liu

ABSTRACT

In the past, a great deal of research has been conducted to determine the fractal properties of river networks, and there are many kinds of methods calculating their fractal dimensions. In this paper, we compare two most common methods: one is geomorphic fractal dimension obtained from the bifurcation ratio and the stream length ratio, and the other is box-counting method. Firstly, synthetic fractal trees are used to explain the role of the junction angle on the relation between two kinds of fractal dimensions. The obtained relationship curves indicate that box-counting dimension is decreasing with the increase of the junction angle when geomorphic fractal dimension keeps constant. This relationship presents continuous and smooth convex curves with junction angle from 60° to 120° and concave curves from 30° to 45°. Then 70 river networks in China are investigated in terms of their two kinds of fractal dimensions. The results confirm the fractal structure of river networks. Geomorphic fractal dimensions of river networks are larger than box-counting dimensions and there is no obvious relationship between these two kinds of fractal dimensions. Relatively good non-linear relationships between geomorphic fractal dimensions and box-counting dimensions are obtained by considering the role of the junction angle.

Key words | box-counting dimension, geomorphic fractal dimension, junction angle, river networks, synthetic fractal tree

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HIGHLIGHTS

- The role of the junction angle on the relation between two kinds of fractal dimensions is explained by synthetic fractal trees.
- Two kinds of fractal dimensions of 70 actual river networks in China are compared.
- Junction angle has a significant influence on fractal dimensions of river networks.

INTRODUCTION

One of the most important issues in the field of river hydrology is the quantitative description of river networks. It is widely accepted for decades that river networks are scaling, possessing self-similar structures over a huge range of scales. Horton (1932, 1945), the forerunner focusing on the underlying laws of the dendritic structures of river networks initially describes river networks in a quantitative way. On that basis, the hierarchical classification of streams came into existence, which underwent a further improvement and modification by Strahler (1952, 1964).

According to the ordering scheme proposed by Horton and Strahler, a first-order stream can be defined as a

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stream which does not receive flow from any other stream. Naturally, as two streams of order w come together, a stream of order $w+1$ is constituted. If two streams of different orders join, the order of resulting stream remains identical to the higher-order stream. Horton's ratios are empirical quantities that describe the scaling structure of a river network.

The bifurcation, length and area ratios are expressed as:

$$R_B = \frac{N_w}{N_{w+1}}$$
$$R_L = \frac{L_{w+1}}{L_w} \quad (1)$$
$$R_A = \frac{A_{w+1}}{A_w}$$

where N_w is the number of streams, L_w is the average length of streams, A_w is the average total contributing area and w is the stream order. The values of R_B, R_L and R_A changing for rivers usually range from 3 to 5, from 1.5 to 3 and from 3 to 6, respectively (Horton 1945; Schumm 1956).

Deepened exploration concerns the empirical relationships between basin and river network (Hack 1957; Gray 1961; Leopold et al. 1964; Eagleson 1970; Mueller 1973). The relationship between the observed length of the mainstream and basin area can be generally expressed as:

$$L \sim A^{\alpha} \quad (2)$$

where L represents the mainstream length, A signifies the basin area and α means a fitted exponent for rivers found to be universally in a small range, $\alpha = 0.6 \pm 0.1$ (Hack 1957; Robert & Roy 1990; Crave & Davy 1997; Paik & Kumar 2011).

Pioneered in fractal geometry, Mandelbrot (1977) introduced fractals in hydrology, which promoted the wide application of fractal approach in studying the geometric structure of river networks (Tarboton et al. 1988; La Barbera & Rosso 1989). A novel feature of the river network, that is, fractal dimension, was subsequently coined. Numerous studies have demonstrated the characterization of the fractal properties and the determination of the fractal dimension of river networks. For instance, based on the bifurcation and length ratios, La Barbera & Rosso (1989) gave a definition of the fractal dimension of river networks. Moreover, box-counting method (Tarboton et al. 1988; Rosso et al. 1991) became a fiercely debated applied technique, which can be manifested in various authors' usage of box-counting fixed-size algorithms (Giorgilli et al. 1986; Liebovitch & Toth 1989; Block et al. 1990; Hou et al. 1990; Barth et al. 1992; Meisel et al. 1992; Molteno 1993; Yamaguti & Prado 1995, 1997; Feeny 2000).

River basins are three-dimensional landscapes organized around dendritic structures that constitute the drainage network whose role is to convey water and sediment from every site of the basin to a common outlet. Study on the fractal structure of river networks has both theoretical significance and practical value (Rodriguez-Iturbe & Rinaldo 2001). For the theoretical aspect, minimum energy principles control the development of drainage basins (Leopold & Langbein 1962; Stevens 1974; Howard 1990). The fractal structure of river networks is possibly a product of least energy dissipation (Rinaldo et al. 1992; Rodriguez-Iturbe et al. 1992a). The exponent of the power law is directly related to a suitable fractal dimension of the boundaries, to the elongation of the basin and to the scaling exponent of mainstream lengths (Maritan et al. 1996). A theory of energy dissipation, runoff production and the three-dimensional structure of river basins has been proposed linking three principles of local and global energy expenditure in naturally aggregated fluvial networks to explain structural characteristics observed in the geomorphology of drainage systems (Rodriguez-Iturbe et al. 1992b). One of the most important roles of the fractal dimension of the river network for applications is improving the capability of modeling the hydrologic response of a catchment to surface runoff by linking hydrology with quantitative geomorphology (Gupta et al. 1980; Rodriguez-lturbe et al. 1982). For instance, on the basis of a probabilistic model of the advance through the catchment of a randomly placed water drop, the formulation of the impulse response function has been given in terms of Horton's order ratios (Rodriguez-lturbe & Valdes 1979). The Nash model of the instantaneous unit hydrograph (IUH) is parameterized in terms of Horton order ratios of a catchment on the basis of a geomorphologic model of catchment (Rosso 1984).

Influenced by the introduction of fractals, fractal dimensions are considered to be widely used and a standard way to quantify self similarity. Taking into consideration diverse concepts in defining the fractal dimension and corresponding distinct techniques, scholars regard the relationship among multiple fractal dimensions of river networks as a hot issue. For instance, Tarboton (1996) expounded how Horton scaling ratios relate to fractal dimensions, providing a reference to the structure of river networks. Starting from analyzing the differences between the values of geomorphic and raster dimensions, Fac-Beneda (2013) confirmed the unfitness of geomorphic fractal dimension in describing a river network. To the best of our knowledge, there has been relatively little discussion of the implications of the relationship between these different fractal dimensions of river networks. Few studies have been conducted on the influencing factors governing these relationships.

The junction angle, referring to the branching angle between two joining stream lines, is one of the main characteristics of river networks related to basin dissection and network orientation (Howard 1990). It has been shown to be correlated with other properties, such as slope and discharge at the junction (Horton 1932, 1945; Howard 1971a, 1971b; Roy 1983). Horton (1932) proposed the first quantitative model for predicting the angle between the overland flow on the hillslope and the stream by relating it to the ratio of the main stream's gradient to that of the hillslope. Horton (1945) utilized the same concept for predicting the angle between a major stream and a tributary. In addition, this angle has long been considered a control on morphological and sedimentological processes within fluvial systems (Mosley 1976; Best 1986, 1988). The observed angles are physically explained as the optimal angles that result in minimum energy dissipation (Hooshyar et al. 2017). As the junction angle is an important feature related to the network topology and contains valuable information about the forming mechanisms of the landscape, it is commonly used to investigate the characterization of river networks (Abrahams & Flint 1983; Ichoku & Chorowicz 1994; Mejia & Niemann 2008).

In this study, we explore the relationship between the two most common fractal dimensions obtained by Horton's order ratios and box-counting methods. The role of the junction angle on the relation between these two fractal dimensions is discussed. This paper is organized as follows. In the 'Methodology' section, two methods used to compute fractal dimensions are described, and junction angle calculation and a model of fractal plane trees are explained. The 'Study area and data' section introduces available dataset over 70 actual river networks in China. The role of the junction angle on the relation between two kinds of fractal dimensions is presented in the 'Results and discussion' section. And results obtained from two methods for calculating fractal dimensions and their comparison of actual river networks are discussed in this section. In the 'Conclusions' section, the main conclusions are presented and future work is discussed.

METHODOLOGY

Two methods for the calculation of fractal dimensions

To address the implication of the relationships between different fractal dimensions of river networks and the influencing factors governing these relationships, two methods are used for the calculation of fractal dimensions. One is the Horton ratios method (La Barbera & Rosso 1989; Rosso et al. 1991; Helmlinger et al. 1993; Nikora 1994) and the other is the box-counting method (Tarboton et al. 1988; Helmlinger et al. 1993).

For the Horton ratios method, the fractal dimension can be expressed as:

$$D_g = \frac{\ln R_B}{\ln R_L} \tag{3}$$

For the box-counting method, a grid with spacing δ is placed on the fractal object, and the number of boxes, $N(\delta)$, in which a part of the fractal falls is counted (Mandelbrot 1982). This method is carried out at varying grid spacings, and points representing grid spacings and corresponding box counts are plotted on a log–log plot. The resulting relationship can be represented as:

$$N(\delta) \sim \delta^{-D_r} \tag{4}$$

and D_r is the fractal dimension.

Junction angle calculation

Figure 1 shows examples of junctions in a catchment. Intersecting valleys do not follow a linear path in most cases as shown in Figure 1; therefore, additional treatment is needed in calculating the junction angle. Internal nodes in river networks are extracted and all these junction angles are calculated using the method proposed by Serres & Roy (1990). In particular, the junction angle is estimated as the angle between the lines connecting the upstream nodes.

A model of fractal plane trees

Synthetic fractal networks are usually used and their properties are exploited to achieve analytical expression as they are strictly self-similar fractals (Claps *et al.* 1996). Fractal plane trees represent fractal (self-similar) objects which, according to the definition, means 'having a shape made of parts similar to the whole in some way' (Mandelbrot 1986). By repeated generation calculation, fractal objects can be obtained. In the procedure, a set of segments, or more precisely, an initiator was adopted at the beginning while a different set of segments named a generator was also involved. The initiator is substituted with the generator and then each segment of the generator becomes an initiator and is substituted again, in a recursive way (Feder 1988). What is worth mentioning is that the approach we adopt is confined to uniform fractals instead of considering those non-uniform ones which can be obtained under the premise that selected segments of the generator become initiators.

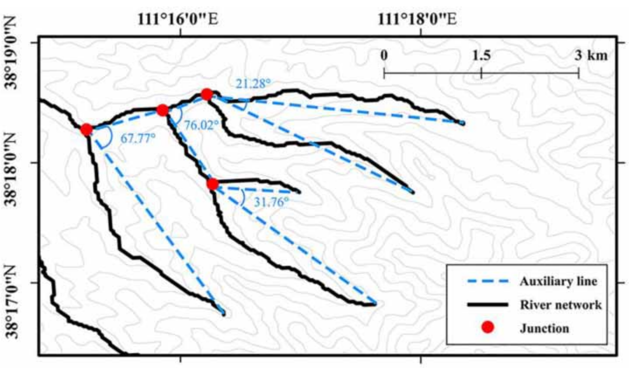

Figure 1 | Junction examples in a catchment located in Weifen watershed. The junctions are marked by red dots and the dashed blue lines represent the lines connecting the upstream nodes. Please refer to the online version of this paper to see this figure in colour: http://dx.doi.org/10.2166/nh.2020.082.

Study area and data

All the computations and analyses in this study use data from 70 natural watersheds. These watersheds range in size from 500 to 2,000 km^2 and are located throughout humid and semi-humid areas in China, with average precipitation ranging between 400 and 2,100 mm $year^{-1}$. All these watersheds are not nested. The general location of the watersheds is shown in Figure 2. Digital elevation models (30 m × 30 m) are downloaded from Geospatial Data Cloud site, Computer Network Information Center, Chinese Academy of Sciences (http://www.gscloud.cn) to delineate upstream watershed boundaries and extract river networks. A digital map (2012; scale 1:250,000) obtained from National Catalogue Service for Geographic Information (http://www.webmap.cn) is used to verify the delineated watershed boundaries and extracted river networks. The main characteristics of these 70 watersheds are summarized in Table 1.

RESULTS AND DISCUSSION

Comparison of two fractal dimensions in geometric fractal trees with different junction angles

Both an initiator composed of a unit-length segment and a generator characterized by a tree-like shape and made up of equal shorter segments are necessary for fractal plane trees. In this paper, generators are taken such that the

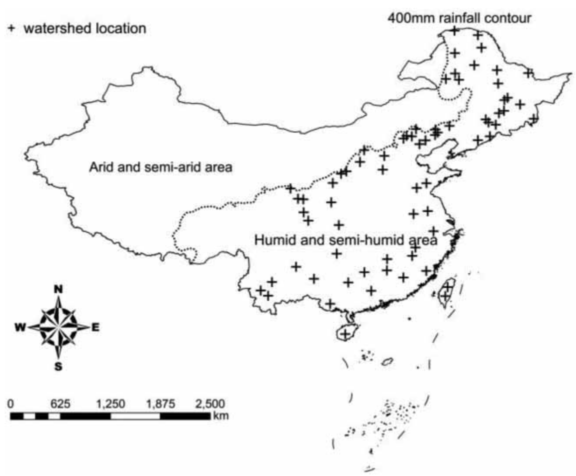

Figure 2 | Distribution of 70 natural watersheds.

Table 1 | Main characteristics of watersheds and river networks analyzed

Number	Name	Area (km²)	Order	R_B	R_L
1	Woluo River	1,248.6	5	4.30	2.25
2	Kuti River	571.3	5	3.60	2.22
3	Yixun River	1,149.6	5	4.01	1.75
4	Erergeqi River	559.4	5	3.76	2.15
5	Qingsongling River	1,494.5	5	4.58	2.46
6	Chao River	1,871.1	6	3.62	1.62
7	Guoen River	822.6	5	4.06	2.14
8	Fabiela River	532.0	5	3.36	1.84
9	Gongbiela River	1,073.3	5	4.18	2.26
10	Ai River	1,506.8	5	4.51	2.65
11	Huai River	1,811.2	6	4.70	2.04
12	Yangjia River	1,004.6	5	4.24	1.64
13	Sha River	1,994.5	6	3.67	1.96
14	Weifen River	1,484.4	5	4.41	2.32
15	Ye River	1,773.2	6	3.55	1.90
16	Fang River	1,319.8	6	3.33	1.61
17	Guangtong River	1,581.6	5	4.46	2.44
18	Lichuan River	589.7	5	3.64	1.68
19	Yao River	1,559.2	5	4.47	2.24
20	Xi River	1,760.5	5	4.82	2.28
21	Tangxi River	1,595.1	5	4.63	2.48
22	Xizhao River	1,364.8	5	4.42	2.11
23	Luxi River	1,268.4	5	4.20	2.15
24	Xiang River	1,465.6	5	4.40	2.24
25	Malong River	1,937.7	5	4.97	2.72
26	Tuolin River	818.4	5	4.17	2.30
27	Gushui River	848.4	5	3.76	2.37
28	Xiaohei River	1,966.5	5	4.66	2.54
29	Laonong River	834.6	4	6.11	3.67
30	Changhua River	783.7	5	4.03	2.06
31	Beijicun River	660.6	5	3.47	2.00
32	Ailin River	707.1	5	3.90	2.16
33	Moke River	620.2	4	5.15	2.94
34	Jiaban River	947.7	5	3.99	2.27
35	Dashi River	806.5	5	3.82	2.06
36	Shanshi River	806.6	5	3.95	2.19
37	Daling River	1,084.0	5	4.12	2.13
38	Hulu River	835.1	5	3.97	2.21
39	Taiping River	1,369.2	5	4.31	2.21
40	Nanting River	867.0	5	4.02	2.06
41	Wangzhuang River	1,077.9	5	4.17	2.12
42	Zuo River	1,007.1	5	4.17	2.30
43	Nanweng River	1,144.1	5	4.30	2.26
44	Yinlong River	1,288.9	5	4.49	2.40
45	Hailun River	997.7	5	4.18	2.31
46	Huolun River	916.1	5	4.11	2.23
47	Dasha River	976.0	5	4.32	2.24
48	Liu River	1,045.7	5	4.17	2.10
49	Sa River	1,133.5	5	4.20	2.22
50	Yan River	1,413.9	5	4.45	2.46
51	Shimo River	833.9	5	3.86	2.18
52	Wangmo River	553.1	5	3.43	1.94
53	Xinquan River	1,305.1	5	4.47	2.44
54	Zhile River	1,131.9	5	4.17	2.56
55	Ahengtixiluoqi River	787.9	5	3.89	2.11
56	Huichun River	989.7	5	4.01	2.17
57	Xiangyang River	697.8	4	5.94	3.34
58	Shidao River	520.9	5	3.46	2.18
59	Jianyucha River	656.3	4	5.48	2.97
60	Taoping River	621.6	4	5.71	2.80
61	Shu River	750.1	5	3.87	2.17
62	Caijia River	648.5	5	3.63	2.24
63	Ting River	678.4	5	3.77	2.04
64	Yangcun River	711.9	5	3.97	2.22
65	Huangshui River	596.6	5	3.93	2.20
66	Mei River	938.0	5	4.03	2.12
67	Tuli River	1,084.8	5	4.16	2.21
68	Xibeicha River	1,214.8	5	4.32	2.20
69	Haiqing River	844.8	5	4.20	2.38
70	Mabai River	1,003.9	5	4.07	2.19

longest path is straight and junction angles of segments are 30°, 45°, 60°, 75°, 90°, 105° and 120°, respectively, as shown in Figure 3.

As fractal plane trees used in the comparison of two fractal dimensions have bifurcation structure, R_b is always equal to 2. Different values of D_g can be obtained by setting R_L, which are then compared with the results from box-counting method under different junction angles. Table 2 shows the results of two fractal dimensions by Horton's order ratios and box-counting method for geometric fractal

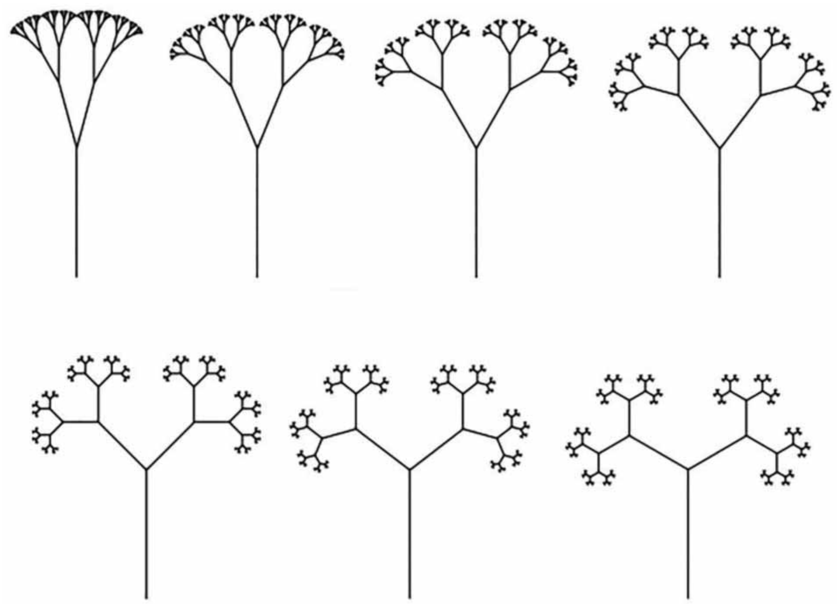

Figure 3 | Fractal plane trees with junction angles 30°, 45°, 60°, 75°, 90°, 105° and 120°, respectively, from left to right.

Table 2 | Two fractal dimensions obtained by Horton's order ratios and box-counting method for geometric fractal trees

		D_r						
R_L	D_g	30°	45°	60°	75°	90°	105°	120°
1.90	1.08	1.08	1.08	1.07	1.07	1.06	1.06	1.05
1.82	1.16	1.16	1.15	1.13	1.11	1.10	1.09	1.07
1.74	1.25	1.24	1.20	1.15	1.14	1.12	1.10	1.08
1.67	1.36	1.32	1.25	1.18	1.15	1.13	1.11	1.09
1.63	1.43	1.35	1.28	1.20	1.17	1.14	1.12	1.10
1.57	1.53	1.38	1.32	1.23	1.19	1.16	1.13	1.11
1.54	1.61	1.40	1.34	1.25	1.21	1.18	1.15	1.12
1.50	1.70	1.42	1.37	1.27	1.24	1.21	1.18	1.15
1.46	1.83	1.44	1.40	1.32	1.29	1.26	1.23	1.20
1.43	1.94	1.46	1.42	1.38	1.36	1.33	1.30	1.27

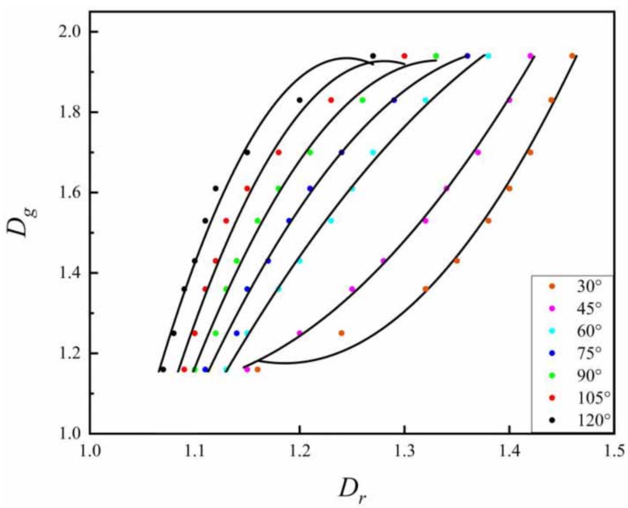

Figure 4 | Relation between two fractal dimensions in geometric fractal trees.

trees. As D_g expressed in Equation (3) is not related to the junction angle, there is no one-to-one corresponding relationship between these two fractal dimensions. The results verify that D_g reflects the topological structure of a stream network and D_r is a measure of the ability of a network to fill a plane (Liu 1992; Schuller *et al.* 2001).

Figure 4 shows the relation between two fractal dimensions in geometric fractal trees. The quadratic equation is applied for function construction of the relation curves. A series of curves is distributed from left to right in Figure 4 with the decrease of junction angle. As it can be seen, D_g is larger than D_r. Analyzing the relation between two fractal dimensions, we can say that D_g is positively correlated with D_r when the junction angle keeps constant. There exists a good non-linear relationship between these two fractal dimensions controlled by the junction angle.

The relationship presents continuous and smooth convex curves with junction angle from 60° to 120° and concave curves from 30° to 45°. D_r is decreasing with the increase of the junction angle when D_g keeps constant. Table 3 illustrates the performance statistics for the quadratic equation. The results show that empirical equations can well reflect the relation between two fractal dimensions. R^2 is the coefficient of determination.

Characteristics of river networks in study areas

In Table 1, we report the estimates of Horton's order ratios for 70 watersheds obtained by Equation (1). As the highest stream order in most of 70 watersheds is 5, Horton's order ratios in these watersheds are statistically analyzed. The relations between the stream order and its corresponding number are shown in Figure 5. The relations between the stream order and average stream length are shown in Figure 6. The results indicate that all the 70 watersheds satisfy Horton's laws and most of the watersheds fall in the ranges of R_b and R_L proposed by Horton (1945) and Schumm (1956). This process verifies that Horton's laws properly describe the structure of river networks in middle and small watersheds.

Comparison of two fractal dimensions in actual river networks

The fractal dimensions of 70 actual river networks are calculated by two different methods. One of them is based on the bifurcation ratio and the stream length ratio expressed in Equation (3). Another method is functional box counting

Table 3 | Empirical relationships between two fractal dimensions and corresponding performance statistics

Junction angle	Empirical equation	R^2
30°	$D_g = 6.8742D_r^2 - 15.406D_r + 9.7345$	0.982
45°	$D_g = 5.4832D_r^2 - 11.256D_r + 6.8498$	0.995
60°	$D_g = -0.4483D_r^2 + 4.1699D_r - 2.9234$	0.985
75°	$D_g = -5.3468D_r^2 + 16.304D_r - 10.32$	0.983
90°	$D_g = -8.0529D_r^2 + 22.701D_r - 13.995$	0.986
105°	$D_g = -14.07D_r^2 + 36.932D_r - 22.294$	0.987
120°	$D_g = -21.084D_r^2 + 52.929D_r - 31.284$	0.989

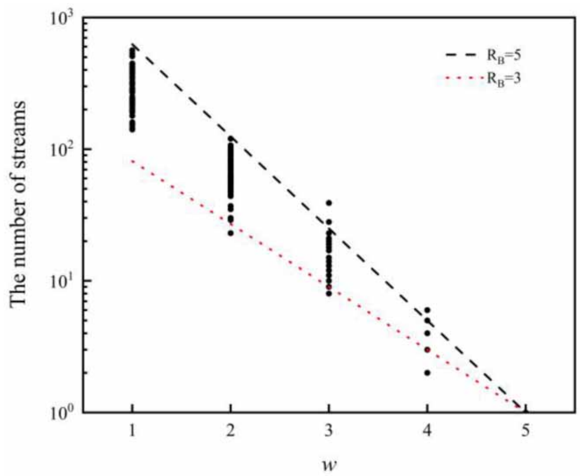

Figure 5 | Relations between the stream order and its corresponding number.

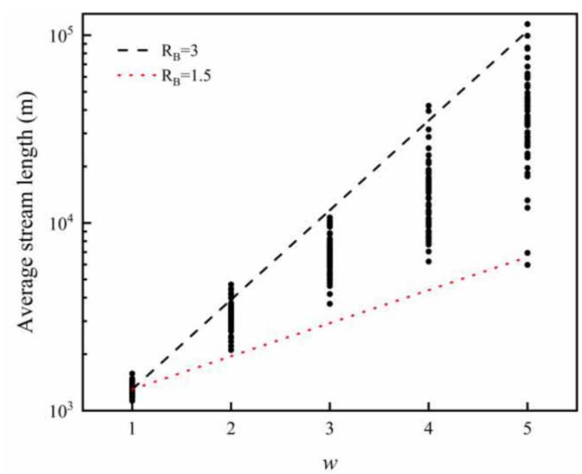

Figure 6 | Relations between the stream order and average stream length.

as described in Equation (4). In the box-counting method, considering that the calculation of fractal dimensions is carried out in middle and small watersheds, small scale-invariant interval (0.1–2.5 km) is chosen. Straight lines for 70 river networks are found in the linear regression of Equation (4) in log–log plot (not shown) with R^2 all above 0.99, meaning that values of fractal dimensions obtained by box-counting method are robust.

The results of fractal dimensions for 70 river networks calculated by two different methods mentioned above are listed in Table 4. For these watersheds, D_g is strictly larger than D_r. The D_g values range from 1.47 to 2.59 with the majority of the values falling between 1.52 and 1.96, and the D_r values range from 1.15 to 1.28 with the

Table 4 | Fractal dimensions of 70 river networks based on box-counting measures and the estimates from Horton's order ratios, respectively

Number	Name	D_r	R^2	D_g	Angle (°)
1	Woluo River	1.196	0.994	1.909	68.42
2	Kuti River	1.184	0.994	1.617	66.76
3	Yixun River	1.167	0.994	**2.288**	75.46
4	Erergeqi River	1.253	0.993	1.749	59.10
5	Qingsongling River	1.228	0.993	1.867	59.02
6	Chao River	1.199	0.994	**2.236**	66.73
7	Guoen River	1.180	0.994	1.896	65.69
8	Fabiela River	1.276	0.993	1.831	60.39
9	Gongbiela River	1.182	0.994	1.855	65.07
10	Ai River	1.179	0.994	1.702	69.29
11	Huai River	1.222	0.994	**2.308**	68.02
12	Yangjia River	1.185	0.994	**2.581**	74.19
13	Sha River	1.189	0.993	1.872	72.09
14	Weifen River	1.180	0.994	1.902	70.35
15	Ye River	1.174	0.994	1.871	71.55
16	Fang River	1.192	0.994	**2.061**	64.08
17	Guangtong River	1.193	0.993	1.826	67.70
18	Lichuan River	1.167	0.993	**2.173**	75.73
19	Yao River	1.216	0.992	1.994	60.77
20	Xi River	1.163	0.994	**2.109**	70.54
21	Tangxi River	1.160	0.997	1.867	70.59
22	Xizhao River	1.202	0.992	**2.097**	58.42
23	Luxi River	1.177	0.993	1.958	71.31
24	Xiang River	1.183	0.993	1.963	68.08
25	Malong River	1.159	0.995	1.829	73.15
26	Tuolin River	1.179	0.994	1.810	69.35
27	Gushui River	1.184	0.994	1.585	67.00
28	Xiaohei River	1.172	0.995	1.835	70.68
29	Laonong River	1.151	0.995	1.667	83.84
30	Changhua River	1.179	0.995	1.961	73.73
31	Beijicun River	1.174	0.989	1.800	72.66
32	Ailin River	1.191	0.992	1.771	68.50
33	Moke River	1.183	0.992	1.519	69.85
34	Jiaban River	1.220	0.991	1.691	57.91
35	Dashi River	1.194	0.991	1.850	69.71
36	Shanshi River	1.178	0.991	1.749	69.41
37	Daling River	1.220	0.989	1.872	63.38
38	Hulu River	1.171	0.989	1.735	74.87
39	Taiping River	1.204	0.988	1.843	69.83
40	Nanting River	1.238	0.989	1.927	60.45
41	Wangzhuang River	1.238	0.988	1.903	62.49
42	Zuo River	1.207	0.989	1.709	64.72
43	Nanweng River	1.209	0.991	1.786	60.80
44	Yinlong River	1.228	0.991	1.717	57.40
45	Hailun River	1.220	0.991	1.705	60.30
46	Huolun River	1.224	0.990	1.763	64.90
47	Dasha River	1.215	0.990	1.821	61.20
48	Liu River	1.223	0.989	1.927	63.00
49	Sa River	1.169	0.952	1.797	70.60
50	Yan River	1.192	0.988	1.657	67.40
51	Shimo River	1.162	0.991	1.729	71.20
52	Wangmo River	1.178	0.991	1.866	71.15
53	Xinquan River	1.217	0.989	1.678	62.95
54	Zhile River	1.163	0.992	1.520	70.30
55	Ahengtixiluoqi River	1.195	0.990	1.825	66.70
56	Huichun River	1.201	0.990	1.796	64.50
57	Xiangyang River	1.201	0.991	1.478	59.10
58	Shidao River	1.206	0.991	1.595	59.20
59	Jianyucha River	1.172	0.989	1.561	67.80
60	Taoping River	1.203	0.990	1.694	69.30
61	Shu River	1.207	0.992	1.745	56.80
62	Caijia River	1.175	0.993	1.601	68.30
63	Ting River	1.214	0.989	1.858	62.60
64	Yangcun River	1.228	0.988	1.732	55.07
65	Huangshui River	1.199	0.991	1.734	61.33
66	Mei River	1.210	0.988	1.860	65.22
67	Tuli River	1.225	0.989	1.797	61.60
68	Xibeicha River	1.225	0.990	1.855	57.80
69	Haiqing River	1.199	0.992	1.658	63.80
70	Mabai River	1.183	0.987	1.790	66.47

bulk of the values between 1.16 and 1.24. According to the concept of fractal dimension of river networks and theoretical analysis (Smart 1972; Marani *et al.* 1991), the values of two kinds of fractal dimensions should fall within the range between 1 and 2. However, the values of the geomorphic fractal dimension for natural river networks obtained from the bifurcation ratio and the stream length ratio do not always fall within the range from 1 to 2

(Magnuszewski 1990, 1993; Bajkiewicz-Grabowska & Olszewski 2001; Fac-Beneda 2013). Therefore, we use bold fonts indicating values not ranging 1–2 (Table 4). These values are not included in the analysis of the relation between two kinds of fractal dimensions. There is no obvious relationship between these two kinds of fractal dimensions and there exist significant distinctions (Figure 7) and the correlation coefficient is only 0.153. For example, D_g values for Huichun River and Xiangyang River are quite different, which are 1.796 and 1.478, respectively, while D_r values for these two rivers are the same. The above results indicate that the relation between these two kinds of fractal dimensions is not linear.

The role of the junction angle on the relation between two kinds of fractal dimensions

The average junction angle of each river network shown in Table 4 is used as a factor for representing the structure of the river network. The results show that all these average junction angles for 70 river networks are acute ones and nearly distributed uniformly in the range 55°–84°. This is similar to that observed in the work of Serres & Roy (1990) indicating the existence of minimum energy principles in the development of drainage basins (Rinaldo et al. 1992; Hooshyar et al. 2017).

As the natural watersheds are influenced by climatic, physiographic and topographic constraints, the structure of river networks can appear remarkably distinct in different regions and various classes of these networks can be determined, such as dendritic, parallel, pinnate, rectangular and trellis networks (Jung et al. 2019). Hence, theoretical curves shown in Figure 4 obtained from fractal plane trees cannot be directly applied to natural river networks. However, the trends of these relationships are similar. Using the average junction angle of river network as a control parameter to represent the relation between these two kinds of fractal dimensions, the original chaotic relationship becomes more regular. When the average junction angle of river network is fixed, the relation between these two kinds of fractal dimensions seems to be a statistically positive correlation (Figure 8), and it may be non-linear. When the average junction angle of river network is small, scatter plots are distributed in the bottom-right corner of the correlation diagram. With the increasing average junction angle, scatter plots begin to move to up-left region. This reflects that the average junction angle has a significant impact on the relation between these two kinds of fractal dimensions. This behavior is the same with that in the geometric fractal tree model. The 70 river networks are divided into 4 groups by the junction angle in the range of 50°–60°, 60°–65°, 65°–70°, 70°–85°, respectively. The points of each group are fitted by the quadratic equation and the equations and R^2 are shown in Table 5.

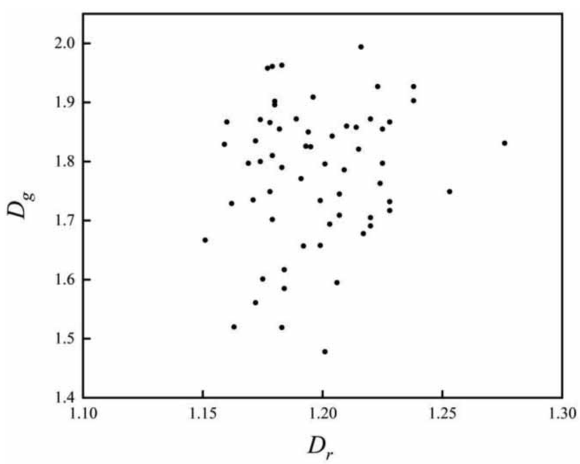

Figure 7 | Correlation analysis for the results obtained with the box-counting method and geomorphic fractal dimension.

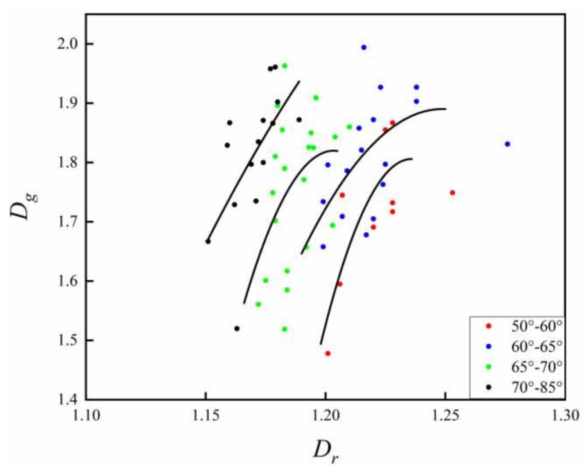

Figure 8 | Correlation analysis of the box-counting method and geomorphic fractal dimension obtained in the study area using the average junction angle as parameter.

Table 5 | Empirical relationships between two fractal dimensions and corresponding performance statistics for actual river networks in the study area

Junction angle	Empirical equation	R^2
50°–60°	$D_g = -217.97D_r^2 + 538.76D_r - 331.1$	0.630
60°–65°	$D_g = -70.56D_r^2 + 176.23D_r - 108.14$	0.296
65°–70°	$D_g = -184.11D_r^2 + 443.1D_r - 264.78$	0.168
70°–85°	$D_g = -24.37D_r^2 + 64.02D_r - 39.73$	0.370

CONCLUSIONS

It is reasonable that the geometric pattern of a drainage basin's river network can be viewed as a fractal with a fractional dimension. Given the diversity of methods for calculating river network fractal dimensions, the existence of multiple fractal dimension values is inevitable. Due to the difficulties in finding 'correct' values for river network fractal dimensions, further exploration is needed in comparing and evaluating the results of those proposed methods. In this paper, we compare two of the most common methods: one is geomorphic fractal dimension obtained from the bifurcation ratio and the stream length ratio, and the other to calculate the fractal dimension of river networks is box-counting method. Firstly, the synthetic fractal trees are used to explain the role of the junction angle on the relation between two kinds of fractal dimensions. Then 70 river networks in China are investigated in terms of their two kinds of fractal dimensions to further verify how the junction angle controls the relation between these two kinds of fractal dimensions. The following conclusions could be obtained:

1. The relationship between these two kinds of fractal dimensions is controlled by the junction angle.
2. When the junction angle remains constant, there exists a relatively good non-linear positive relationship between these two fractal dimensions. D_g is larger than D_r.
3. D_r is decreasing with the increase of the junction angle when D_g keeps constant.

These results provide a linkage between these two kinds of fractal dimensions and offer the possibility for better understanding the geometry and composition of river networks. Further work is required to refine procedures for the construction of the relationship between different kinds of fractal dimensions and compare the results produced by different models of fractal plane trees.

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DATA AVAILABILITY STATEMENT

All relevant data are included in the paper or its Supplementary Information.

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