

Comparison between two generalized Nash models with a same non-zero initial condition

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ABSTRACT

Initial condition can impact the forecast precision especially in a real-time forecasting stage. The discrete linear cascade model (DLCM) and the generalized Nash model (GNM), though expressed in different ways, are both the generalization of the Nash cascade model considering the initial condition. This paper investigates the relationship and difference between DLCM and GNM both mathematically and experimentally. Mathematically, the main difference lies in the way to estimate the initial storage state. In the DLCM, the initial state is estimated and not unique, while that in the GNM is observed and unique. Hence, the GNM is the exact solution of the Nash cascade model, while the DLCM is an approximate solution and it can be transformed to the GNM when the initial storage state is calculated by the approach suggested in the GNM. As a discrete solution, the DLCM can be directly applied to the practical discrete streamflow data system. However, the numerical calculation approach such as the finite difference method is often used to make the GNM practically applicable. Finally, a test example obtained by the solution of the Saint-Venant equations is used to illustrate this difference. The results show that the GNM provides a unique solution while the DLCM has multiple solutions, whose forecast precision depends upon the estimate accuracy of the current state.

Key words: discrete linear cascade model, generalized Nash model, initial storage state, unique solution

HIGHLIGHTS

- The main difference lies in the way to estimate the initial storage state.
- The GNM is the unique solution of the Nash cascade model.
- The DLCM is an approximate solution of the Nash cascade model.
- The DLCM can be transformed to the GNM.

1. INTRODUCTION

In hydrology, the concept of linear reservoir cascade suggested by Nash (1957) is widely used in connection with the mathematical modeling of surface runoff. Many Nash cascade based models have been developed to model the rainfall-runoff process, e.g., the urban parallel cascade model proposed by Diskin *et al.* (1978), the hybrid and extended hybrid model, respectively, represented by Bhunya *et al.* (2005) and Singh *et al.* (2007), the two-reservoir variable storage coefficient model formulated by Bhunya *et al.* (2008), the cascade of submerged reservoirs model developed by Kurnatowski (2017), the inter-connected linear reservoir model (ICLRM) introduced by Khaleghi *et al.* (2018), and the linear combination model of Nash model and ICLRM recently developed by Monajemi *et al.* (2021). Most of these models introduced the concept of the Nash model – cascade of linear reservoirs. In fact, the Nash model is also applicable to river flow routing (Yan *et al.* 2015, 2019), which has been done independently by Kalinin & Milyukov (1957), also known as a Characteristic Reach method. In the river flood forecasting, the initial state is usually thought to be insignificant as its effect will vanish after a sufficiently long simulation time. But for some short time prediction situations, just like the identification of the impulse-response function and the real-time forecasting, the initial state will produce a relatively great impact. Szollosi-Nagy (1982) formulated a state-space description of the Nash cascade model, i.e., the discrete linear cascade model (DLCM) in a matrix form whereby the initial state was included that can be thought of a generalization of the Nash cascade model. The determination of the initial state of the DLCM was then proposed by Szollosi-Nagy (1987) via observability analysis. The DLCM was discretized originally in a pulse-data system framework which seems more suitable for the irregularly

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changing precipitation but not necessarily for the gradually changing streamflow. Under a linear change assumption of the input, the DLCM was extended by Szilagyi (2003) to a sample-data system framework. Since then, Szilagyi and his team have made great efforts to develop this model (Szilagyi 2006; Szilagyi & Laurinyecz 2014). With so many advantages that have been summarized by Szilagyi (2006), the DLCM has been in operational use for over 30 years in Hungary. However, it has not yet been applied more broadly except in Hungary. One possible reason may be due to the complicated mathematical expression and calculation. The development of a simpler expression of the DLCM is necessary to make it more popular and applicable in practice.

Yan *et al.* (2015) applied the Laplace transform and the principle of mathematical induction to solve the n th order non-homogeneous linear ordinary differential equation (NLODE) of the Nash cascade model with a non-zero initial condition, and obtained the generalized Nash model (GNM) with a simpler expression. What's more, the GNM has been physically interpreted, which makes it a conceptual model and not only a mathematical formulation. Compared with the DLCM, the GNM is also obtained from the Nash cascade model with the same initial condition. But whether the expressions or the simulation results of these two models are differently exhibited, there may be some confusion to the model users. It is necessary to distinguish these two models for the users. Hence, the relationship and difference between DLCM and GNM are studied in the following sections both mathematically and experimentally.

2. RELATIONSHIP BETWEEN THE DLCM AND THE GNM

In the derivation of the DLCM, a state-space matrix approach was used. The state and output equations of the Nash cascade model are formulated as follows (Szollosi-Nagy 1982; Szilagyi 2003):

$$S(t) = \Phi(t, t_0)S(t_0) + \int_{t_0}^t \Phi(t, \tau)GI(\tau)d\tau \quad (1)$$

$$O(t) = HS(t) \quad (2)$$

where $S(t)$ is the storage state vector and denotes the stored water volumes of the n linear cascade reservoirs,

$$\Phi(t, t_0) = \begin{bmatrix} e^{-(t-t_0)/K} & 0 & \dots & 0 \\ \frac{t-t_0}{K}e^{-(t-t_0)/K} & e^{-(t-t_0)/K} & \dots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ \frac{(t-t_0)^{n-1}}{K^{n-1}(n-1)!}e^{-(t-t_0)/K} & \dots & \frac{t-t_0}{K}e^{-(t-t_0)/K} & e^{-(t-t_0)/K} \end{bmatrix} \quad (3)$$

is the state transition matrix, t_0 is the initial time, K is the storage coefficient, $G = [1, 0, \dots, 0]^T$, $I(\cdot)$ is the instantaneous inflow of the first reservoir, $O(t)$ is the outflow, and $H = [0, 0, \dots, 1/K]$.

Combining Equations (1) and (2) and assuming $t_0 = 0$, one obtains (Szilagyi 2006)

$$O(t) = H\Phi(t, 0)S(0) + \int_0^t u(t-\tau)I(\tau)d\tau \quad (4)$$

where $u(\cdot)$ is the instantaneous unit hydrograph.

Equation (4) is the basic formula for DLCM. For the discrete streamflow data system, assuming that both input and output are sampled at equidistant sampling intervals Δt , the recursive form of the DLCM can be written as follows (Szilagyi 2006):

$$O(t + \Delta t) = H\Phi(\Delta t, 0)S(t) + \int_t^{t+\Delta t} u(t + \Delta t - \tau)I(\tau)d\tau \quad (5)$$

Once the initial storage state vector $S(0)$ is obtained, the outflow can be estimated recursively by using Equations (1) and (5). The identification of $S(0)$ is an inverse problem, which can be computed by inverting Equation (4) and from the first n

input-output pairs, as originally proposed by Szollosi-Nagy (1987) and Szilagyi (2003), i.e.,

$$S(0) = \Omega_n^{-1}(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)^T \tag{6}$$

where

$$\Omega_n = [H\Omega(\Delta t, 0), H\Omega^2(\Delta t, 0), \dots, H\Omega^n(\Delta t, 0)]^T \tag{7}$$

and

$$\varepsilon_i = O(i\Delta t) - \int_0^{i\Delta t} u(t - \tau)I(\tau)d\tau, \quad i = 1, \dots, n. \tag{8}$$

That's the complete procedure of the DLCM. It is a discrete solution of the Nash cascade model. Actually, the initial storage state vector $S(0)$ can be calculated by another simpler approach that has been proposed by Yan *et al.* (2015). From the linear storage-outflow relationship suggested in the Nash cascade model, we have

$$S(0) = [KO_1(0), KO_2(0), \dots, KO_n(0)]^T \tag{9}$$

where $O_j(0)(j = 1, \dots, n)$ is the initial outflow of the j th reservoir and can be computed by (Yan *et al.* 2015)

$$O_j(0) = \sum_{i=0}^{n-j} C_{n-j}^i K^i O^{(i)}(0) \tag{10}$$

where $O^{(i)}(0)$ is the i th derivative of $O(0)$, and

$$C_n^r = \frac{n!}{r!(n-r)!} \tag{11}$$

is the combination formula. Note that $O_n(0)$ is equal to the initial downstream outflow $O(0)$. Then,

$$\begin{aligned} H\Phi(t, 0)S(0) &= \sum_{j=1}^n \frac{e^{\frac{t}{K}}}{(n-j)!} \left(\frac{t}{K}\right)^{n-j} \sum_{i=0}^{n-j} C_{n-j}^i K^i O^{(i)}(0) \\ &= \sum_{j=0}^{n-1} \frac{e^{\frac{t}{K}}}{j!} \left(\frac{t}{K}\right)^j \sum_{i=0}^j C_j^i K^i O^{(i)}(0) \end{aligned} \tag{12}$$

Note that $H\Phi(t, 0)S(0)$ is just equal to $R_0(t)$ that has been defined in the GNM (Yan *et al.* 2015). Substituting Equations (3) and (9) into Equation (4) gives

$$O(t) = e^{\frac{t}{K}} \sum_{j=0}^{n-1} \sum_{i=0}^j O^{(i)}(0) \frac{t^j}{j!(j-i)!K^{j-i}} + \int_0^t u(t - \tau)I(\tau)d\tau \tag{13}$$

That's just the GNM that has been proposed by Yan *et al.* (2015). Hence, the DLCM can be transformed to the GNM when the initial storage state is calculated by the linear storage-outflow relationship.

3. DIFFERENCE BETWEEN THE DLCM AND THE GNM

The main difference between the two models lies in the estimation of the initial storage state. In the DLCM, the initial storage state $S(0)$ is expressed as a function of the first n input-output pairs, while, in the GNM, it is expressed as a function of the i th derivative of the initial outflow.

To further illustrate this difference, take the special case of $n = 1$ as an example. In the DLCM, the initial storage $S(0)$ can be estimated by (Szollosi-Nagy 1987)

$$S(0) = Ke^{\frac{\Delta t}{K}}[O(\Delta t) - (1 - e^{-\frac{\Delta t}{K}})I(0)] \quad (14)$$

It is suggested that the initial storage is estimated by the input/output pair $[I(0), O(\Delta t)]$. In fact, when $n = 1$, the river flow routing system can be described by the following NLODE (Szollosi-Nagy 1982; Yan *et al.* 2015)

$$KO'(t) = I(t) - O(t) \quad (15)$$

It is easy to get the solution of this NLODE with a result of

$$O(t) = O(0)e^{-\frac{t}{K}} + \int_0^t u(t - \tau)I(\tau)d\tau. \quad (16)$$

Provided that $I(t)$ is taken to be constant at the value it obtains at time t , in the $[t, t + \Delta t]$ interval (Szilagyi 2003), for one step ahead, we obtain

$$O(\Delta t) = O(0)e^{-\frac{\Delta t}{K}} + (1 - e^{-\frac{\Delta t}{K}})I(0) \quad (17)$$

Then, the initial outflow can be estimated by

$$O_{\text{est}}(0) = e^{\frac{\Delta t}{K}}[O(\Delta t) - (1 - e^{-\frac{\Delta t}{K}})I(0)] \quad (18)$$

where $O_{\text{est}}(0)$ is the estimated initial outflow.

Combing Equations (14) and (18) yields

$$S(0) = KO_{\text{est}}(0) \quad (19)$$

On the contrary, in the GNM, the initial state is directly obtained by Equation (9) based on the concept of linear reservoir with a result of

$$S(0) = KO_{\text{obs}}(0) \quad (20)$$

where $O_{\text{obs}}(0)$ is the observed initial outflow.

Comparison of Equations (19) and (20) shows that the difference between the two models in the case of $n = 1$ lies in the fact that whether the initial outflow is estimated or observed. In the DLCM, the initial outflow $O(0)$ is estimated by the observed input/output pair $[I(0), O(\Delta t)]$, as shown in Equation (18). In fact, at the initial time, the outflow for the next time step $O(\Delta t)$ is still unknown, while the observed initial outflow $O(0)$ is available at that time and does not need to be estimated. Instead, this observed value is directly used in the GNM. Theoretically, they are equivalent with the same numerical values if no predict error exists. However, the predict error is virtually inevitable. Though this error may be ignored in some cases, it's at least a truth for a real-time forecasting that the estimation of the current state $S(t)$ depends on the current inflow $I(t)$ and the outflow for the next time step $O(t + \Delta t)$ when the recursive DLCM is employed according to Equation (5). It seems paradoxical because the outflow for the next time step $O(t + \Delta t)$ is still unknown at the current time and is to be predicted by using the current state $S(t)$. The approach used in the DLCM to deal with this paradox is to estimate the current state $S(t)$ by applying the transition matrix to the initial state from Equation (1). In the case of $n = 1$, $S(t)$ calculated from Equation (1) can be simplified as follows:

$$S(t) = S(0)e^{-\frac{t}{K}} + \int_0^t e^{-\frac{(t-\tau)}{K}}I(\tau)d\tau \quad (21)$$

where $S(0)$ is estimated by Equation (14). Then, the recursive DLCM can be written as

$$\begin{aligned} O(t + \Delta t) &= \frac{1}{K} S(t + \Delta t) \\ &= \frac{1}{K} \left[S(t) e^{-\frac{\Delta t}{K}} + \int_t^{t+\Delta t} e^{-\frac{(t+\Delta t-\tau)}{K}} I(\tau) d\tau \right] \\ &= \frac{1}{K} S(t) e^{-\frac{\Delta t}{K}} + (1 - e^{-\frac{\Delta t}{K}}) I(t). \end{aligned} \quad (22)$$

It is suggested from Equation (21) that the current state $S(t)$ depends to some extent on the initial state $S(0)$, or equivalently, the current state is not unique since any time before current time can be taken as the initial time. As a result, the outflow for the next time step $O(t + \Delta t)$ determined by $S(t)$ and $I(t)$ from Equation (22) will have multiple solutions. While in the recursive GNM, the current state is uniquely determined by the current outflow $O(t)$ according to Equation (9) in which the initial time is set to the current time. In this case, the recursive GNM has the following unique expression:

$$O(t + \Delta t) = O(t) e^{-\frac{\Delta t}{K}} + (1 - e^{-\frac{\Delta t}{K}}) I(t) \quad (23)$$

Comparison of Equations (22) and (23) suggests that the only difference between the recursive form of the two models in the case of $n = 1$ lies in whether the current outflow is estimated or observed. Similarly, for $n > 1$ in the GNM, the current storage state is estimated by current outflow and its derivatives, as shown in Equations (9) and (10), or equivalently current and antecedent observed outflows but not estimated ones used in the DLCM. Hence, the DLCM is an approximate solution but not the exact solution of the Nash cascade model. As an analytical solution, the GNM is applicable to the natural continuous streamflow system. However, in practice, the streamflow data are usually discretely measured. The derivative term in the GNM does not exist in the discrete streamflow data system. To make the GNM practically applicable, Yan *et al.* (2019) defined a variable S_n -curve to simplify the expression of the derivative term and further discretized the GNM. While the DLCM, as a discrete solution, can be directly applied to the discrete streamflow data system.

4. AN ILLUSTRATIVE EXAMPLE

A test example was used to further illustrate this difference. This example was obtained by numerically integrating the Saint-Venant equations of open channel flow over a rectangular channel of $L = 120$ km in length, $B = 20$ m in width, and a constant channel slope $S_0 = 0.0002$. The Manning's roughness parameter n_0 was set to 0.004 for the entire length of the channel. The upstream boundary condition was defined by the following inflow hydrograph (Camacho & Lees 1999)

$$I(t) = I_b + (I_p - I_b) \left(\frac{t}{t_p} \right)^{\frac{1}{\gamma-1}} \exp\left(\frac{1-t/t_p}{\gamma-1} \right) \quad (24)$$

where I_b is the initial steady flow ($100 \text{ m}^3/\text{s}$) in the reach; I_p is the peak flow ($300 \text{ m}^3/\text{s}$); t_p is the time to peak (20 h), and γ is the skewness factor (1.2). The downstream boundary condition, fixed at 120 km downstream, was defined by a looped-rating curve based on the Manning equation for normal flow.

The hydrograph was routed to distances of 20, 40, 60, 80, 100, and 120 km from the inflow section. To minimize the somewhat artificial nature of the upper and lower boundary conditions (Szilagyi 2006), the middle reach between 40 and 80 km was selected for flow routing, i.e., the flowrate values given by the Saint-Venant equations at 40 and 80 km served as the 'observed' upstream and downstream flow values, respectively. The SCE-UA global optimization algorithm (Duan *et al.* 1994) was used to optimize parameters in the two models by directly minimizing the root mean squared error, with same optimized values of $n = 1$ and $K = 4.6$ h. In the real-time forecasting, for example, take $t = 16$ h as the current time, then any time before $t = 16$ h can be taken as the initial time t_0 in the DLCM. If $t_0 = 1$ h, the current state can be estimated by combing Equations (14) and (21), and further the current outflow, i.e., $O(t = 16 \text{ h})$ can be calculated by linear storage-outflow relationship, with a result of $126.13 \text{ m}^3/\text{s}$. If $t_0 = 15$ h, the current outflow $O(t = 16 \text{ h})$ was estimated by the same procedure with a result of $118.58 \text{ m}^3/\text{s}$. Then, this value was used to estimate the following outflow by using Equation (22). While for the GNM, the 'observed' value of $O(t = 16 \text{ h}) = 113.92 \text{ m}^3/\text{s}$ was directly used to estimate the outflow

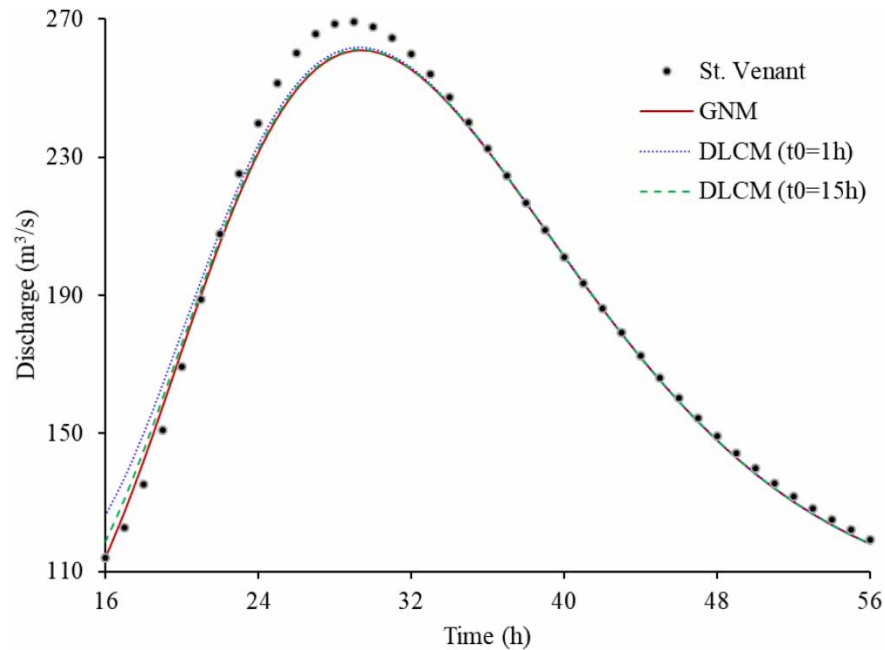


Figure 1 | Routing results for the discrete linear cascade model (DLCM) and the generalized nash model (GNM).

recursively by using Equation (23). The hydrographs obtained by the DLCM with different initial time as well as the GNM are illustrated in Figure 1. With different current outflow, the DLCM correspondingly provided different forecasted discharge values, especially the first few ones. The Nash–Sutcliffe efficiency coefficient (E_{NS}) values for $t_0 = 1$ h and $t_0 = 15$ h were 0.9882 and 0.9919, respectively. The GNM provided the unique and also the best forecasted results, with a result of $E_{NS} = 0.9928$. It is shown from this example that the DLCM has multiple solutions and the forecast precision depends upon the estimate accuracy of the current state.

5. CONCLUSION

Both the DLCM and GNM are derived from the Nash cascade model with a same non-zero initial condition. The DLCM formulated the Nash cascade model in a matrix form based on the principles of state-space analysis, while the GNM was written in a simpler algebraic expression after the complicated theoretical derivation. To clarify these two models, the relationship and difference have been investigated mathematically and experimentally. The main conclusions are summarized as follows:

- (1) The DLCM can be transformed to the GNM when the initial storage state is directly calculated by the linear storage-outflow relationship suggested in the Nash cascade model.
- (2) The essential difference between these two models lies in the identification of the initial state. In the DLCM, the initial state is estimated, while that in the GNM is observed.
- (3) The DLCM is an approximate solution of the Nash cascade model but not the exact solution due to its nonuniqueness of the initial estimated state. The GNM is the unique analytical solution of the Nash cascade model, whose initial state is implicitly written in a form of derivative and does not need to be estimated separately.

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DATA AVAILABILITY STATEMENT

All relevant data are included in the paper or its Supplementary Information.

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