Helicity Injection Fluxes of Twist and Writhe

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Abstract

We consider helicity flux equations proposed by previous authors and offer interpretations of these helicity injection equations with twist and writhe. At first, comparing Pariat’s helicity flux equation (2005, A&A, 439, 1191) with Longcope’s (2007, ApJ, 668, 571), we find that the Pariat’s helicity injection equation shows helicity evolution of continuous magnetic field lines, which is an extreme case of the Longcope’s multiple flux tube model. Next, we investigated Berger’s helicity flux density (1984, J. Fluid Mech., 147, 133) and Pariat’s on the positive and negative polarities, and confirmed that these equations produce different helicity fluxes on the positive and negative polarities. Under the idea of the multiple flux tube model, we propose two interpretations of helicity injection fluxes of twist and writhe; the single flux tube model and the continuous field line model. Recent results of helicity flux injections need to be reanalyzed.

Key words: Sun: magnetic fields — Sun: magnetic helicity

1. Introduction

Magnetic helicity is a conserved quantity in ideal magnetohydrodynamics (Berger & Field 1984), and is expected to be a key parameter in the storage and release processes of magnetic field energy. At present, there are two major issues concerning the helicity flux calculation. One is how to obtain velocity fields on a boundary (e.g., Welsch et al. 2007), and another is which of helicity injection equations is appropriate. The latter is the main issue in this paper. We propose interpretations of helicity flux equations in this paper.

Berger and Field (1984) firstly derived a helicity flux equation with the idea of relative magnetic helicity. (We precisely introduce helicity injection equations in the next section.) Chae (2001) attempted to obtain velocity fields from magnetograms, and firstly obtained helicity injection fluxes on an active region. Since Chae (2001), many authors have studied how to estimate velocity fields. For more precise descriptions, see Welsch et al. (2007). On the other hand, helicity flux equations are another major issue. Welsch and Longcope (2003) studied helicity injection fluxes of magnetic patches in a quiet region, and showed a helicity injection equation divided into twist and writhe components. Longcope, Ravindra, and Barnes (2007a) studied helicity injection fluxes in active regions with a more generalized equation. Pariat, Démoulin, and Berger (2005) showed a different equation from that of Berger and Field (1984). Their purpose is to reduce fake helicity flux densities (e.g., see their figure 2). There is no paper discussing relations between these equations.

We are motivated by two recent papers, Tian and Alexander (2009) and Fan, Alexander, and Tian (2009). Tian and Alexander (2009) obtained helicity injection fluxes in emerging regions with an equation proposed by Pariat, Démoulin, and Berger (2005), and reported helicity injection asymmetries between the positive and negative polarities. Here, we have a question: what do these helicity injection asymmetries indicate? A magnetic helicity injection flux is originally defined as an area-integrated value (Berger & Field 1984). Can we divide an area-integrated helicity flux into those on the positive and negative polarities?

Fan, Alexander, and Tian (2009) performed three-dimensional anelastic MHD simulations, and calculated helicity injection fluxes on the positive and negative polarities with the original helicity flux equation (Berger & Field 1984). They interpreted these helicity fluxes with the single flux tube model proposed by Longcope and Welsch (2000), and compared them with the helicity fluxes of Tian and Alexander (2009). There is another question: can helicity fluxes obtained from different equations be compared on the positive and negative polarities? Figure 6 of Pariat, Démoulin, and Berger (2005) shows that different helicity equations produce different helicity fluxes on the positive and negative polarities.

In this work, we investigate the helicity flux equations, and discuss interpretations of these equations. In section 2, the helicity flux equations are introduced. In section 3, helicity flux densities estimated from different motions are shown. We propose interpretations of the helicity flux equations in section 4. Discussions are given in section 5. Our results are summarized in section 6.

2. Helicity Injection Equations

In the following, we introduce helicity flux equations. The original helicity flux equation derived by Berger and Field (1984) is as follows:

\[
H_R = 2 \int A_P \times E \ dS,
\]

\[
= 2 \int [(A_P \cdot B_i) V_n - (A_P \cdot V_i) B_n] \cdot n \ dS, \tag{1}
\]

where \(H_R\) is the relative magnetic helicity, \(A\) is the vector potential, \(V\) is the velocity field vector, \(B\) is the magnetic
field vector, subscript $P$ means the potential field component, subscripts $t$ and $n$ indicate transverse and normal components, respectively, and $n$ is the unit normal vector on the boundary directed into the region. For the precise derivation of this equation, see Berger and Field (1984).

Démoulin and Berger (2003) derived the following equation from equation (1):

$$\dot{H}_R = -2 \int [(A_p \cdot u) B_n] \cdot n \, dS,$$

(2)

$$u = V_t - \frac{V_n}{B_n} B_t.$$  

(3)

They argued that the local correlation tracking method can estimate $u$, and the helicity injection flux is calculated only from $u$.

In order to calculate the helicity fluxes of twist and writhe on magnetic-field patches, Longcope, Ravindra, and Barnes (2007a) derived the following equation:

$$\dot{H}_R = -2 \sum_i \int_{S_i} [(A_p \cdot u) B_n] \, dS$$

$$- \frac{1}{\pi} \sum_i \sum_{j \neq i} \int_{S_i} \int_{S_j} \frac{r \times u}{r^2} \cdot n B_n B'_n \, dS' \, dS,$$  

(4)

where $S$ is the area of a magnetic patch, and $r = (x - x')$ is the relative position vector of magnetic patches. The first term on the right-hand side of this equation shows temporal evolution of twist in each patch, and the second term that of writhe.

In order to obtain more appropriate helicity flux densities, Pariat, Démoulin, and Berger (2005) proposed the following equation:

$$\dot{H}_R = -1 \int \frac{r \times [u - u']}{r^2} \cdot n B_n B'_n \, dS' \, dS.$$  

(5)

Here, one can find that, if the relative velocity $(u - u')$ is applied in equation (4), the right-hand side of equation (5) is very similar to the second term of equation (4). These terms show (relative) rotation rates at two points $(x$ and $x')$. When multiple flux tubes are considered on the photosphere, we should use equation (4). Then, what situation do we assume in applying equation (5)? Here, we propose that equation (5) should be applied when continuous magnetic field lines are assumed.

This is because, if we consider many flux tubes in the solar atmosphere, equation (4) becomes equivalent to equation (5). When there are many flux tubes, we can assume that an extreme case in the cross section of each flux tube is infinitely small. Each flux tube is considered to be a magnetic field line; also, a vector potential at a field line can be calculated only from the other field lines. We cannot divide the cross section of each field line into several parts, and cannot calculate a vector potential from a field line, itself. This means that the twist cannot be defined in a field line, and the first term on the right-hand side of equation (4) vanishes. Then, the second term of equation (4) is equal to equation (5), if the relative velocity is applied. It seems that Pariat, Démoulin, and Berger (2005) implicitly assumed continuous field lines having no twist. They did not consider twist components in magnetic flux tubes, as is shown in their figure 3.

### 3. Helicity Flux Densities

Here, we show similar helicity flux density maps to those in Pariat, Démoulin, and Berger (2005). These helicity flux densities are produced by separating, spinning, and braiding motions. In order to obtain more information, we divide $A_p$ into two components, as follows:

$$A_p(x, y) = A^+(x, y) + A^-(x, y),$$

(6)

where superscripts “+” and “-” indicate the values integrated from the positive and negative magnetic polarities, respectively. With this equation, the helicity flux densities of equation (1) are also divided into two components, as follows:

$$\dot{H}_R = \int 2(A^+ \times E + A^- \times E) \, dS,$$

(7)

$$= \int (G_A^+ + G_A^-) \, dS.$$  

(8)

We also divided the helicity flux densities of equation (5) into two components, $G_A^+$ and $G_A^-$. In the following, for calculation of $A$, we applied the integration approach, and not the Fourier approach. Owing to this integration approach, we can compare the helicity flux densities with great accuracy.

For the sake of simplicity, we consider a flux tube having constant field strength in each polarity,

$$B(x, y) = (0, 0, \pm B_0),$$  

(9)

where $B_0$ is set to be unity. The radii of the flux tubes ($R_0$) are 3. The distances between the center positions of the negative and positive polarities are 10. The pixel sizes are 0.1.

#### 3.1. Separating Motion

In this subsection, we consider a separating motion as follows:

$$V_- (x, y) = (V_0, 0, 0),$$

(10)

where subscript “-” indicates the physical quantity on the negative polarity, and $V_0$ is a constant value (here $V_0 = 10$). Hereafter, we briefly consider only motions of the negative polarity.

The left panels of figure 1 show $G_A^+$ and $G_A^-$, and the right panels $G_0^+$ and $G_0^-$. In the left panels, the helicity fluxes appear only on the negative polarity region, because the velocity field is given on the negative polarity. In the right panels, $G_0^+$ on the positive polarity and $G_0^-$ on the negative polarity are zero, because relative velocities are zero. The integrated helicity fluxes in this figure are almost zero. As is reported by Pariat, Démoulin, and Berger (2005), the unsigned flux densities of $G_A^+$ and $G_A^-$ are larger than those of $G_0^+$ and $G_0^-$.  

#### 3.2. Braiding and Spinning Motions

In figure 6 of Pariat, Démoulin, and Berger (2005), they showed helicity flux densities estimated from one polarity rotation motion around another polarity. However, as they described, this rotation motion includes spinning and braiding motions simultaneously. In this subsection we show flux densities due to braiding and spinning motions.

A braiding motion is given as
V− = (0, V0, 0).

The left panels of figure 2 show that this braiding motion produces a net helicity flux \( G_π^+ \) only on the negative polarity. On the other hand, the right panels show that the braiding motion produces helicity flux densities both on the negative and positive polarities, because of the relative velocities.

Spinning motion is given as

\[
V_− = r \times (ω₀ z),
\]

where \( ω₀ \) is the constant angular velocity (\( ω₀ = 1 \)). Here, \( r = x - x₀ \) is the relative position vector, and \( x₀ \) is the center of the negative polarity. In the right panels of figure 3, unsigned densities of \( G_π^+ \) on the positive polarity are not zero and much smaller than those on the negative polarity. The values of \( G_π^+ \) on the negative polarity are constant.

Here, we can reproduce figure 6 of Pariat, Démoulin, and Berger (2005) by summing the flux densities shown in the right panels of figures 2 and 3. After summing, as explained in the Appendix of Pariat, Démoulin, and Berger (2005), the flux densities on the negative polarity are almost zero, and those on the positive polarity show positive values.

Note that there are resolution dependencies in the integrated helicity fluxes of \( G_π^+ \) on the negative polarity, \( G_π^− \) on the negative polarity, and \( G_A^− \) on the positive polarity. This is due to the left-right asymmetries of these helicity flux densities. As the spatial resolution increases, the integrated helicity fluxes of these components are closer to zero, while the integrated helicity fluxes of \( G_π^+ \) and \( G_A^− \) on the negative polarity are closer to a constant value \( (ω₀ Φ^2 / 2π) \), where \( Φ \) is the magnetic flux.

4. Helicity Flux Interpretations

Under the idea of the multiple flux tube model (Welsch & Longcope 2003), there are three interpretations of helicity fluxes...
Fig. 2. Helicity flux densities due to a braiding motion. The panels are the same as in figure 1. The dynamic range of display is within ±6.4. The helicity flux densities on the left panels are made saturated in this dynamic range.

In the corona, where plasma $\beta$ is less than unity, the shape of magnetic fields is dynamically and approximately decided only from magnetic fields (e.g., nonlinear force free fields: Schrijver et al. 2008). Therefore, the helicity partition between writhe and twist is decided by the dynamic balance in the corona. On the other hand, the helicity injection rates on the photosphere would physically reflect the shape of flux tubes below the photosphere, where plasma $\beta$ is more than unity. Below the photosphere, magnetic flux tubes are believed to be buffeted by surrounding convective motions (e.g., Longcope et al. 1998).

4.1. The Single Flux Tube Model

In this model, helicity injection fluxes are interpreted by using a single flux tube. As shown in figures 2 and 3, temporal evolutions of twist ($T_w$) and writhe ($W_r$) are divided as follows:

$$\dot{H}_{T_w}^+ = 2 \int_{B>0} A^+ \times E \, dS,$$

$$\dot{H}_{W_r}^+ = 2 \int_{B>0} A^- \times E \, dS,$$

$$\dot{H}_{W_r}^- = 2 \int_{B<0} A^+ \times E \, dS,$$

$$\dot{H}_{T_w}^- = 2 \int_{B<0} A^- \times E \, dS.$$

Applying the results of Longcope and Welsch (2000), we can estimate the average angular velocities ($\omega$) on the positive and negative polarities as

$$\dot{H}_{T_w}^i = \frac{d^2}{d\omega^2} \Omega^i,$$

where superscript $i$ indicates + or −. Also, the rotation rate ($\Omega$) of each polarity can be estimated as
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Fig. 3. Helicity flux densities due to a spinning motion. The panels are the same as in figure 1. The dynamic range of display is within $\pm 6.4$. The helicity flux densities on the left panels are made saturated in this dynamic range.

\[ \hat{H}_{Wr} = \frac{\Phi^2}{2\pi} \Omega_i, \]

(18)

Note that we can apply this single flux tube model for all active regions, and can make a comparison of $\omega$ and $\Omega$ between all active regions. One of our future subjects is to recalculate helicity injection fluxes in the events studied by Yamamoto and Sakurai (2009) with a single flux tube model.

4.2. The Multiple Flux Tube Model

Welsch and Longcope (2003) analyzed helicity injection fluxes of magnetic patches in a quiet-Sun region. Longcope, Ravindra, and Barnes (2007a) developed an area partitioning method, and analyzed six active regions using equation (4). They reported that there is no tendency among the helicity fluxes in these active regions. Applying equation (4), Longcope et al. (2007b) studied quantitative relations between an injected helicity around flare loop footpoints and an ejected helicity by a coronal mass ejection.

This model would be useful for understanding trigger processes of solar flares and other active phenomena. Some results reported in previous papers should be reanalyzed by using this model: flare occurrences (e.g., Moon et al. 2002); sigmoid loop formations (e.g., Yamamoto et al. 2005). In order to understand the true linkage between the magnetic patches of an active region, we need to study nonlinear force free field calculations (e.g., Schrijver et al. 2008).

4.3. The Continuous Field Line Model

Equation (4) proposed by Pariat, Démoülin, and Berger (2005) is applied when continuous field lines are assumed:

\[ \hat{H}_{Wr} = -\frac{1}{2\pi} \int \int \frac{r \times [u - u'] \cdot n B_n B'_n}{r^2} dS' dS. \]

(19)

As pointed out in section 2, this equation shows temporal evolution of writhe in continuous field lines having no twist. At present, we cannot propose any useful application of this model. This is because it is difficult to physically interpret helicity flux densities estimated using equation (4).

Pariat, Démoülin, and Berger (2005) calculated helicity flux densities in their figure 6 from the rotation motion of
the negative polarity, as shown in the right panels of figures 2 and 3. However, one can easily imagine from figure 3 that a constant angular velocity \( \omega \) on the positive polarity reproduces these helicity flux densities. Therefore, helicity injection asymmetries that Tian and Alexander (2009) found would include some uncertainties, and need to be investigated more precisely.

5. Discussion

5.1. Magnetic Flux Balance

Here, we must point out that the magnetic flux balance between the positive and negative polarities is a precondition in the helicity flux calculation. This condition is introduced in derivation of the relative magnetic helicity (Berger & Field 1984), and so far has not yet been frequently confirmed in helicity flux calculations. Tian and Alexander (2009), however, discussed physical relations between the helicity flux asymmetries and the magnetic flux imbalances in active regions. Unfortunately, they did not consider these magnetic flux imbalances as error sources in the helicity flux calculation.

Magnetic flux conservation is introduced as follows. Originally, the magnetic helicity, \( H = \int A \cdot B dV \), is defined in a closed space including closed magnetic fields (Berger & Field 1984). In order to apply magnetic helicity in an open space (e.g., solar corona), Berger and Field (1984) introduced the relative magnetic helicity, \( H_R = \int (A + A_P) \cdot (B - B_P) dV \) [equation (20) of Berger 1999]. In this derivation, Berger and Field (1984) divided a closed space including closed magnetic fields into two (or multiple) spaces. Therefore, magnetic flux balances on boundaries are included here. The original helicity flux equation [equation (1)] is derived from time derivation of the relative magnetic helicity, and magnetic flux balance is also a precondition of this equation.

We can understand that it is difficult to set an appropriate area where the magnetic helicity flux is integrated. However, it is hard to accept large magnetic flux imbalances in helicity flux calculations without any error estimation. In appendix B of Yamamoto and Sakurai (2009), they discussed errors of helicity injection fluxes due to magnetic flux imbalance with a simple equation.

5.2. Helicity Fluxes in Welsch et al. (2007)

Table 1 of Welsch et al. (2007) shows area-integrated helicity fluxes estimated from seven velocity fields by using equations (1) and (5), hereafter, \( dh_A/dt \) and \( dh_\theta/dt \). Here, note that \( dh_A/dt \) and \( dh_\theta/dt \) in this table are not equal, and they have opposite signs even in three cases. There are not such differences between \( dh_A/dt \) and \( dh_\theta/dt \), because equations (1) and (5) are mathematically equivalent. Hereafter, we show the values of \( dh_A/dt \) and \( dh_\theta/dt \) calculated from the limited area. Here, temporally averaged data (AVBX, AVBY, AVBZ, AVVXPP, AVVYPP, and AVVZPP in the data package) were used. The data sizes were \( 288 \times 288 \) pix\(^2\). The pixel sizes were 348.36 km. As a result, \( dh_A/dt \) and \( dh_\theta/dt \) from the whole area show the correct value, and the difference between these values is on the order of 0.001%.

Second, we calculated \( dh_A/dt \) and \( dh_\theta/dt \) from limited pixels. Table 1 of Welsch et al. (2007) gives the pixel numbers in which \( dh_A/dt \) and \( dh_\theta/dt \) were calculated. We selected an area \( 91 \times [x_i, y_i] \leq 191 \), where \( x_i \) and \( y_i \) are the pixel positions in individual data) similar to an area shown in figure 1 of Welsch et al. (2007), and applied the threshold of field strengths of Welsch et al. (2007). They selected pixels where \(|B_z| > 5\% \) of max \(|B_y|\). We selected 4160 pixels, and calculated \( dh_A/dt \) and \( dh_\theta/dt \) from these pixels. The obtained fluxes are \( -1.66 \times 10^{37} \text{Mx}^2 \text{s}^{-1} \). The difference between these values is the order of rounding errors.

Here, one may consider that these ignorable differences between \( dh_A/dt \) and \( dh_\theta/dt \) are due to numerical data, and noisy observation data could produce a large difference. Therefore, third, we calculated \( dh_A/dt \) and \( dh_\theta/dt \) from the noisy magnetic and velocity fields given by pseudo random numbers. The data sizes and the pixel sizes are the same as those of Welsch et al. (2007). The pseudo random numbers were estimated from the normal distribution. The one-sigma values were set to be 1 km s\(^{-1}\) for the velocity field and 10\(^3\) G for the magnetic field. We selected pixels where \(|B_z| > 5\% \) of max \(|B_y|\). As a result, the obtained helicity fluxes are \( 2.21 \times 10^{36} \text{Mx}^2 \text{s}^{-1} \), and the difference of the helicity fluxes is also on the order of the rounding errors.

From these results, we concluded that some mistakes made the differences between \( dh_A/dt \) and \( dh_\theta/dt \) such as are given in table 1 of Welsch et al. (2007). We doubt that helicity flux densities may be integrated from all pixels, not from the limited ones. If so, \( dh_A/dt \) and \( dh_\theta/dt \) integrated from the limited pixels would show different values.

6. Summary

In this paper, we studied relations between the helicity injection equations (Berger & Field 1984; Pariat et al. 2005; Longcope et al. 2007a). From the relation between Longcope’s and Pariat’s helicity flux equations, it is found that the latter means temporal evolution of writhe in the continuous field lines. We also investigated the helicity flux densities estimated by using Berger’s and Pariat’s helicity injection equations, and proposed three interpretations of the helicity flux of twist and writhe (the single flux tube, multiple flux tube, and continuous field line models). Some previous results of helicity injection fluxes should be reanalyzed according to these interpretations.

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