Resonant Excitation of Disk Oscillations in Deformed Disks. V.
Effects of Dissipative Process

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Abstract

It is suggested that a set of positive- and negative-energy oscillations can be resonantly excited in the inner region of deformed (warped or eccentric) relativistic disks. In this paper we examine how a dissipative process affects this wave excitation. The results show that when the resonant condition in frequency is roughly satisfied, and thus the oscillations are excited, the introduction of a dissipative process works so as to decrease the growth rate of the oscillations. When the frequency difference between the two oscillations is larger than a certain amount that is required by the resonant condition, however, the oscillations are excited by introducing a dissipative process. This excitation by the dissipative process can be understood as a special example of a double-diffusive instability.

Key words: accretion, accretion disks — resonance — stability — warp — X-rays: stars

1. Introduction

Investigating the origin of high-frequency quasi-periodic oscillations (HF QPOs) observed in neutron-star and black-hole low-mass X-ray binaries is one of the important subjects concerning astrophysical disks. It is because the investigation will provide clues for understanding the structure of the innermost region of relativistic disks, and for estimating the spins of the central sources. One of the possible causes of HF QPOs is disk oscillations in relativistic disks [see Kato (2001) or Kato et al. (2008) for review of disk oscillations in relativistic disks]. Related to this subject, Kato (2004, 2008a, 2008b) suggested that a pair of positive- and negative-energy oscillations can be excited in deformed (warped or eccentric) disks by their resonant interaction through disk deformation. The presence of this excitation mechanism seems to be confirmed by numerical calculations (Ferreira & Ogilvie 2008; Oktariani et al. 2010) and by analytical examinations (Kato et al. 2011, hereafter Paper I) different from Kato’s original one.

There remains, however, a serious disagreement between the results by Kato’s group (Kato 2004, 2008a, 2008b; Paper I) and numerical ones by Ferreira and Ogilvie (2008). In Ferreira and Ogilvie’s ones, a dissipative process acting on one of the oscillations is necessary to the excitation of the pair of oscillations, while in Kato’s ones any energy dissipative process is unnecessary to the excitation of the set of positive- and negative-energy oscillations.

Let us consider two oscillation modes, the sets of whose frequency and azimuthal wave number are, respectively, \((\omega_1, m_1)\) and \((\omega_2, m_2)\). Disk deformation is denoted by \((m_1, m_2)\), and \(m_1\) is fixed in the present problem. In the case where disk deformation is a stationary warp, for example, \(\omega_D = 0\) and \(m_D = 1\). The necessary conditions of a resonant interaction between the two above oscillations through the disk deformation are:

\[
m_2 = m_1 \pm m_D \quad \text{and} \quad \omega_2 \sim \omega_1 \pm \omega_D.
\]  

(1)

In Kato’s original studies (2004, 2008a, 2008b), he showed that a set of oscillations satisfying conditions (1) is resonantly excited when their wave energies have opposite signs, assuming that a Lindblad resonance occurs in the propagation region of the oscillations. In a subsequent study by Paper I, it is shown that the presence of a Lindblad resonance in wave-propagation region is not essential for excitation. The essentials are opposite signs of the wave energy of resonantly interacting oscillations with a sufficient accuracy of \(\omega_2 \sim \omega_1 \pm \omega_D\). Numerical calculations made by Ferreira and Ogilvie (2008), however, suggest that in addition to the above necessary conditions, a dissipative process in one of the oscillations is necessary to the excitation of the set of oscillations. As the dissipative process, they consider corotation damping. Except for p-mode oscillations with no node in the vertical direction, nonaxisymmetric oscillations are known to be damped by the presence of a corotation resonance, even if the corotation point (radius) appears in the evanescent region of the oscillations. To follow this corotation damping, they take into account an artificial resistive force acting on oscillations in their numerical calculations.

In this paper we take into account a resistive force, as Ferreira and Ogilvie (2008) did, and examine how in our formulation the excitation of the set of positive- and negative-energy oscillations is affected by the presence of the dissipative process. The results of analyses show that the difference between Kato et al. (Kato 2004, 2008a, 2008b; Paper I) and Ferreira and Ogilvie (2008) comes from seeing different aspects of the same resonant excitation process.
2. Basic Hydrodynamical Equations and Resistive Damping

We follow the same procedure as that in Paper I, except that in the present study a resistive force acting on a fluid element is additionally considered. The resistive force is assumed to work in perturbed states in the direction opposite to the Lagrangian velocity perturbation, \( \delta \mathbf{u} \). In this case, the hydrodynamical equation describing perturbations over the equilibrium state is

\[
\delta \left[ \frac{D \mathbf{u}}{Dt} + \nabla \psi + \frac{1}{\rho} \nabla p \right] = -\beta \delta \mathbf{u},
\]

where \( \delta \) denotes the Lagrangian variation and \( D/Dt \) is the Lagrangian time derivative along the perturbed flow \( \mathbf{u} \), and \( \beta \) is the coefficient of the resistive force, which is taken to be a positive constant. The other notations in equation (2) have their usual meanings. We are interested in relativistic disks, but, for the sake of simplicity, the Newtonian formulation is adopted. The effects of the relativity are taken into account by using the pseudo-Newtonian potential introduced in Paczyński and Wiita (1980).

Now, we introduce the Lagrangian displacement vector, \( \xi \), and express equation (2) in terms of \( \xi \). Since \( \delta \mathbf{u} \) is related to the Lagrangian time derivative along the unperturbed flow \( \mathbf{u}_0(r) \), i.e., \( D_0/Dt \), by \( \delta \mathbf{u} = D_0 \xi /Dt \), and \( \delta \) and \( D/Dt \) are commutative (Lynden-Bell & Ostriker 1967), we have

\[
\delta \left( \frac{D \mathbf{u}}{Dt} \right) = \frac{D_0}{Dt} \delta \mathbf{u} = \frac{D_0^2}{Dt^2} \xi.
\]

Hence, equation (2) is written as

\[
\frac{D_0^2}{Dt^2} \xi + \delta \left( \nabla \psi + \frac{1}{\rho} \nabla p \right) = -\beta \frac{D_0}{Dt} \xi.
\]

A general expression for the right-hand side of equation (4) in terms of the displacement vector \( \xi \) has been obtained by Lynden-Bell and Ostriker (1967) in the case where the perturbations are adiabatic and have small amplitudes. The results show that \( \rho_0(r) \) times the left-hand side of equation (4) can be expressed as

\[
\rho_0 \frac{\partial^2 \xi}{\partial t^2} + 2 \rho_0 (u_0 \cdot \nabla) \frac{\partial \xi}{\partial t} + L(\xi).
\]

where \( \partial/\partial t \) is the Eulerian time derivative, \( \rho_0(r) \) is the density in the unperturbed state, and \( L(\xi) \) is a linear Hermitian operator (see in detail Lynden-Bell & Ostriker 1967). A detailed expression for \( L(\xi) \) is unnecessary here, except that it is a Hermitian.

Now, we consider the case where perturbations are weakly nonlinear. In this case, equation (4) is written as

\[
\rho_0 \frac{\partial^2 \xi}{\partial t^2} + 2 \rho_0 (u_0 \cdot \nabla) \frac{\partial \xi}{\partial t} + L(\xi) = -\beta \rho_0 \left( \frac{\partial}{\partial t} + (u_0 \cdot \nabla) \right) \xi + \rho_0 C(\xi, \xi),
\]

where \( C(\xi, \xi) \) is a weakly nonlinear term. No detailed expression for \( C(\xi, \xi) \) is given here [see Kato (2004, 2008a) for detailed expressions], but an important characteristic of \( C(\xi, \xi) \) is that we have commutative relations (Kato 2008a) for an arbitrary set of \( \eta_1(r), \eta_2(r), \text{and} \eta_3(r), \text{e.g.,} \)

\[
\int \rho_0 \eta_1 C(\eta_2, \eta_3) dV = \int \rho_0 \eta_2 C(\eta_3, \eta_1) dV = \int \rho_0 \eta_3 C(\eta_1, \eta_2) dV.
\]

where the integration is made over the whole volume. As shown later, the presence of these commutative relations leads to a simple expression of the instability criterion. We suppose that the presence is a general property of conservative systems beyond the assumption of weak nonlinearity.

Equation (6) is the basic equation to be treated in this paper, which is reduced to equation (1) in Paper I in the limiting case of \( \beta = 0 \). Before examining effects of the \( \beta \)-term on resonant excitation of oscillations, we briefly check whether in the case \( C = 0 \) the \( \beta \)-term of equation (6) really leads to the damping of oscillations. To do so, let us take the time dependence of oscillations to be \( \exp(i \omega t) \). Then, in the case of no damping (i.e., \( C = 0 \)), equation (6) becomes

\[
-\omega^2 \rho_0 \xi + 2i \omega \rho_0 (u_0 \cdot \nabla) \xi + L(\xi) + \beta \rho_0 [i \omega + (u_0 \cdot \nabla)] \xi = 0.
\]

This equation is multiplied by the complex conjugate of \( \xi \) and integrated over the whole volume of the disks, assuming that \( \rho_0 \) vanishes on the disk surface. Taking the imaginary part of the resulting equation [using the fact that \( L \) and \( i \rho_0 (u_0 \cdot \nabla) \) are Hermitian operators], we have

\[
\Im \omega = \frac{1}{2} \beta.
\]

This shows that the \( \beta \)-term acts so as to dampen the oscillations, as expected.
3. Coupling of Two Oscillations through Disk Deformation

As in Paper I, we consider the case where two normal modes of disk oscillations, \( \xi_1(r, t) \) and \( \xi_2(r, t) \), resonantly couple through disk deformation, \( \xi_{1D}(r, t) \). Their sets of frequency and azimuthal wavenumber are denoted by \((\omega_1, m_1), (\omega_2, m_2)\), and \((\omega_D, m_D)\), as mentioned in section 1. For example, \((\omega_1, m_1)\) means that \( \xi_1(r, t) \) is proportional to \( \exp[i(\omega_1t - m_1\varphi)] \), where \( \varphi \) is the azimuthal coordinate of the cylindrical ones \((r, \varphi, z)\), whose origin is at the disk center, and whose \( z \)-axis is in the direction of disk rotation. The resonant conditions are then given by equation (1). In general, through the nonlinear coupling term, \( \rho_0C(\xi, \xi) \) [see equation (6)], some modes other than \( \xi_1 \) and \( \xi_2 \) appear, and their amplitudes as well as those of \( \xi_1 \) and \( \xi_2 \) become time-dependent. The purpose here is to examine the conditions under which the amplitudes of \( \xi_1 \) and \( \xi_2 \) increase with time when they interact with each other through a given disk deformation. In order to examine this problem, we assume that normal modes of oscillations form a complete set, and expand the whole oscillations, \( \xi(r, t) \), resulting from the coupling in the following form:

\[
\xi(r, t) = A_1(t)\xi_1(r, t) + A_2(t)\xi_2(r, t) + \sum_\alpha A_\alpha(t)\xi_\alpha(r, t),
\]

(10)

where \( \xi_\alpha(r, t) \), which is proportional to \( \exp[i(\omega_\alpha t - m_\alpha\varphi)] \), is a normal mode other than \( \xi_1 \) and \( \xi_2 \), and \( \sum_\alpha \) represents the summation over such modes. In equation (10) the disk deformation is not included, since it is assumed to have a large amplitude so that its time variation can be neglected in the present study of excitation of small-amplitude oscillations.

Substitution of equation (10) for \( \xi \) in equation (6) leads to

\[
\sum_{i=1,2} 2\rho_0\frac{dA_i}{dt} \left[ i\omega_i + (u_0 \cdot \nabla) \right] \xi_i + \sum_{i=1,2} \beta_i\rho_0A_i[i\omega_i + (u_0 \cdot \nabla)]\xi_i
+ \sum_\alpha 2\rho_0\frac{dA_\alpha}{dt} \left[ i\omega_\alpha + (u_0 \cdot \nabla) \right] \xi_\alpha + \sum_\alpha \beta_\alpha\rho_0A_\alpha[i\omega_\alpha + (u_0 \cdot \nabla)]\xi_\alpha
= \sum_{i=1,2} \frac{1}{2} A_i \left[ \rho_0C(\xi_1, \xi_D) + \rho_0C(\xi_D, \xi_1) \right] + A_D^* \left[ \rho_0C(\xi_1, \xi_D) + \rho_0C(\xi_D, \xi_1) \right]
+ \sum_\alpha \frac{1}{2} A_\alpha \left[ \rho_0C(\xi_\alpha, \xi_D) + \rho_0C(\xi_D, \xi_\alpha) \right] + A_D^* \left[ \rho_0C(\xi_\alpha, \xi_D) + \rho_0C(\xi_D, \xi_\alpha) \right].
\]

(11)

In deriving this equation, such small terms as \( d^2A_i/dt^2 \) and \( \beta_i dA_i/dt \) have been neglected, since we are interested in slow secular evolutions of \( A \)'s due to the \( \beta \)-term and the coupling term. On the right-hand side of equation (11), the coupling terms that are not related to disk deformation are neglected. Furthermore, the coefficients of the damping term, \( \beta \), of different normal modes of oscillations have been distinguished by attaching a subscript. This is because, as mentioned in section 1, they are introduced to follow the corotation damping, which depends on the modes of oscillations. The asterisk, *, denotes the complex conjugate.

As discussed in Paper I, the normal modes of oscillations can be regarded approximately as a set of orthogonal functions. In particular, in vertically isothermal disks, the orthogonality of the normal modes approximately holds with the weight function, \( \rho_0(r) \) (Paper I; see also Okazaki et al. 1987). That is, for the two normal modes, \( \xi_i \) and \( \xi_j \), with moderately short radial wavelengths, we have

\[
\langle \rho_0 \xi_i^* \xi_j \rangle = \langle \rho_0 \xi_i^* \xi_j \rangle \delta_{ij},
\]

(12)

where \( \langle \rho_0 \xi_i^* \xi_j \rangle \) is the integration of \( \rho_0 \xi_i^* \xi_j \) over the whole volume of the disk, and the subscripts, \( i \) and \( j \), are labels distinguishing between modes.

Equation (11) is now multiplied by \( \xi_i^* \) and integrated over the whole volume. Then, using the orthogonal relation and the resonant conditions (1), we have an equation describing the time evolution of \( A_i(t) \), which is (for detailed procedures see Paper I)

\[
4i\frac{E_i}{\omega_1} \left( \frac{dA_i}{dt} + \frac{1}{2}\beta_1A_i \right)
= A_2A_D \hat{W}_{12}\exp(-i\Delta_+)\delta_{m_1m_2+m_D} + A_2A_D^* \hat{W}_{12*}\exp(-i\Delta_-)\delta_{m_1m_2-m_D},
\]

(13)

where

\[
\Delta_+ = \omega_1 - \omega_2 - \omega_D \quad \text{and} \quad \Delta_- = \omega_1 - \omega_2 + \omega_D.
\]

(14)

The coupling terms, \( \hat{W}_{12} \) and \( \hat{W}_{12*} \), in equation (13) are time-independent quantities defined by

\[
\hat{W}_{12} = \left\langle \frac{1}{2} \left[ \langle \rho_0 \xi_1^* \cdot C(\xi_2, \xi_D) \rangle + \langle \rho_0 \xi_2^* \cdot C(\xi_D, \xi_2) \rangle \right] \exp(i\Delta_+) \right\rangle
\]

(15)

It is noted that \( \hat{W}_{12} \) and \( \hat{W}_{12*} \) are time-independent quantities, since, for example, \( \langle \rho_0 \xi_1^* \cdot C(\xi_2, \xi_D) \rangle \) and \( \langle \rho_0 \xi_2^* \cdot C(\xi_D, \xi_2) \rangle \) in equation (15) are proportional to \( \exp(-i\Delta_+) \).
and
\[ \dot{W}_{1_{2s}} = \frac{1}{2} \left[ \langle \rho_0 \xi_1^* \cdot C(\xi_2, \xi_1^*) \rangle + \langle \rho_0 \xi_2^* \cdot C(\xi_1, \xi_1^*) \rangle \right] \exp(i \Delta_+ t). \] (16)

In equation (13), \( E_1 \) is the wave energy of mode 1, and is defined by (e.g., Kato 2001)
\[ E_1 = \frac{1}{2} \omega_1 \left[ \omega_1 \langle \rho_0 \xi_1^* \xi_1 \rangle - i \langle \rho_0 \xi_1^* (u_0 \cdot \nabla) \xi_1 \rangle \right]. \] (17)
which is a real quantity, since \( i \rho_0 (u_0 \cdot \nabla) \) is a Hermitian operator. The symbol \( \delta_{a,b} \) in equation (13) means that it is the delta-function, i.e., unity when \( a = b \), while zero when \( a \neq b \).

Similarly, multiplying equation (11) by \( \xi_2^* \) and integrating it over the whole volume, we have an equation describing the time evolution of \( A_2(t) \), which is
\[ 4i \frac{E_2}{\omega_2} \left( \frac{dA_2}{dt} + \frac{1}{2} \beta_2 A_2 \right) = A_1 A_D \dot{W}_{21} \exp(i \Delta_- t) \delta_{m_2, m_1 + m_1} + A_1 A_D^{*} \dot{W}_{21}^{*} \exp(i \Delta_+ t) \delta_{m_2, m_1 - m_1}, \] (18)
where \( \dot{W}_{21} \) and \( \dot{W}_{21}^{*} \) are time-independent quantities defined by
\[ \dot{W}_{21} = \frac{1}{2} \left[ \langle \rho_0 \xi_2^* \cdot C(\xi_1, \xi_1^*) \rangle + \langle \rho_0 \xi_2^* \cdot C(\xi_D, \xi_1^*) \rangle \right] \exp(-i \Delta_- t), \] (19)
\[ \dot{W}_{21}^{*} = \frac{1}{2} \left[ \langle \rho_0 \xi_2^* \cdot C(\xi_1, \xi_1^*) \rangle + \langle \rho_0 \xi_2^* \cdot C(\xi_D, \xi_1^*) \rangle \right] \exp(-i \Delta_+ t), \] (20)
and \( E_2 \) is the wave energy of mode 2 defined by
\[ E_2 = \frac{1}{2} \omega_2 \left[ \omega_2 \langle \rho_0 \xi_2^* \xi_2 \rangle - i \langle \rho_0 \xi_2^* (u_0 \cdot \nabla) \xi_2 \rangle \right]. \] (21)

In the limiting case of \( \beta_1 = \beta_2 = 0 \), equations (13) and (18) are reduced, respectively, to equations (29) and (36) in Paper I.

Here, it is important to note that we have the following identical relations:
\[ \dot{W}_{21} = (\dot{W}_{1_{2s}})^{*}, \] (22)
\[ \dot{W}_{21}^{*} = (\dot{W}_{1_{2s}})^{*}. \] (23)
These relations come from the fact that the order of \( \eta_1, \eta_2, \) and \( \eta_3 \) in \( \langle \rho_0 \eta_1 \cdot C(\eta_2, \eta_3) \rangle \) can be arbitrarily changed [see equation (7), and appendix 1 of Kato (2008a)].

4. **Equation Describing Growth Rate**

Equations (13) and (18) show that in the limiting case of no coupling, \( A_1 \) and \( A_2 \) are damped, respectively, at the rates of \( \beta_1 / 2 \) and \( \beta_2 / 2 \), as expected. The next problem is to examine how \( A_1 \) and \( A_2 \) evolve with time by resonant coupling. This is studied in the two cases of \( m_2 = m_1 + m_D \) and \( m_2 = m_1 - m_D \), separately.

4.1. **Case of** \( m_2 = m_1 + m_D \)

New variables defined by
\[ \tilde{A}_1(t) = A_1(t) \exp \left( i \frac{1}{2} \Delta_- t \right) \] (24)
and
\[ \tilde{A}_2(t) = A_2(t) \exp \left( -i \frac{1}{2} \Delta_- t \right) \] (25)
are introduced. Then, from equations (13) and (18), we have
\[ 4i \frac{E_1}{\omega_1} \left( \frac{d}{dt} - i \frac{1}{2} \Delta_- + \frac{1}{2} \beta_1 \right) \tilde{A}_1 = \tilde{A}_2 A_D \dot{W}_{1_{2s}} \] (26)
and
\[ 4i \frac{E_2}{\omega_2} \left( \frac{d}{dt} + i \frac{1}{2} \Delta_- + \frac{1}{2} \beta_2 \right) \tilde{A}_2 = \tilde{A}_1 A_D \dot{W}_{21}. \] (27)

\(^2\) In Paper I, we have introduced \( \tilde{A}_1(t) \) defined by \( \tilde{A}_1(t) = A_1(t) \exp(\Delta_- t) \), different from equation (24). The definition of \( \tilde{A}_1(t) \) by equation (24) is, however, better, since the resulting equations (26) and (27) are symmetric.
Equations (26) and (27) are easily solved by taking $\hat{A}_1$ and $\hat{A}_2$ to be proportional to $\exp(i \sigma t)$. The equation describing $\sigma$ is then found to be

$$\sigma^2 - i \frac{1}{2} (\beta_1 + \beta_2) \sigma - \frac{1}{4} \Delta^2 + i \frac{1}{4} \Delta \cdot (\beta_2 - \beta_1) - \frac{1}{4} \beta_1 \beta_2 - \frac{\omega_1 \omega_2}{16 E_1 E_2} |A_D|^2 |\dot{W}_{21}|^2 = 0,$$  

(28)

where equation (22) has been used.

In the limiting case of $\beta_1 = \beta_2 = 0$, equation (28) is reduced to

$$\sigma^2 = \frac{1}{4} \Delta^2 + \frac{\omega_1 \omega_2}{16 E_1 E_2} |A_D|^2 |\dot{W}_{21}|^2,$$  

(29)

and the oscillations are unstable ($\sigma^2 < 0$) when

$$\Delta^2 + \frac{\omega_1 \omega_2}{4 E_1 E_2} |A_D|^2 |\dot{W}_{21}|^2 < 0.$$  

(30)

This is the instability condition obtained in Paper I, and $(\omega_1 \omega_2/E_1 E_2) < 0$ is a necessary condition for instability. In another limiting case of $\beta_1 = \beta_2 \equiv \beta$, equation (28) gives

$$\left(\sigma - i \frac{1}{2} \beta\right)^2 = \frac{1}{4} \Delta^2 + \frac{\omega_1 \omega_2}{16 E_1 E_2} |A_D|^2 |\dot{W}_{21}|^2.$$  

(31)

This equation shows that the term of $\beta$ always works so as to dampen the oscillations, since $\beta$ appears in the set of $\sigma - i \beta/2$. An interesting, important effect of $\beta$, however, appears when $\beta_2 \neq \beta_1$, which is considered in the next section.

4.2. Case of $m_2 = m_1 - m_D$

In this case we introduce variables $\tilde{A}_1$ and $\tilde{A}_2$, defined by

$$\tilde{A}_1(t) = A_1(t) \exp\left(i \frac{1}{2} \Delta_+ t\right)$$  

(32)

and

$$\tilde{A}_2(t) = A_2(t) \exp\left(-i \frac{1}{2} \Delta_+ t\right).$$  

(33)

Then, from equations (13) and (18), we have

$$4i \frac{E_1}{\omega_1} \left(\frac{d}{dt} - i \frac{1}{2} \Delta_+ + \frac{1}{2} \beta_1\right) \tilde{A}_1 = \tilde{A}_2 A_D \dot{W}_{12}$$  

(34)

and

$$4i \frac{E_2}{\omega_2} \left(\frac{d}{dt} + i \frac{1}{2} \Delta_+ + \frac{1}{2} \beta_2\right) \tilde{A}_2 = \tilde{A}_1 A_D^* \dot{W}_{21*}.$$  

(35)

As in the case of $m_2 = m_1 + m_D$, the time dependences of $\tilde{A}_1(t)$ and $\tilde{A}_2(t)$ are taken to be $\exp(i \sigma t)$. Then, from equations (34) and (35) we have an equation describing $\sigma$ as

$$\sigma^2 - i \frac{1}{2} (\beta_1 + \beta_2) \sigma - \frac{1}{4} \Delta^2 + i \frac{1}{4} \Delta \cdot (\beta_2 - \beta_1) - \frac{1}{4} \beta_1 \beta_2 - \frac{\omega_1 \omega_2}{16 E_1 E_2} |A_D|^2 |\dot{W}_{12}|^2 = 0,$$  

(36)

where equation (23) has been used. This equation is the same as equation (28), except that $\Delta_+$ and $\dot{W}_{21}$ in equation (28) are now changed to $\Delta_+$ and $\dot{W}_{12}$, respectively.

5. Numerical Calculations of Growth Rate

Equations (28) and (36) have similar forms. Considering this, we numerically solve the following equation:

$$\sigma^2 - i \frac{1}{2} \beta (1 + 2) \sigma - \frac{1}{4} \Delta^2 + i \frac{1}{4} \Delta \beta (1 + 2) - \frac{1}{4} \beta^2 + W = 0.$$  

(37)

where $W$ is an abbreviation of $-(1/16)(\omega_1 \omega_2/E_1 E_2)|A_D|^2 |\dot{W}_{21}|^2$ or $-(1/16)(\omega_1 \omega_2/E_1 E_2)|A_D|^2 |\dot{W}_{12}|^2$ and taken to be positive, since we are interested in the case of $(\omega_1 \omega_2/E_1 E_2) < 0$. Furthermore, $\beta$ and $\delta$ represent $\beta_2$ and $\beta_1/\beta_2$, respectively. In the case of equation (28), $\Delta_+$ in equation (37) is $\Delta_+$, while it is $\Delta_+$ in the case of equation (36).

As mentioned in the previous section, an interesting case of $\beta_2 \neq \beta_1$ is that one is small and negligible as compared with the other. It is a supposable case. For example, let us consider the case where an axisymmetric g-mode oscillation ($m_1 = 0$) with one
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node in the vertical direction \((n_1 = 1)\) resonantly interacts with an one-armed g-mode oscillation \((m_2 = 1)\) with two nodes in the vertical direction \(n_2 = 2\) through a warp \((m_D = 1, n_D = 1)\).\footnote{This is one of the cases numerically examined by Ferreira and Ogilvie (2008).} In this case, the resonant condition of \(m_2 = m_1 + m_D\) is satisfied. Furthermore, the resonant condition of \(\omega_1 = \omega_2 + \omega_D\) can also be satisfied with \(\omega_D \approx 0\). This is because the propagation region of the former oscillation is \(\omega^2 < \kappa^2\) (\(\kappa\) being the epicyclic frequency), while that of the latter is \((\omega - \Omega)^2 < \kappa^2\); these two propagation regions overlap in relativistic disks, since \(\kappa(r)\) has the maximum near \(r = 4r_g\) \((r_g\) being the Schwarzschild radius). The former oscillation has no corotation damping \((\beta_1 = 0)\), since it is axisymmetric, while the corotation damping occurs in the latter oscillation \((\beta_2 \neq 0)\). Hence, this case corresponds to \(\delta = 0\) \((\beta_1 = 0 \text{ and } \beta_2 \neq 0)\), and equation (37) is reduced to

\[
\sigma^2 - i\frac{1}{2} \beta \sigma - \frac{1}{4} \Delta^2 + i \frac{1}{4} \beta \Delta + W = 0.
\]  

The opposite case of \(\beta_2 = 0\) and \(\beta_1 \neq 0\) may occur, if we consider a different set of oscillations. Such cases are formally involved in equation (38) by changing the sign of \(\Delta\). Considering these situations, we solve equation (38) for various sets of parameters. Dimensionless parameters that we adopt are \(\Delta / W^{1/2}\) and \(\beta / W^{1/2}\). If we obtain the dimensionless frequency, \(\sigma / W^{1/2}\), whose imaginary part is negative, the sets of oscillations considered are amplified (overstable).

First, we consider the range of parameters to be examined in order to understand the characteristics of equation (38). In the case of \(\beta = 0\), equation (38) gives \(\sigma^2 = \Delta^2 / 4 - W\), and the sign of \(\sigma^2\) changes at \(\Delta / W^{1/2} = \pm 2\). Hence, since regions both in the case of \(\sigma^2 < 0\) (growing oscillations) and in the case of \(\sigma^2 > 0\) (constant amplitude oscillations) are involved in the range of \(-4 < \Delta / W^{1/2} < 4\), this range will be enough to examine the basic behavior of equation (38). The contours of constant value of \(-3\sigma / W^{1/2}\) (i.e., a normalized growth rate of oscillations) in the parameter plane of the \(\Delta / W^{1/2} - \beta / W^{1/2}\) are shown in figure 1 by solving equation (38), i.e., \(\delta = 0\).

Figure 1 shows that the growth rate of oscillations is the strongest at the point of \(\Delta = 0\) and \(\beta = 0\) on the \(\Delta / W^{1/2} - \beta / W^{1/2}\) diagram, and decreases as departing from the point on the diagram. Next, let us see how the growth rate changes as \(\beta\) increases from zero, while keeping \(\Delta\) fixed. Approximately speaking, when \(|\Delta| / W^{1/2} < 2\), the growth rate decreases as \(\beta\) increases. In the case of \(|\Delta| / W^{1/2} > 2\), however, the situation changes. That is, in the latter case we have \(\sigma^2 = 0\) and the oscillations are purely periodic (there is neither growth nor damping) in the limiting case of \(\beta = 0\), but the oscillations are amplified as \(\beta\) increases from zero. In other words, \(\beta\) acts so as to excite oscillations.

In order to see in more detail the effects of \(\beta\) on the growth rate of oscillations, the \(\beta\)-dependence of the growth rate is shown in figure 2 for some values of \(\delta\), while fixing \(|\Delta| / W^{1/2}\). The figure shows that when \(\delta = 0\) and \(|\Delta| / W^{1/2} > 2\), the \(\beta\) terms in equation (38) act so as to excite oscillations. If \(\beta / W^{1/2}\) is sufficiently large, however, the growth rate tends to zero.

As mentioned in section 4, the simultaneous presence of \(\beta_1\) and \(\beta_2\) acts so as to dampen the oscillations. To numerically demonstrate this, the cases of \(\delta = 0.3\) and \(\delta = 1.0\) are added in figure 2, by solving equation (37) [not equation (38)].

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**Fig. 1.** Contours of constant (dimensionless) growth rate, \(-3\sigma / W^{1/2}\), on the \(\beta / W^{1/2} - \Delta / W^{1/2}\) plane. The values labeled to curves denote dimensionless growth rate, \(-3\sigma / W^{1/2}\). The parameter, \(\delta = (\beta_1 / \beta_2)\), is taken to be zero.

**Fig. 2.** Dimensionless growth rate, \(-3\sigma / W^{1/2}\), as functions of parameter, \(\beta / W^{1/2}\), for some values of \(\Delta / W^{1/2}\) and \(\delta = (\beta_1 / \beta_2)\). The thick curves are for \(\delta = 0\) (i.e., \(\beta_1 = 0\)), and the four curves are for \(\Delta / W^{1/2} = 0, 1.0, 2.0, \text{ and } 3.0\) from the top to the bottom. The thin curves in the middle region of the figure are for \(\delta = 0.3\), and the values of \(\Delta / W^{1/2}\) are 0, 1.0, 2.0, and 3.0 from the top to the bottom. The three curves in the left-lower corner of the figure are for \(\delta = 1.0\), and the values of \(\Delta / W^{1/2}\) are 0, 1.0, and 2.0 from the top to the bottom (the curve of \(\Delta / W^{1/2} = 3.0\) cannot be distinguished from that of \(\Delta / W^{1/2} = 2.0\) on the diagram).
6. Summary of Growth Rate and Interpretation

First, let us consider the case of no dissipation ($\beta = \delta = 0$). In this case the two oscillations are independent and have their own frequencies, $\omega_1$ and $\omega_2$, when their frequencies are far from the resonant conditions (1). Here, we examine only the case where the resonant conditions are $\omega_1 \approx \omega_2 + \omega_D$ and $m_1 = m_2 + m_D$ with $\omega_D > 0$, in order to avoid any unnecessary complication arising from considering other cases simultaneously. If $|\Delta_+|$ decreases, for example, by an increase of $\omega_D$, the frequencies of two oscillation modes, $\omega_1$ and $\omega_2$, move toward $\omega_1 - \Delta_+/2$ and $\omega_2 + \Delta_+/2$, respectively, so that the exact resonant condition of the frequencies is satisfied by these shifted frequencies, i.e., $(\omega_1 - \Delta_+/2) = (\omega_2 + \Delta_+/2) + \omega_D$.4 This is realized finally at $|\Delta_+| = 2W^{1/2}$. If $|\Delta_+|$ decreases further, the frequencies of oscillations no longer change (frequency rocking), but the oscillations become overstable (amplified). This involves the results of Paper I. The cause of this overstability is an energy exchange between positive- and negative-energy oscillations by resonance.

In the case of $|\Delta_+| > 2W^{1/2}$ with no dissipation, the oscillations are purely time-dependent, although the frequencies shift from their proper ones, $\omega_1$ and $\omega_2$, as mentioned above. Here, we can consider what happens if dissipative processes are added (i.e., $\beta's \neq 0$) to the above case. The results show that the oscillations are amplified if a dissipative process works mainly on one of the oscillations (see figure 2). This result can be understood in the following way.

Let us consider the behavior of the oscillation of $\xi_1$ (mode 1). Two forces act on mode 1. One is a restoring force (hereafter, force A), which acts so as to make mode 1 oscillate with its proper frequency, $\omega_1$. The other is a force (hereafter, force B) that works so that a fluid element of the oscillation departs from the oscillation with proper frequency, $\omega_1$. The latter force (force B) comes from the resonance coupled with the oscillation of mode 2 [see the right-hand side of equation (34)]. In the case of $|\Delta_+| > 2W^{1/2}$, the former force (force A) is stronger than the latter one (force B), and mode 1 is a constant-amplitude oscillation although the frequency shifts from $\omega_1$, when there is no dissipation. If a dissipative force works on mode 2 (not mode 1), force B working on mode 1 becomes weak during the oscillation, since the resonant term depends on the amplitude of mode 2. This means that the net restoring force (i.e., force A minus force B) at a phase of the oscillation increases as compared with that at the phase in one cycle before, due to a decrease of force B. In other words, a fluid element associated with the oscillations returns to the equilibrium position with a faster velocity when compared with that in one cycle before, and the amplitude of oscillations increases with time. If this amplification of mode 1 is stronger than the weakening of mode 2 in dissipative process, the whole system of the two oscillations of modes 1 and 2 are amplified. This is the reason why the dissipative process amplifies the set of oscillations of modes 1 and 2.

If a dissipative process also works on mode 1 (i.e., $\beta_1 \neq 0$), the restoring force acting on mode 1 (force A) decreases during one cycle of the oscillation of mode 1, and thus the net restoring force acting on mode 1 decreases, leading to damping the whole system of oscillations of modes 1 and 2. It is noted that in the above-mentioned argument the role of mode 1 can be changed by that of mode 2.

This excitation mechanism of oscillations is quite similar to double-diffusive instability known in the fields of other astrophysics and oceanography. For example, the overstable convection due to double-diffusion processes is known in magnetized or rotating media. Let us consider, for instance, a stellar atmosphere where the temperature stratification is super-adiabatic, but convection is suppressed by restoring force of magnetic fields. If the thermal convection works on fluid elements, and its time scale is faster than that of the diffusion process of the magnetic field by magnetic diffusivity, the net restoring force working on the fluid elements (magnetic restoring force minus buoyancy force) increases during an oscillation, since the decrease of the buoyancy force by thermal conduction is faster than that of the magnetic restoring force by magnetic diffusivity. Then, we can expect oscillations whose amplitude increases with time (overstable convection). This mechanism of overstable convection was first recognized by Cowling (1958), and later by Chandrasekhar (see Chandrasekhar 1961). Semiconvection also occurs in a chemically stratified medium by thermal conduction (Kato 1966), since the time scale of chemical element diffusion is much slower than that of thermal diffusion. All of them are the excitation of overstable oscillations by a double-diffusive instability. As recent applications of double-diffusive instability, see, e.g., Charbonnel and Zahn (2007) and Zaussing and Spruit (2011) for elements mixing in stars and Latter, Bonart, and Balbus (2010) for protostellar disks.

7. Discussion

In order to explain one of possible origins of high-frequency quasi-periodic oscillations observed in neutron-star and black-hole low-mass X-ray binaries, Kato (2004, 2008a, 2008b) suggested on analytical calculations that in deformed (warp or eccentric deformation) disks, a set of positive- and negative-energy oscillations can be resonantly excited. Ferreira and Ogilvie (2008) and Oktarian, Okazaki, and Kato (2010) made numerical calculations and examined whether the excitation really exists. Subsequently, Paper I made an analytical examination by a different method from Kato’s original ones (Kato 2004, 2008a, 2008b).

The above-mentioned analytical and numerical investigations now seem to confirm that if positive- and negative-energy oscillations resonantly interact with each other through disk deformation, they are excited. However, there remains a serious difference between the results by Kato’s group and those by Ferreira and Ogilvie. That is, in the results by Ferreira and Ogilvie (2008),
an additional ingredient is necessary for excitation. This is a dissipative process acting on one of the oscillations. On the other hand, such a dissipative process is unnecessary to the excitation of the oscillations in analytical and semianalytical calculations made by Paper I and Oktariani, Okazaki, and Kato (2010).

Our results presented in this paper seem to resolve the problem of this disagreement. Our results show that when the frequency deviation from the exact resonant condition is small, i.e., $|\Delta|/W^{1/2} < 2$, the oscillations are excited without the help of any dissipation process. When the deviation is large, i.e., $|\Delta|/W^{1/2} > 2$, however, a dissipative process ($\beta \neq 0$) is necessary to the excitation of the oscillations. In this context, we think that the case examined by Ferreira and Ogilvie (2008) corresponds to the case of $|\Delta|/W^{1/2} > 2$. The following seems to also support this conjecture. In the numerical results by Ferreira and Ogilvie (2008), the growth rate increases from zero along with an increase of the damping factor, $\beta$, but saturates at a certain level (see figure 4 of their paper). This is qualitatively consistent with our result shown in figure 2.5

It is noted that the analyses made by Kato (2004, 2008a, 2008b) and those done by Paper I and this paper are rather different. In the former studies, the presence of a Lindblad resonance in the propagation region of oscillations is assumed, and the condition of excitation of the oscillations is derived by considering the behavior of oscillations around the resonance. In the present study as well as in Paper I, such an assumption is not involved in the analyses. In this sense, the latter analyses are more general. However, it should be remembered that the magnitude of the growth rate depends on that of the coupling terms, such as $|\hat{W}_1|^2$ and $|\hat{W}_2|^2$, and they are large when a Lindblad resonance appears in the propagation region of the oscillations.

It is also noted that for the present resonant excitation mechanism to work efficiently, the propagation regions of the positive- and negative-energy oscillations must overlap in the radial direction. Otherwise, the terms of resonant couplings are small and there is practically little excitation. In non-self-gravitating disks, the overlapping of the propagation regions is easy to occur in relativistic disks.

In our excitation mechanism of disk oscillations, disk deformation (e.g., warp or eccentric deformation) is necessary. What kinds of origins of such disk deformation are conceivable? There are some possibilities. For example, the angular momentum axis of disks will not always be expected to be aligned with the spin axis of the central source. Then, the disks are tilted and have precession. In such disks, the excitation of a set of disk oscillations with positive- and negative-energies is expected. It is noted that the direct cause of this excitation is not the spin of the central source, but a disk deformation. As some other possible causes of disk deformation, a disk instability due to irradiation from central sources (Pringle 1992) is conceivable. The Papaloizou-Pringle instability also makes the inner region of disks deformed at the phase where inner tori are formed (Machida & Matsumoto 2008).

Finally, let us briefly mention whether our excitation mechanism can describe the spin-frequency relation observed in QPOs of millisecond pulsars. When a burst occurs on the surface of spinning central neutron stars, a nonaxisymmetric pattern rotating with the spin frequency, $\omega_s$, will be induced on the disks, i.e., $\omega_D = \omega_s$. In such cases, the excitation of oscillations that satisfy $\omega_1 - \omega_2 \sim \omega_s$ is expected, and QPOs with such a frequency relation are really observed (e.g., 4U 1702–43). In some other millisecond pulsars, however, the observed frequency relation is $\omega_2 - \omega_1 \simeq \omega_s/2$ (e.g., SAX J1808.4–3658). At a glance, it seems to be difficult to describe the latter by the present resonant model, although more considerations will be worthwhile.

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5 In our results the growth rate decreases and tends to zero, when the damping factor increases further. In figure 4 of Ferreira and Ogilvie (2008), such a trend seems to be also present.