Real extensions of distal minimal flows and continuous topological ergodic decompositions

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Abstract

We prove a structure theorem for topologically recurrent real skew product extensions of distal minimal compact metric flows with a compactly generated Abelian acting group (for example, \( \mathbb{Z}^d \)-flows and \( \mathbb{R}^d \)-flows). The main result states that every such extension apart from a coboundary can be represented by a perturbation of a so-called Rokhlin skew product. We obtain as a corollary that the topological ergodic decomposition of the skew product extension into prolongations is continuous and compact with respect to the Fell topology on the hyperspace. The right translation acts minimally on this decomposition, therefore providing a minimal compact metric analogue to the Mackey action. This topological Mackey action is a distal (possibly trivial) extension of a weakly mixing factor (possibly trivial), and it is distal if and only if perturbation of the Rokhlin skew product is defined by a topological coboundary.

1. Introduction and main results

The study of real-valued topological cocycles and real skew product extensions has been initiated by Besicovitch, Gottschalk and Hedlund. Besicovitch [4] proved the existence of point transitive real skew product extensions of an irrational rotation on the one-dimensional torus. Furthermore, he proved that none of them is minimal, that is, there are always non-transitive points for a point transitive real skew product extension. The main result in Chapter 14 of [13] can be rephrased to the dichotomy that a topologically conservative real skew product extension of a minimal rotation on a torus (finite- or infinite-dimensional) is either point transitive or it is defined by a topological coboundary and almost periodic. This result and a generalization to skew product extensions of a Kronecker transformation (cf. [20]) exploit the isometric behaviour of a minimal rotation. A corresponding result apart from isometries is based on homotopy conditions for the class of distal minimal homeomorphisms usually called Furstenberg transformations (cf. [14]). However, in general, this dichotomy is not valid, and counterexamples can be provided by the Rokhlin skew products of the so-called topological type III_0. In this paper, we shall study recurrent real skew product extensions of distal minimal flows with Abelian compactly generated acting groups, including a description of the orbit structure in terms of a topological version of the Mackey range.

Throughout this paper, we shall denote by \( T \) a compactly generated Abelian Hausdorff topological group acting continuously on a compact metric phase space \((X, d)\) so that \((X, T)\) is a compact metric flow. In the monograph [13] such an acting group \( T \) is called generative, and notions of recurrence are provided for such Abelian acting groups apart from \( \mathbb{Z} \) and \( \mathbb{R} \).
For a \( \mathbb{Z} \)-action on \( X \), we let \( T \) be the self-homeomorphism of \( X \) generating the action by \( (n, x) \mapsto T^n x \), while in the case of a real flow we shall use the notation \( \{ \phi^t : t \in \mathbb{R} \} \) for the acting group. We call a flow \textit{minimal} if the whole phase space is the only non-empty invariant closed subset, and then for every \( x \in X \) the orbit closure \( \overline{O}_T(x) = \{ \tau x : \tau \in T \} \) is all of \( X \). A point \( x \in X \) with a minimal orbit closure \( \overline{O}_T(x) \) is called an \textit{almost periodic}. If all points in \( X \) are almost periodic, then the flow \((X, T)\) is \textit{pointwise almost periodic} and admits a partition of \( X \) into minimal sets. A flow \((X, T)\) is \textit{topologically transitive} if for arbitrary non-empty open sets \( U, V \subset X \) there exists some \( \tau \in T \) with \( \tau U \cap V \neq \emptyset \), and it is \textit{weakly mixing} if the flow \((X \times X, T)\) with the diagonal action is topologically transitive. For a topologically transitive flow \((X, T)\) with a complete separable metric phase space there exists a dense \( G_\delta \)-set of \textit{transitive points} \( x \) with \( \overline{O}_T(x) = X \), and a flow with transitive points is \textit{point transitive}. If \((X, T)\) and \((Y, T)\) are flows with the same acting group \( T \) and \( \pi : X \to Y \) is a continuous \textit{mapping} with \( \pi(\tau x) = \tau \pi(x) \) for every \( \tau \in T \) and \( x \in X \), then \((Y, T)\) is called a \textit{factor} of \((X, T)\) and \((X, T)\) is called an \textit{extension} of \((Y, T)\). Such a mapping \( \pi \) is called a \textit{homomorphism} of the flows \((X, T)\) and \((Y, T)\), and a bijective homomorphism is an \textit{isomorphism}. The set of isomorphisms of a flow \((X, T)\) onto itself is the topological group \( \text{Aut}(X, T) \) of \textit{automorphisms} of \((X, T)\). Two points \( x, y \in X \) are called \textit{distal} if

\[
\inf_{\tau \in T} d(\tau x, \tau y) > 0,
\]

otherwise they are called \textit{proximal}. For a general compact Hausdorff flow \((X, T)\) distality of two points \( x, y \in X \) is defined by the absence of any nets \( \{ \tau_n \}_{n \in I} \subset T \) with \( \lim \tau_n x = \lim \tau_n y \). A flow is called distal if any two distinct points are distal, and an extension of flows is called distal if any two distinct points in the same fibre are distal. A compact distal extension of a \textit{minimal} flow is pointwise almost periodic and the homomorphism is an \textit{open mapping} (cf. [3, 5]).

Suppose that \( \mathcal{A} \) is an Abelian locally compact second countable (Abelian l.c.s.c.) group with zero element \( 0_\mathcal{A} \), and let \( \mathcal{A}_\infty \) denote its one point compactification with the convention that \( g + \infty = \infty + g = \infty \) for every \( g \in \mathcal{A} \). A \textit{cocycle} of a compact metric flow \((X, T)\) is a continuous mapping \( f : T \times X \to \mathcal{A} \) with the identity

\[
f(\tau, \tau' x) + f(\tau', x) = f(\tau \tau', x)
\]

for all \( \tau, \tau' \in T \) and \( x \in X \). Given a compact metric \( \mathbb{Z} \)-flow \((X, T)\) and a continuous function \( f : X \to \mathcal{A} \), we can define a cocycle \( f : \mathbb{Z} \times X \to \mathcal{A} \) with \( f(1, \cdot) \equiv f \) by

\[
f(n, x) = \begin{cases} 
\sum_{k=0}^{n-1} f(T^k x) & \text{if } n \geq 1, \\
0_\mathcal{A} & \text{if } n = 0, \\
-f(-n, T^n x) & \text{if } n < 0.
\end{cases}
\]

Moreover, there is a natural occurrence of cocycles of \( \mathbb{R} \)-flows as solutions to ODEs. Suppose that \( (M, \{ \phi^t : t \in \mathbb{R} \}) \) is a smooth flow on a compact manifold \( M \) and \( A : M \to \mathbb{R} \) is a continuous function. Then a continuous real-valued cocycle \( g(t, m) \) of the flow \((M, \{ \phi^t : t \in \mathbb{R} \})\) is given by the fundamental solution to the ODE

\[
\frac{dg(t, m)}{dt} = A(\phi^t(m))
\]

with the initial condition \( g(0, m) = 0 \). We shall denote by \( \mathbb{1} \) the linear cocycle of an \( \mathbb{R} \)-flow \((M, \{ \phi^t : t \in \mathbb{R} \})\) with \( \mathbb{1}(t, m) = t \) for every \( (t, m) \in \mathbb{R} \times M \). The \textit{skew product extension} of the flow \((X, T)\) by the cocycle \( f : T \times X \to \mathcal{A} \) is defined by the homeomorphisms

\[
\tilde{\tau}_f(x, a) = (\tau x, f(\tau, x) + a)
\]
of \( X \times \mathbb{A} \) for all \( \tau \in T \), which provide a continuous action \((\tau, x, a) \mapsto \tilde{T}_f(x, a)\) of \( T \) on the product space \( X \times \mathbb{A} \) by the cocycle identity. We shall denote the orbit closure of \((x, a) \in X \times \mathbb{A} \) under \( \tilde{T}_f \) by \( \tilde{\mathcal{O}}_{T,f}(x, a) = \{ \tilde{T}_f(x, a) : \tau \in T \} \). For a \( \mathbb{Z} \)-flow \((X, T)\) the skew product action is generated by the homeomorphism of \( X \times \mathbb{A} \) with

\[
\tilde{T}_f(x, a) = (Tx, f(x) + a) \quad \text{for every} \ (x, a) \in X \times \mathbb{A}.
\]

The prolongation \( \mathcal{D}_T(x) \) of \( x \in X \) under the group action of \( T \) is defined by

\[
\mathcal{D}_T(x) = \bigcap \{ \tilde{\mathcal{O}}_{T,U} : U \text{ is an open neighbourhood of} \ x \},
\]

and we shall denote by \( \mathcal{D}_{T,f}(x, a) \) the prolongation of \((x, a) \in X \times \mathbb{A} \) under the skew product action \( \tilde{T}_f \). The essential property of a skew product is that the \( T \)-action on \( X \times \mathbb{A} \) commutes with the right translation action of \( \mathbb{A} \) on \( X \times \mathbb{A} \) defined by

\[
R_b(x, a) = (x, a - b) \quad \text{for every} \ (x, a) \in X \times \mathbb{A} \text{ and } b \in \mathbb{A}.
\]

It follows that \( R_b(\tilde{\mathcal{O}}_{T,f}(x, a)) = \tilde{\mathcal{O}}_{T,f}(R_b(x, a)) \) and \( \mathcal{D}_{T,f}(x, a) = \mathcal{D}_{T,f}(R_b(x, a)) \) for every \((x, a) \in X \times \mathbb{A} \) and \( b \in \mathbb{A} \), and the set of orbit closures (prolongations) is invariant under \( R_b \).

While the inclusion of the orbit closure in the prolongation is obvious, the coincidence of these sets is generic by a result from the paper [12]. This result is usually referred to as ‘topological ergodic decomposition’.

**FACT 1.1.** For every compact metric flow \((X, T)\), there exists a \( T \)-invariant dense \( G_\delta \) set \( \mathcal{F} \subset X \) so that for every \( x \in \mathcal{F} \) holds

\[
\tilde{\mathcal{O}}_T(x) = \mathcal{D}_T(x).
\]

For a skew product extension \( \tilde{T}_f \) of \((X, T)\) by a cocycle \( f : T \times X \to \mathbb{A} \) there exists a \( T \)-invariant dense \( G_\delta \) set \( \mathcal{F} \subset X \) so that for every \( x \in \mathcal{F} \) and every \( a \in \mathbb{A} \) holds

\[
\tilde{\mathcal{O}}_{T,f}(x, a) = \mathcal{D}_{T,f}(x, a).
\]

This assertion holds as well for the extension of \( \tilde{T}_f \) to \( X \times \mathbb{A}_\infty \), which is defined by \((x, \infty) \mapsto (\tau x, \infty) \) for every \( x \in X \), and given an \( \mathbb{R}^2 \)-valued topological cocycle \( g = (g_1, g_2) : T \times X \to \mathbb{R}^2 \) for the extension of \( \tilde{T}_g \) to \( X \times (\mathbb{R}_\infty)^2 \) which is defined by \((x, s, \infty) \mapsto (\tau x, s + g_1(x), \infty), (x, \infty, t) \mapsto (\tau x, \infty, t + g_2(x)) \), and \((x, \infty, \infty) \mapsto (\tau x, \infty, \infty) \), for every \( x \in X \) and \( s, t \in \mathbb{R} \).

**Proof.** The statement for a compact metric phase space and a general acting group is according to [1, Theorem 1]. The other statements can be verified by means of the extension of \( \tilde{T}_f \) onto the compactification of \( X \times \mathbb{A} \). The coincidence of \( \tilde{\mathcal{O}}_{T,f}(x, a) \) and \( \mathcal{D}_{T,f}(x, a) \) for some \((x, a) \in X \times \mathbb{A} \) implies this coincidence for all \((x, a') \in \{x\} \times \mathbb{A}_\infty \), since the extension of \( \tilde{T}_f \) to \( X \times \mathbb{A}_\infty \) commutes with the right translation on \( X \times \mathbb{A}_\infty \).

**Remark 1.** If \( y \in \tilde{\mathcal{O}}_T(x) \), then \( \tilde{\mathcal{O}}_T(y) \subset \tilde{\mathcal{O}}_T(x) \). The corresponding statement for prolongations is not necessarily valid, however, if \( y \in \tilde{\mathcal{O}}_T(x) \) then \( \mathcal{D}_T(x) \subset \mathcal{D}_T(y) \).

We shall consider more general Abelian acting groups than \( \mathbb{Z} \) and \( \mathbb{R} \), hence the definition of recurrence requires the notions of a replete semigroup and an extensive subset of the Abelian compactly generated group \( T \) (cf. [13]). We recall that a semigroup \( P \subset T \) is replete if for every compact subset \( K \subset T \) there exists a \( \tau \in T \) with \( \tau K \subset P \), and a subset \( E \subset T \) is extensive if it intersects every replete semigroup. Therefore, a subset \( E \) of \( T = \mathbb{Z} \) or \( T = \mathbb{R} \) is extensive if and only if \( E \) contains arbitrarily large positive and arbitrarily large negative elements.
**Definition 1.** We call a cocycle \( f(\tau, x) \) of a minimal compact metric flow \((X, T)\) **topologically recurrent** if for an arbitrary non-empty open set \( U \subset X \) and an arbitrary neighbourhood \( U(0_\mathbb{A}) \subset \mathbb{A} \) of \( 0_\mathbb{A} \) there exists an extensive set of elements \( \tau \in T \) with
\[
U \cap \tau^{-1}(U) \cap \{ x \in X : f(\tau, x) \in U(0_\mathbb{A}) \} \neq \emptyset.
\]
Since \( \bar{\tau}_f \) and the right translation on \( X \times \mathbb{A} \) commute, this is equivalent to the **regional recurrence** of the skew product action \( \bar{\tau}_f \) on \( X \times \mathbb{A} \), that is, for every non-empty open set \( U \subset X \times \mathbb{A} \) there exists an extensive set of elements \( \tau \in T \) with \( \bar{\tau}_f(U) \cap U \neq \emptyset \). A non-recurrent cocycle is called **transient**.

A point \((x, a) \in X \times \mathbb{A}\) is \( \bar{\tau}_f \)-**recurrent** if for every neighbourhood \( U \subset X \times \mathbb{A} \) of \((x, a)\) the set of \( \tau \in T \) with \( \bar{\tau}_f(x, a) \in U \) is extensive. Moreover, a point \((x, a) \in X \times \mathbb{A}\) is **regionally \( \bar{\tau}_f \)-recurrent** if for every neighbourhood \( U \) of \((x, a)\) the set of \( \tau \in T \) with \( \bar{\tau}_f(U) \cap U \neq \emptyset \) is extensive.

**Remark 2.** If \( f(\tau, x) \) is recurrent, then by \cite[Theorems 7.15 and 7.16]{schmidt} there exists a dense \( G_\delta \) set of \( \bar{\tau}_f \)-recurrent points in \( X \times \mathbb{R} \).

Given a regionally \( \bar{\tau}_f \)-recurrent point \((x, a) \in X \times \mathbb{A}\), every point in \( \{x\} \times \mathbb{A}\) is regionally \( \bar{\tau}_f \)-recurrent. The minimality of \((X, T)\) and \cite[Theorem 7.13]{schmidt} imply that every point in \( X \times \mathbb{A}\) is regionally \( \bar{\tau}_f \)-recurrent, hence \( f(\tau, x) \) is recurrent.

A cocycle \( f(n, x) \) of a \( \mathbb{Z} \)-flow is topologically recurrent if and only if \( \tilde{T}_f \) is topologically conservative, that is, for every non-empty open set \( U \subset X \times \mathbb{A} \) there exists an integer \( n \neq 0 \) so that \( T_f^n(U) \cap U \neq \emptyset \).

One of the most important concepts in the study of cocycles is the essential range, originally introduced in the measure theoretic category by Schmidt \cite{schmidt}.

**Definition 2.** Let \( f(\tau, x) \) be a cocycle of a minimal compact metric flow \((X, T)\). An element \( a \in \mathbb{A} \) is in the set \( E(f) \) of **topological essential values** if for an arbitrary non-empty open set \( U \subset X \) and an arbitrary neighbourhood \( U(a) \subset \mathbb{A} \) of \( a \) there exists an element \( \tau \in T \) so that
\[
U \cap \tau^{-1}(U) \cap \{ x \in X : f(\tau, x) \in U(a) \}
\]
is non-empty. The set \( E(f) \) is also called the **topological essential range**. The cocycle identity implies that \( f(1_T, x) = 0_\mathbb{A} \) for all \( x \in X \) and hence \( 0_\mathbb{A} \in E(f) \).

**Fact 1.2.** The essential range is always a closed **subgroup** of \( \mathbb{A} \) (cf. \cite[Proposition 3.1]{schmidt}), which carries over from a minimal \( \mathbb{Z} \)-action to a general Abelian group acting minimally).

If \( f(\tau, x) \) is a cocycle with full topological essential range \( E(f) = \mathbb{A} \), then \( \{x\} \times \mathbb{A} \subset D_{T,f}(x,a) \) for every \((x,a) \in X \times \mathbb{A}\). By Fact 1.1, there exists a \( T \)-invariant dense \( G_\delta \) set \( \mathcal{F} \subset X \) with \( \{x\} \times \mathbb{A} \subset \bar{\mathcal{O}}_{T,f}(x,a) \) for every \((x,a) \in \mathcal{F} \times \mathbb{A}\), and for every \( \tau \in T \) we can conclude that \( \{\tau x\} \times \mathbb{A} \subset \bar{\mathcal{O}}_{T,f}(x,a) \). By the minimality of the flow \((X,T)\) every \((x,a) \in \mathcal{F} \times \mathbb{A}\) is a transitive point for \( \bar{\tau}_f \).

Throughout this paper we shall use a notion of ‘relative’ triviality of cocycles.

**Definition 3.** Let \( f_1(\tau, x) \) and \( f_2(\tau, x) \) be \( \mathbb{R} \)-valued cocycles of a minimal compact metric flow \((X, T)\). We shall call the cocycle \( f_2(\tau, x) \) **relatively trivial** with respect to \( f_1(\tau, x) \) if for
every sequence \( \{(\tau_k, x_k)\}_{k \geq 1} \subset T \times X \) with \( d(x_k, \tau_k x_k) \to 0 \) and \( f_1(\tau_k, x_k) \to 0 \) it holds also that \( f_2(\tau_k, x_k) \to 0 \) as \( k \to \infty \).

**Remark 3.** Suppose that the cocycle \( f_2(\tau, x) \) is relatively trivial with respect to \( f_1(\tau, x) \). Let \( \{\tau_k\}_{k \geq 1} \subset T \) be a sequence and \( \bar{x} \in X \) be a point so that the sequence \( \{\tau_k \bar{x}\}_{k \geq 1} \) is convergent in \( X \) and the sequence \( \{f_1(\tau_k, \bar{x})\}_{k \geq 1} \) is convergent to a real number. Then the sequence \( \{f_2(\tau_k, \bar{x})\}_{k \geq 1} \) is also convergent to a real number.

By the following lemma it suffices to verify an essential value condition ‘locally’.

**Lemma 1.3.** Let \((X, T)\) be a minimal compact metric flow with an Abelian group \( T \) acting, and let \( f(\tau, x) \) be a cocycle of \((X, T)\) with values in an Abelian l.c.s.c. group \( \mathbb{A} \). If there exists a sequence \( \{(\tau_k, x_k)\}_{k \geq 1} \subset T \times X \) with \( d(x_k, \tau_k x_k) \to 0 \) and \( f(\tau_k, x_k) \to a \) in \( \mathbb{A}_\infty \) (\( \mathbb{R}_\infty \times \mathbb{R}_\infty \) for \( \mathbb{A} = \mathbb{R}^2 \), respectively) as \( k \to \infty \), then for every \( x \in X \) it holds that \( (x, a) \in D_{T, f}(x) \).

Proof. We let \( \{(\tau_k, x_k)\}_{k \geq 1} \subset T \times X \) be a sequence with the properties above, and we may assume that \( x_k \to x' \in X \) as \( k \to \infty \). For an arbitrary non-empty open set \( U \subset X \) and an arbitrary neighbourhood \( U(a) \) of \( a \) in \( \mathbb{A}_\infty \) we can fix an element \( \tau \in T \) with \( \tau x' \in U \), and since the group \( T \) is Abelian it follows that \( \tau x_k \to \tau x' \) and \( \tau_k x_k = \tau \tau_k x_k \to \tau x' \) as \( k \to \infty \). From the cocycle identity and the continuity of \( f(\tau, \cdot) \) it follows that

\[
    f(\tau_k, \tau x_k) = f(\tau, \tau_k x_k) + f(\tau, x_k) + f(\tau^{-1}, \tau x_k)
    = f(\tau, \tau_k x_k) + f(\tau, x_k) - f(\tau, \tau x_k) \to a
\]

as \( k \to \infty \), and for all \( k \) large enough it holds that \( \tau x_k, \tau_k \tau x_k \in U \) and \( f(\tau_k, \tau x_k) \in U(a) \). Since the non-empty open sets \( U \) and \( U(a) \) were arbitrary, we have \( (x, a) \in D_{T, f}(x) \) for every \( x \in X \) and \( a \in E(f) \) if \( a \neq \infty \).

If the sequence \( g_2(\tau_k, x_k) \) is unbounded, then by the first statement \( (x, 0, \infty) \in D_{T, g}(x, 0, 0) \) for every \( x \in X \). Otherwise, the sequence \( g_2(\tau_k, x_k) \) is bounded however not convergent to zero. The closed subspace \( E(g) \subset \mathbb{R}^2 \) has then an element \((0, c)\) with \( c \neq 0 \), and it follows again that \( (x, 0, \infty) \in D_{T, g}(x, 0, 0) \) for every \( x \in X \). By Fact 1.1 there exist \( \bar{x} \in X \) and a sequence \( \{\tau_k\}_{k \geq 1} \subset T \) so that \( \tau_k \bar{x} \to \bar{x} \) and \( g(\tau_k, \bar{x}) \to (0, \infty) \). For an arbitrary point \( \bar{y} \in \pi^{-1}(\bar{x}) \), we can select an increasing sequence of positive integers \( \{\ell_k\}_{k \geq 1} \) with \( d(y, (\tilde{\tau}_{k+1} \bar{y}, \tilde{\tau}_k \bar{y})) \to 0 \) and \( (g \circ \sigma)(\tilde{\tau}_{k+1} (\tilde{\tau}_k)^{-1}, \tilde{\tau}_k \bar{y}) \to (0, \infty) \) and put \( \{((\tau_k, y) = (\tilde{\tau}_{k+1} (\tilde{\tau}_k)^{-1}, \tilde{\tau}_k \bar{y})\}_{k \geq 1} \).

**Definition 4.** Let \( f(\tau, x) \) be a cocycle of a minimal compact metric flow \((X, T)\) with values in an Abelian l.c.s.c. group \( \mathbb{A} \), and let \( b : X \to \mathbb{A} \) be a continuous function. Another cocycle of the flow \((X, T)\) can be defined by the \( \mathbb{A} \)-valued function

\[
    g(\tau, x) = f(\tau, x) + b(\tau x) - b(x).
\]

The cocycle \( g(\tau, x) \) is called topologically cohomologous to the cocycle \( f(\tau, x) \) with the transfer function \( b(x) \). A cocycle \( g(\tau, x) = b(\tau x) - b(x) \) topologically cohomologous to zero is bounded on \( T \times X \) and called a topological coboundary.
REMARK 4. If \( f(\tau, x) \) and \( g(\tau, x) = f(\tau, x) + b(\tau x) - b(x) \) are topologically cohomologous cocycles, then the homeomorphism \( \ell(x, a) = (x, a - b(x)) \) of \( X \times \mathbb{A} \) is an isomorphism of skew product transformations \( \hat{\ell}_f \) and \( \hat{\ell}_g \) so that \( \hat{\ell}_f \circ \ell = \ell \circ \hat{\ell}_g \) for all \( \tau \in T \).

The Gottschalk–Hedlund theorem (cf. [13, Theorem 14.11]) characterizes topological coboundaries of a minimal \( \mathbb{Z} \)-action as cocycles bounded on at least one semi-orbit. The generalization to an Abelian group \( T \) acting minimally is natural.

FACT 1.4. A real-valued topological cocycle \( f(\tau, x) \) of a minimal compact metric flow \( (X, T) \) with an Abelian acting group \( T \) is a coboundary if and only if there exists a point \( \bar{x} \in X \) so that the function \( \tau \mapsto f(\tau, \bar{x}) \) is bounded on \( T \). For the groups \( T = \mathbb{Z} \) and \( T = \mathbb{R} \) acting, the boundedness on a semi-orbit is sufficient.

A real-valued cocycle \( f(\tau, x) \) is also a topological coboundary if for every sequence \( \{(\tau_k, x_k)\}_{k \geq 1} \subset T \times X \) with \( d(\tau_k, x_k) \to 0 \) the set \( \{f(\tau_k, x_k)\}_{k \geq 1} \subset \mathbb{R} \) is bounded.

Proof. Suppose that the mapping \( \tau \mapsto f(\tau, \bar{x}) \) is bounded on \( T \). From the cocycle identity it follows for all \( \tau, \tau' \in T \) that

\[
f(\tau, \tau' \bar{x}) = f(\tau \tau', \bar{x}) - f(\tau', \bar{x}),
\]

and the density of the \( T \)-orbit of \( \bar{x} \) implies that \( f(\tau, x) \) is bounded on \( T \times X \) so that \( E(f) = \{0\} \).

Also by the density of the \( T \)-orbit of \( \bar{x} \) and the boundedness of \( \tau \mapsto f(\tau, \bar{x}) \), the intersection \( \{x\} \times \mathbb{R} \cap \hat{O}_{T,f}(\bar{x}, 0) \) is non-empty for every \( x \in X \). For every \( x \in X \) this intersection is a singleton, since otherwise Lemma 1.3 proves a non-zero element in \( E(f) \). Hence, the compact set \( \hat{O}_{T,f}(\bar{x}, 0) \) is the graph of a continuous function \( b : X \to \mathbb{R} \) with \( f(\tau, \bar{x}) = b(\tau \bar{x}) \), and therefore \( f(\tau, x) = b(\tau x) - b(x) \) holds for every \( (\tau, x) \in T \times X \). For \( T = \mathbb{Z} \) and \( T = \mathbb{R} \) the set of limit points of a semi-orbit is a \( T \)-invariant closed subset of \( X \), which is non-empty by compactness and equal to \( X \) by minimality. We can conclude the proof as above, however, using the corresponding semi-orbit instead of the full orbit.

Now suppose that \( f(\tau, x) \) is not a topological coboundary and let \( \bar{x} \in X \) be arbitrary. Then there exists a sequence \( \{\tau'_l\}_{l \geq 1} \subset T \) with \( |f(\tau'_l, \bar{x})| \to \infty \), and we may assume that \( \tau'_l \bar{x} \to x' \) as \( l \to \infty \). Since \( (X, T) \) is minimal, there exists sequence \( \{\tau'_k\}_{k \geq 1} \subset T \) with \( \tau'_k x' \to \bar{x} \) as \( k \to \infty \).

A diagonalization with a sufficiently increasing sequence of positive integers \( \{l_k\}_{k \geq 1} \) yields for \( \tau_k = \tau'_k \tau'_l \) that \( \tau_k \bar{x} \to \bar{x} \) and \( |f(\tau_k, \bar{x})| = |f(\tau'_k, \tau'_l \bar{x}) + f(\tau'_l, \bar{x})| \to \infty \) as \( k \to \infty \).

The following lemma appeared originally in the paper [2] in a setting for \( \mathbb{R}^d \)-valued cocycles of a minimal rotation on a torus.

**Lemma 1.5.** Let \( f(\tau, x) \) be a real-valued topological cocycle of a minimal compact metric flow \( (X, T) \) with an Abelian acting group \( T \). If the skew product action \( \hat{\ell}_f \) is not point transitive on \( X \times \mathbb{R} \), then for every neighbourhood \( U \subset \mathbb{R} \) of \( 0 \) there exist a compact symmetric neighbourhood \( K \subset U \) of \( 0 \) and an \( \varepsilon > 0 \) so that for every \( \tau \in T \) holds

\[
\{x \in X : d(x, \tau x) < \varepsilon \text{ and } f(\tau, x) \in K \setminus K^0\} = \emptyset.
\]

**Proof.** Suppose that \( f(\tau, x) \) is real-valued and \( \hat{\ell}_f \) is not point transitive. By Fact 1.2 the essential range \( E(f) \) is a proper closed subgroup of \( \mathbb{R} \), and thus there exists a compact symmetric neighbourhood \( K \subset U \) of \( 0 \) with \( (K \setminus K^0) \cap E(f) = \emptyset \). If the assertion is false for the neighbourhood \( K \), then there exists a sequence \( \{(\tau_k, x_k)\}_{k \geq 1} \subset T \times X \) with \( d(x_k, \tau_k x_k) \to 0 \).
and \( f(\tau_k, x_k) \rightarrow t \in K \setminus K^0 \) as \( k \rightarrow \infty \). However, Lemma 1.3 implies that \( t \in E(f) \cap (K \setminus K^0) \), in contradiction to the choice of \( K \).

We shall commence the study of cocycles of distal minimal flows by the generalization of the results for minimal rotations in \([13, 20]\).

**Proposition 1.6.** Let \((X, T)\) be a minimal compact isometric flow with a compactly generated Abelian acting group \( T \), and let \( f(\tau, x) \) be a topologically recurrent real-valued cocycle of \((X, T)\). Then the cocycle \( f(\tau, x) \) is either a coboundary or its skew product extension \( \tilde{\tau}_f \) is point transitive on \( X \times \mathbb{R} \).

**Proof.** Suppose that the cocycle \( f(\tau, x) \) is not a coboundary and \( \tilde{\tau}_f \) is not point transitive. By Lemma 1.5 there exist a compact symmetric neighbourhood \( K \) of 0 and an \( \varepsilon > 0 \) so that equality (1.2) holds for every \( \tau \in T \). Furthermore, if \( L \subset T \) is a compact generative subset, then \( \varepsilon > 0 \) can be chosen small enough so that for all \( \tau' \in L \) and \( x, x' \in X \) with \( d(x, x') < \varepsilon \) it holds that

\[
\left| f(\tau', x) - f(\tau', x') \right| < \varepsilon.
\]

By Fact 1.4 we can fix a pair \((\tilde{\tau}, \tilde{x}) \in T \times X \) with \( d(\tilde{x}, \tilde{\tau} \tilde{x}) < \varepsilon \) and \( f(\tilde{\tau}, \tilde{x}) \notin K \), since \( f(\tau, x) \) is not a coboundary. The Abelian group \( T \) acts on \( X \) isometrically, and thus \( d(\tilde{x}, \tilde{\tau} \tilde{x}) < \varepsilon \) implies that \( d(\tilde{\tau}' \tilde{x}, \tilde{\tau} \tilde{x}) = d(\tilde{\tau}' \tilde{x}, \tilde{\tau} \tilde{x}) < \varepsilon \). Together with equality (1.2) we can conclude for every \( \tau' \in L \) that

\[
\left| f(\tilde{\tau}', \tilde{x}) - f(\tilde{\tau}, \tilde{x}) - f(\tilde{\tau}', \tilde{x}) \right| + \left| f(\tilde{\tau}', \tilde{x}) \notin K, \right|
\]

and hence both of the real numbers \( f(\tilde{\tau}, \tilde{x}) \) and \( f(\tilde{\tau}, \tilde{x}') \) are elements of the one and the same of the disjoint sets \( \mathbb{R}^+ \setminus K \) and \( \mathbb{R}^- \setminus K \). Since the set \( L \) is generative in the Abelian group \( T \) acting minimally on \( X \), it follows by induction that \( f(\tilde{\tau}, x) \) is in the closure of one of the sets \( \mathbb{R}^+ \setminus K \) and \( \mathbb{R}^- \setminus K \) for every \( x \in X \). Thus, we have a constant \( c > 0 \) with \( |f(\tilde{\tau}^k, x)| > |k|c \) for every integer \( k \), and we define a subset \( P \subset T \) by

\[
P = \bigcup_{k \geq 1} \tilde{\tau}^k \cdot \{ \tau \in T : f(\tau, \cdot) < |k|c/2 \}.
\]

Given two integers \( k, k' \geq 1 \) and \( \tilde{\tau}^k \tau, \tilde{\tau}^k \tau' \in P \) with \( f(\tau, \cdot) < |k|c/2 \) and \( f(\tau', \cdot) < |k'|c/2 \), we can conclude that \( \tilde{\tau}^k \tilde{\tau}^k \tau, \tilde{\tau}^k \tau' \in P \) with \( f(\tau \tau', \cdot) < |k+k'|c/2 \), hence \( P \) is a semigroup. Moreover, the semigroup \( P \) contains a translate of every compact set \( L \subset T \), since for large enough \( k \geq 1 \) the inequality \( f(\tau, x) < |k|c/2 \) holds for every \( \tau \in L \) and every \( x \in X \). Therefore, \( P \) is a replete semigroup in \( T \) so that \( |f(\tau, x)| > c/2 \) holds for every \((\tau, x) \in P \times X \), which contradicts the existence of a dense \( G_\delta \) set of \( \tilde{\tau}_f \)-recurrent points (cf. Remarks 2).

The Rokhlin extensions and the Rokhlin skew products have been studied in the measure theoretic setting in \([19, 21]\). We shall introduce the notion of a perturbed Rokhlin skew product, which will be inevitable in our main result.

**Definition 5.** Suppose that \((X, T)\) and \((M, \{\phi^t : t \in \mathbb{R}\})\) are distal minimal compact metric flows. Let \( f : T \times X \rightarrow \mathbb{R} \) be a cocycle of \((X, T)\) with a point transitive skew product \( \tilde{\tau}_f \) on \( X \times \mathbb{R} \). We define the Rokhlin extension \( \tau_{\phi,f} \) on \( X \times M \) by

\[
\tau_{\phi,f}(x, m) = (x, \phi^{f(\tau,x)}(m)).
\]

**Fact 1.7.** A Rokhlin extension \( \tau_{\phi,f} \) which is defined by a distal minimal compact metric flow \((X, T)\), a cocycle \( f : T \times X \rightarrow \mathbb{R} \) of \((X, T)\) with a point transitive skew product \( \tilde{\tau}_f \) on
X × R, and a distal minimal compact metric R-flow \((M, \{\phi^t : t \in R\})\) turns out to be a distal minimal \(T\)-action on \(X \times M\).

**Proof.** The mapping \((\tau, x, m) \mapsto \tau_{\phi,f}(x, m)\) is continuous on \(T \times X \times M\) by the joint continuity of the group actions on \(X\) and \(M\) and the continuity of the cocycle \(f(\tau, x)\) on \(T \times X\). If \(\tau = 1_T\), then \(f(\tau, x) = 0\) for all \(x \in X\) so that \(\tau_{\phi,f}(x, m) = (x, m)\) for all \((x, m) \in X \times M\). Moreover, for arbitrary \(\tau, \tau' \in T\) and \((x, m) \in X \times M\), the cocycle identity \(f(\tau \tau', x) = f(\tau, \tau' x) + f(\tau', x)\) implies that

\[
(\tau \tau')_{\phi,f}(x, m) = ((\tau \tau')_x, \phi^f(\tau \tau', x)(m)) = (\tau(\tau' x), \phi^f(\tau, \tau' x) \circ \phi^f(\tau', x)(m)) = \tau_{\phi,f} \circ \tau'_{\phi,f}(x, m).
\]

Therefore, the group \(T\) acts jointly continuous on \(X \times M\).

Now let \((x, m) \neq (x', m')\) be arbitrary distinct points in \(X \times M\). If \(x \neq x'\), then by the distality of \((X, T)\) the points \((x, m)\) and \((x', m')\) are distal for the Rokhlin extension action. If \(x = x'\) and \(m \neq m'\), then \(\tau_{\phi,f}(x', m') = (\tau x, \phi^f(\tau, x)(m'))\) so that \((x, m)\) and \((x', m')\) are distal by the distality of the \(R\)-flow \((M, \{\phi^t : t \in R\})\). Thus, the Rokhlin extension action on \(X \times M\) is distal and pointwise almost periodic.

Let \((\bar{x}, 0)\) be a transitive point for the skew product \(\bar{\tau}_f\) on \(X \times R\), and let \(\bar{m} \in M\) and \((x, m) \in X \times M\) be arbitrary. By the minimality of \((M, \{\phi^t : t \in R\})\) there exists a sequence \(\{\bar{t}_k\}_{k \geq 1} \subset R\) so that \(d_M(\phi^{\bar{t}_k}(\bar{m}), m) \to 0\) as \(k \to \infty\), and since \((\bar{x}, 0)\) is transitive for \(\bar{\tau}_f\) there exists a sequence \(\{t_k\}_{k \geq 1} \subset T\) so that \(\tau_k \bar{x} \to x\) and \(d_M(\phi^{f(\tau_k \bar{x})}(\bar{m}), \phi^{\bar{t}_k}(\bar{m})) \to 0\) as \(k \to \infty\). Therefore, \((\tau_k)_{\phi,f}(\bar{x}, \bar{m}) \to (x, m)\) as \(k \to \infty\). Since \((x, m) \in X \times M\) was arbitrary, it follows that \((\bar{x}, \bar{m})\) is a transitive point for the pointwise almost periodic (and hence minimal) flow \((X \times M, T)\).

**Definition 6.** The skew product extension of the Rokhlin extension \(\tau_{\phi,f}\) on \((X \times M, T)\) by the cocycle \((\tau, x, m) \mapsto f(\tau, x)\) is the Rokhlin skew product \(\bar{\tau}_{\phi,f}\) on \(X \times M \times R\) with

\[
\bar{\tau}_{\phi,f}(x, m, t) = (\tau x, \phi^f(\tau, x)(m), t + f(\tau, x)).
\]

**Fact 1.8.** Let \(g : R \times M \to R\) be a cocycle of the flow \((M, \{\phi^t : t \in R\})\) and let \(1(t, m) = t\) be the linear cocycle of the flow \((M, \{\phi^t : t \in R\})\). Then we can define a real-valued topological cocycle of the Rokhlin extension action \(\tau_{\phi,f}\) on \(X \times M\) by

\[
(\tau, x, m) \mapsto f(\tau, x) + g(f(\tau, x), m) = (1 + g)(f(\tau, x), m).
\]

**Proof.** From the cocycle identities for \(f(\tau, x)\) and \(g(t, m)\), we can conclude that

\[
(1 + g)(f(\tau, x), m) = (1 + g)(f(\tau', x) + f(\tau, x), m) = (1 + g)(f(\tau', x), \phi^f(\tau, x)(m)) + (1 + g)(f(\tau, x), m).
\]

The term \((1 + g)(f(\tau', x), \phi^f(\tau, x)(m))\) coincides with \((1 + g)(f(\tau', x'), m')\) for \((x', m') = \tau_{\phi,f}(x, m)\), and hence \((1 + g)(f(\tau, x), m)\) is a cocycle of \(\tau_{\phi,f}\).

**Definition 7.** The skew product extension of the Rokhlin extension \(\tau_{\phi,f}\) on \((X \times M, T)\) by the cocycle in Fact 1.8 is given by

\[
\bar{\tau}_{\phi,f,g}(x, m, t) = (\tau x, \phi^f(\tau, x)(m), t + f(\tau, x) + g(f(\tau, x), m)).
\]

Such a skew product is called a perturbed Rokhlin skew product \(\bar{\tau}_{\phi,f,g}\) on \(X \times M \times R\).
We shall present at first the basic example of a topological Rokhlin skew product of topological type III$_0$, that is, recurrent with the trivial topological essential range but not a topological coboundary.

**Example 1.** Let $f : T \to \mathbb{R}$ be a continuous function with a point transitive skew product extension $\tilde{T}_f$ of the irrational rotation $T$ by $\alpha$ on the torus, and let $\beta \in (0, 1)$ be irrational so that the $\mathbb{R}$-flow $(T^2, \{\phi^t : t \in \mathbb{R}\})$ defined by $\phi^t(y, z) = (y + t, z + \beta t)$ is minimal and distal. The minimal and distal Rokhlin extension $T_{\phi, f}$ on $T^3$ is

$$T_{\phi, f}(x, y, z) = (x + \alpha, y + f(x), z + \beta f(x)),$$

and putting $h(x, y, z) = f(x)$ for all $(x, y, z) \in T^3$ gives a topological type III$_0$ cocycle $h(n(x, y, z))$ of the homeomorphism $T_{\phi, f}$ with the skew product extension $\tilde{T}_{\phi, f}$. Indeed, since $\tilde{T}_f$ is point transitive, the cocycle $h(n(x, y, z))$ is recurrent, but it is not bounded and therefore no topological coboundary. Let $c \in E(h) \subset \mathbb{R}$ be a an essential value and let $\{(n_k, x_k, y_k, z_k)\}_{k \geq 1} \subset \mathbb{Z} \times T^3$ be a sequence with $\lim_{k \to \infty} (T_{\phi, f}^{n_k})(x_k, y_k, z_k) = 0$ and $h(n_k, x_k, y_k, z_k) = f(n_k, x_k) \to c$ as $k \to \infty$. Since

$$T_{\phi, f}^{n_k}(x, y, z) = (x + n_k \alpha, y + f(n_k, x), z + \beta f(n_k, x)),$$

it follows that $c \mod 1 = 0$ and $\beta c \mod 1 = 0$ so that $c = 0$. For a point $\bar{x} \in T$ so that $(\bar{x}, 0) \in T \times \mathbb{R}$ is transitive under $\tilde{T}_f$ and arbitrary $y, z \in T$ the orbit closure of $(\bar{x}, y, z, 0)$ under the skew product extension of $T_{\phi, f}$ by $h$ is of the form:

$$\tilde{O}_{T_{\phi, f}}((\bar{x}, y, z, 0)) = \tilde{O}_{T_{\phi, f}, h}((\bar{x}, y, z, 0)) = \lim_{k \to \infty} ((\phi^t(y, z), t) \in T^2 \times \mathbb{R} : t \in \mathbb{R}).$$

The collection of these sets is a partition of $T^3 \times \mathbb{R}$ into $\tilde{T}_{\phi, f}$-orbit closures.

The next example makes clear that the perturbation of a Rokhlin skew product by a cocycle is an essential component, which in general cannot be eliminated by continuous cohomology.

**Example 2.** Let $T$, $f$, $h$ and $\{\phi^t : t \in \mathbb{R}\}$ be defined as in Example 1. By [16, Theorem 4.4], there exists a residual set $C_1$ in $C_0(T^2)$ (continuous functions of Lebesgue integral zero) so that for every $A \in C_1$ the solution of the differential equation (1.1) is a cocycle of the flow $\{\phi^t : t \in \mathbb{R}\}$ with an unbounded function $t \mapsto g(t, m)$ for every $m \in T^2$. Since $g(t, m) = \int_0^t A(\phi^s(m)) \, ds$, it follows from Fubini’s theorem and the $\phi^s$-invariance of the Lebesgue measure that $\int_{T^2} g(t, m) \, d\lambda(m) = 0$. Hence, for every $t \in \mathbb{R}$, the mapping $m \mapsto g(t, m)$ takes the value zero on the connected space $T^2$. Let $\{(t_k, m_k)\}_{k \geq 1} \subset \mathbb{R} \times T^2$ be a sequence so that $\phi^{t_k}$ converges uniformly to the identity mapping on $T^2$ and $g(t_k, m_k) = 0$. For every cluster point $\bar{m}$ of the sequence $\{m_k\}_{k \geq 1}$ the point $(\bar{m}, 0) \in M \times \mathbb{R}$ is a regionally recurrent point of $\tilde{\phi}^t$, and thus the cocycle $g(t, m)$ is topologically recurrent (cf. Remarks 2). Moreover, by Proposition 1.6, the unbounded cocycle $g(t, m)$ defines a point transitive skew product extension $\tilde{\phi}^t_g$, and by choosing a function $A \in C_1$ with $\sup_{m \in T^2} |A(m)| < 1/2$, we can ensure that $|g(t, m)| < |t|/2$ for all $t \in \mathbb{R}$ and $m \in T^2$. We define a function

$$\tilde{h}(x, y, z) = f(x) + g(f(x), (y, z)),$$

and the cocycle of $T_{\bar{\phi}, f}$ is $\tilde{h}(n(x, y, z)) = f(n(x) + g(f(n(x)), (y, z)))$ for every integer $n$. Since the perturbation $g(f(n(x)), (y, z))$ is unbounded, there cannot be a continuous transfer function defined on $T^3$ so that $\tilde{h}$ and $h$ are cohomologous. However, by the condition $|g(t, (y, z))| < |t|/2$, the set

$$T \times \{(\phi^t(y, z), t + g(t, (y, z))) \in T^2 \times \mathbb{R} : t \in \mathbb{R}\}$$

is a topological type III$_0$, that is, recurrent with the trivial topological essential range but not a topological coboundary.
is closed, and it coincides with \( \mathcal{O}_{f_{\phi,f,g}}((\bar{x},y,z),0) \) if the point \((\bar{x},0) \in T \times \mathbb{R} \) is transitive under \( \tilde{T}_f \). Thus, the structure of the orbit closures is preserved as well as these sets provide a partition of \( T^3 \times \mathbb{R} \).

**Remark 5.** The structure of Example 1 can be revealed from the toral extensions of \( T_{\phi,f} \) by the function \((\gamma \hat{h}) \mod 1 \) for all \( \gamma \in \mathbb{R} \). This distal homeomorphism of \( T^4 \) is transitive and hence minimal for rationally independent \( 1, \beta \) and \( \gamma \). However, for \( \gamma = 1 \) and \( \beta \) the orbit closures collapse to graphs representing the dependence of \( h \) and the action on the coordinates of the torus. The same approach will not be successful with respect to Example 2. It can be verified that for every \( \gamma \in \mathbb{R} \) the toral extension of \( T_{\phi,f} \) by the function \((\gamma \hat{h}) \mod 1 \) is minimal on \( T^4 \).

The main result of this paper puts these examples into a structure theorem.

**Structure theorem.** Suppose that \((X,T)\) is a distal minimal compact metric flow with a compactly generated Abelian acting group \( T \) and that \( f : T \times X \to \mathbb{R} \) is a topologically recurrent cocycle which is not a coboundary. Then there exist a factor \((X_\alpha, T) = \pi_\alpha(X,T), \) a topological cocycle \( f_\alpha : T \times X_\alpha \to \mathbb{R} \) of \((X_\alpha, T), \) and a distal minimal compact metric \( \mathbb{R} \)-flow \((M, \{\phi^t : t \in \mathbb{R}\})\), so that the Rokhlin extension \((X_\alpha \times M, T)\) with the action \( \tau_{\phi,f_\alpha} \) is a factor \((Y,T) = \pi_Y(X,T) \) of \((X,T)\). The cocycle \( f(\tau,x) \) is topologically cohomologous to

\[
(f_Y \circ \pi_Y)(\tau,x) = f(\tau,x) + b(\tau x) - b(x) \tag{1.3}
\]

with a topological cocycle \( f_Y : T \times Y \to \mathbb{R} \) of the flow \((Y,T)\) so that

\[
\mathcal{D}_{T,f_{Y \circ \pi_Y}}(x,0) \cap (\pi_\alpha^{-1}(\pi_\alpha(x)) \times \{0\}) = \pi_Y^{-1}(\pi_Y(x)) \times \{0\} \tag{1.4}
\]

holds for all \( x \in X \). Moreover, there exists a topological cocycle \( g : \mathbb{R} \times M \to \mathbb{R} \) of the \( \mathbb{R} \)-flow \((M, \{\phi^t : t \in \mathbb{R}\})\) so that the cocycle \((1 + g)(t,m) = t + g(t,m) \) is topologically transient and

\[
f_Y(\tau,(x_\alpha,m)) = f_\alpha(\tau,x_\alpha) + g(\alpha(\tau,x_\alpha),m) = (1 + g)(\alpha(\tau,x_\alpha),m) \tag{1.5}
\]

holds for every \( \tau \in T \) and \((x_\alpha,m) \in Y = X_\alpha \times M \). Thus, the skew product \( \tilde{\tau}_{f_Y} \) on \( Y \times \mathbb{R} \) is the perturbed Rokhlin skew product \( \tilde{\tau}_{\phi,f_\alpha,g} \), and the skew products \( \tilde{\tau}_f \) and \( \tilde{\tau}_{f_Y \circ \pi_Y} \) are isomorphic on \( X \times \mathbb{R} \) by the cohomology of the cocycles. While the prolongations in the skew product \( \tilde{\tau}_{f_Y \circ \pi_Y} \) are saturated with respect to the equivalence relation \( \{(x,t),(x',t'): \pi_Y(x) = \pi_Y(x'), t \in \mathbb{R}\} \) on \( X \times \mathbb{R} \) (cf. (1.4)), the prolongations in the skew product \( \tilde{\tau}_f \) are saturated with respect to the equivalence relation \( \{(x,t),(x',t'): \pi_Y(x) = \pi_Y(x'), b(x) + t = b(x') + t'\} \) on \( X \times \mathbb{R} \) (cf. (1.3)).

We shall conclude the proof of this theorem in the next section of this paper.

The application of the structure theorem for a topological ergodic decomposition requires a suitable topology on the hyperspace of the non-compact space \( X \times \mathbb{R} \). We shall use the Fell topology on the hyperspace of non-empty closed subsets of a locally compact separable metric space. Given finitely many non-empty open sets \( U_1, \ldots, U_k \) and a compact set \( K \), an element of the Fell topology base consists of all non-empty closed subsets which intersect each of the open sets \( U_1, \ldots, U_k \) while being disjoint from \( K \). This topology is separable, metrizable, and \( \sigma \)-compact (cf. [15]). The Fell topology was introduced in [8] as a compact topology on the hyperspace of all closed subsets, with the empty set as infinity.

**Decomposition theorem.** Suppose that \( f : T \times X \to \mathbb{R} \) is a topologically recurrent cocycle of a distal minimal compact metric flow \((X,T)\) with a compactly generated Abelian acting group \( T \). The prolongations \( \mathcal{D}_{T,f}(x,s) \subset X \times \mathbb{R} \) of the skew product action \( \tilde{\tau}_f \) with \((x,s) \in X \times \mathbb{R} \) define a partition of \( X \times \mathbb{R} \). The mapping \((x,s) \mapsto \mathcal{D}_{T,f}(x,s) \) is continuous.
with respect to the Fell topology on the hyperspace of non-empty closed subsets of $X \times \mathbb{R}$, and the right translation on $X \times \mathbb{R}$ is a minimal continuous $\mathbb{R}$-action on the set of prolongations. If the cocycle $f(\tau, x)$ is not a topological coboundary, then the set of all prolongations in the skew product is Fell compact.

**Topological Mackey Action.** A recurrent real-valued cocycle $f(\tau, x)$ apart from a coboundary has a minimal compact metric flow as a topological version of the Mackey action. Its phase space is the set of prolongations in the skew product with the Fell topology, with the right translation of $\mathbb{R}$ acting minimally on the set of prolongations. This flow is a distal extension (possibly the trivial extension) of a weakly mixing compact metric flow (possibly the trivial flow). The Mackey action is distal if and only if the perturbation cocycle $g(t, m)$ in the structure theorem is cohomologous to $ct$ for some real number $c$.

**Uniqueness of the Representation.** For a given recurrent real-valued cocycle $f(\tau, x)$ apart from a coboundary, the distal minimal compact metric $\mathbb{R}$-flow $(M, \{\phi^t : t \in \mathbb{R}\})$ is unique up to isomorphism and a rescaling time change, that is, if the flows $(M_i, \{\phi^t_i : t \in \mathbb{R}\})$ and cocycles $g_i(t, m_i)$ for $i \in \{1, 2\}$ have all the properties in structure theorem, then there exists a homeomorphism $\iota : M_1 \rightarrow M_2$ and $c \in \mathbb{R} \setminus \{0\}$ so that $\iota \circ \phi^t_1 = \phi^{ct}_2 \circ \iota$ for all $t \in \mathbb{R}$. Moreover, the cocycles $(1 + g_1)(t, m_1)$ and $(1 + g_2)(ct, \iota(m_1))$ are topologically cohomologous.

While most of the properties of the topological Mackey action are part of the decomposition theorem, its structure as a distal extension of a weakly mixing flow and the uniqueness properties will be verified in the next section of this paper. The proof of the decomposition theorem depends on the following general lemma on transient cocycles of minimal $\mathbb{R}$-flows, which might be of independent interest.

**Lemma 1.9.** Let $(M, \{\phi^t : t \in \mathbb{R}\})$ be a minimal compact metric $\mathbb{R}$-flow and let $h(t, m)$ be a transient real-valued cocycle of $(\tilde{M}, \{\phi^t : t \in \mathbb{R}\})$. Then there are no non-trivial prolongations under the skew product extension $\tilde{\phi}_h$, and for every point $(m, s) \in M \times \mathbb{R}$ the orbit $\mathcal{O}_{\phi, h}(m, s)$, the orbit closure $\overline{\mathcal{O}_{\phi, h}(m, s)}$, and the prolongation $\mathcal{D}_{\phi, h}(m, s)$ coincide. The mapping from points to their orbits in $M \times \mathbb{R}$ is continuous with a compact range with respect to the Fell topology, and the right translation of $M \times \mathbb{R}$ provides a minimal continuous $\mathbb{R}$-action on the set of orbits. Moreover, for every $m \in M$, the mapping $t \mapsto h(t, m)$ maps $\mathbb{R}$ onto $\mathbb{R}$.

**Proof.** Suppose that there exists a non-trivial prolongation, that is, $(m, s), (m', s') \in M \times \mathbb{R}$ and a sequence $\{(t_k, m_k)\}_{k \geq 1} \subset \mathbb{R} \times M$ so that $(t_k, m_k) \rightarrow (+\infty, m)$ and

$$\tilde{\phi}^{t_k}_h(m_k, s) = (\phi^{t_k}(m_k), s + h(t_k, m_k)) \rightarrow (m', s').$$

If there exists a compact set $L \subset \mathbb{R}$ with $h([0, t_k), m_k) \subset L$ for all $k \geq 1$, then $h([0, \infty), m) \subset L$ since $m_k \rightarrow m$, and by Fact 1.4 the cocycle $h(t, m)$ is a coboundary in contradiction to its transience. Therefore, we have an increasing sequence of integers $\{k_l\}_{l \geq 1}$, a sequence $\{t'_l\}_{l \geq 1} \subset \mathbb{R}$ with $t'_l \in [0, t_k]$, and $S \in \{+1, -1\}$ so that

$$S \cdot h(t'_l, m_k) = \max_{t \in [0, t_k]} S \cdot h(t, m_k) \rightarrow +\infty$$

as $l \rightarrow \infty$. For every limit point $\tilde{m}$ of the sequence $\{\phi^{t'_l}(m_k)\}_{l \geq 1}$ it holds that $S \cdot h(t, \tilde{m}) \leq 0$ for all $t \in \mathbb{R}$, and the mapping $t \mapsto h(t, \tilde{m})$ maps each of the sets $\mathbb{R}^+$ and $\mathbb{R}^-$ onto $S \cdot \mathbb{R}$. Hence, for every $t \in \mathbb{R}^+$, there exists a $t' \in \mathbb{R}^-$ with $h(t, \tilde{m}) = h(t', \tilde{m})$, and by the density of the semi-orbit $\{\phi^t(\tilde{m}) : t \in \mathbb{R}^+\}$ (cf. the proof of Fact 1.4) and the cocycle identity of the
open set

\[ M_k = \{ m \in M : |h(t, m)| < 2^{-k} \text{ for some } t < -k \} \]

is dense for every integer \( k \geq 1 \). For a point \( m_k \) in the dense \( G_\delta \) set \( \bigcap_{t \in \mathbb{Q}} \phi^t(M_k) \), we can find rational numbers \( t_1, \ldots, t_k < -k \) so that \( \phi^{t_1 + \cdots + t_k}(m_k) \in M_k \) and \( |h(t_1 + \cdots + t_k, m_k)| < k2^{-k} \) for all \( 1 \leq l \leq k \). Since \( M \) is compact, there exists a point \( \tilde{m} \in M \) in each neighborhood of which contains at least two different points out of the finite sequence \( \phi^{t_1}(m_k), \ldots, \phi^{t_1 + \cdots + t_k}(m_k) \in M_k \) for an infinite set of integers \( k \geq 1 \). Thus, \( (\tilde{m}, 0) \) is a regionally recurrent point of the skew product \( \phi^t_h \), in contradiction to Remark 2 and the transience of the cocycle \( h(t, m) \). The absence of non-trivial prolongations clearly implies the coincidence of orbit, orbit closure and prolongation for every \((m, s) \in M \times \mathbb{R}\).

Suppose for some \( m \in M \) the mapping \( t \mapsto h(t, m) \) is not onto \( \mathbb{R} \), then either there exists a compact set \( L \subset M \) so that one of the inclusions \( h([0, \infty), m) \subset L, h((\infty, 0], m) \subset L \) holds, or there exists a point \( \tilde{m} \in M \) as above. In each of these cases, we can obtain a contradiction to the transience of \( h(t, m) \) as above.

Let \((m, s) \in M \times \mathbb{R}\) be arbitrary, and let a Fell neighbourhood of its closed \( \sim \phi^t_h \)-orbit be defined by the non-empty open sets \( U_1, \ldots, U_k \) and a compact set \( K \). Obviously the \( \sim \phi^t_h \)-orbit of every point in a suitable neighbourhood of \((m, s) \) intersects each of the neighbourhoods \( U_1, \ldots, U_k \). Moreover, the \( \sim \phi^t \)-orbit is disjoint from \( K \) for every point in a suitable neighbourhood of \((m, s) \), since otherwise the \( \sim \phi^t_h \)-prolongation of \((m, s) \) intersects the compact set \( K \).

This contradicts the coincidence of orbits and prolongations of \( \sim \phi^t_h \), and hence the mapping of a point \((m, s) \in M \times \mathbb{R}\) to its \( \sim \phi^t_h \)-orbit is Fell continuous. By the surjectivity of the mapping \( t \mapsto h(t, m) \), every \( \sim \phi^t_h \)-orbit intersects the set \( \{(m, 0) : m \in M \} \). The Fell compactness of the set of \( \sim \phi^t_h \)-orbits in \( M \times \mathbb{R} \) follows, and the right translation on \( M \times \mathbb{R} \) is a Fell continuous \( \mathbb{R} \)-action by the definition of the Fell neighbourhoods. Since the points \((\phi^t(m), 0) \) and \((m, -h(m, t)) \) are within the same \( \sim \phi^t_h \)-orbit, the minimality of \((M, \{ \phi^t : t \in \mathbb{R} \}) \) implies the minimality of the right translation action.

**Proof of the decomposition theorem.** For a topological coboundary \( f(\tau, x) \) the assertions are trivial, since the prolongations are just the right translates of the graph of the transfer function. If the cocycle \( f(\tau, x) \) is not a coboundary, then we apply the structure theorem. Given a point \( \bar{x}_0 \in X_0 \) with \( \mathcal{O}_{T, f_0}(\bar{x}_0, 0) = X_0 \times \mathbb{R} \) and arbitrary \( m \in M \), we can conclude from equality (1.5) that \( X_0 \times \mathcal{O}_{\phi_1+g}(m, 0) \subset \mathcal{O}_{T, f}(\bar{x}_0, m, 0) \subset X_0 \times \mathcal{O}_{\phi_1+g}(m, 0) \). By Lemma 1.9, it follows that \( \mathcal{O}_{\phi_1+g}(m, 0) = \mathcal{O}_{\phi_1+g}(m, 0) = \pi_{\phi_1+g}(m, 0) \). The set of points \( \bar{x}_0 \in X_0 \) with \( \mathcal{O}_{T, f_0}(\bar{x}_0, 0) = X_0 \times \mathbb{R} \) is a dense \( G_\delta \) set, and for arbitrary points \((x, m) \in X_0 \times M \) and \((x', m', t') \in D_{T, f}(x, m, 0) \) we can find a sequence \( \{(\tau_k, \bar{x}_k, m_k) \} \subset T \times X_0 \times M \) with \( (\bar{x}_k, m_k) \rightarrow (x, m), \tau_k(\bar{x}_k, m_k) \rightarrow (x', m'), f_T(\tau_k, (\bar{x}_k, m_k)) \rightarrow t' \), and \( X_0 \times \mathcal{O}_{\phi_1+g}(m, 0) = \pi_{T, f_0}(\bar{x}_k, m_k, 0) \). Hence, \((m', t') \in D_{\phi_1+g}(m, 0) \) so that \( D_{T, f}(x, m, 0) \subset X_0 \times \mathcal{O}_{\phi_1+g}(m, 0) \). We conclude for every \((x, m, s) \in X_0 \times M \times \mathbb{R} \) that
\[
D_{T, f}(x, m, s) = X_0 \times \{(\phi^t(m), s + t + g(t, m)) : t \in \mathbb{R}\},
\]
and these sets define a partition of \( X_0 \times M \times \mathbb{R} \). The Fell continuity of the mapping \((x, m, s) \mapsto D_{T, f}(x, m, s) \), the compactness of its range, and the minimality of the right translation follow directly from Lemma 1.9.

Suppose that \((x, s) \in X \times \mathbb{R} \) and \((x', m', s') \in D_{T, f}(\pi_Y(x), s) \), and let \( \{\tau_k\}_{k \geq 1} \subset T \) and \( \{y_k\}_{k \geq 1} \subset X_0 \times M \) be sequences with \( y_k \rightarrow \pi_Y(x), \tau_k y_k \rightarrow (x', m'), \) and \( f_T(\tau_k, y_k) \rightarrow s' - s \). Since \( \pi_Y \) is a distal homomorphism, it is an open onto mapping, and thus the mapping \( y \mapsto \pi_Y^{-1}(y) \) is continuous with respect to the Hausdorff metric \( d_H \) (cf. [18, p. 68, Theorem 1, and p. 47]). Therefore, we can define a sequence \( \{x_k \in \pi_Y^{-1}(y_k)\}_{k \geq 1} \subset X \) so that \( x_k \rightarrow x, \tau_k x_k \rightarrow x' \in \pi_Y(x', m') \), and \( f_T(\tau_k y_k) \rightarrow s' - s \). By equality (1.4),
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for every point \( x'' \) in the fibre \( \tilde{\pi}_Y^{-1}(\pi_Y(x')) \) there exists a sequence \( \{(\tilde{\tau}_k, \tilde{x}_k)\}_{k \geq 1} \) with \( \tilde{x}_k \) sufficiently close to \( x_k \) so that \( \tilde{x}_k \to x, \tau_k \tilde{x}_k \to x' \), \( (f_Y \circ \pi_Y)(\tau_k, \tilde{x}_k) \to s' - s \), \( d(\tilde{\tau}_k, \tau_k, x''') < d_H(\pi_Y^{-1}(\pi_Y(\tau_k x_k)), \pi_Y^{-1}(\pi_Y(x')) + 2^{-k} \) and \( (f_Y \circ \pi_Y)(\tau_k, \tau_k \tilde{x}_k) < 2^{-k} \). From \( d_H(\pi_Y^{-1}(\pi_Y(\tau_k x_k)), \pi_Y^{-1}(\pi_Y(x')) \to 0 \), it follows that \( \tilde{\tau}_k \tau_k \tilde{x}_k \to x'' \) and \( (f_Y \circ \pi_Y)(\tilde{\tau}_k, \tau_k \tilde{x}_k) \to s' - s \), hence \( (x'', s') \in \mathcal{D}_{T,f_Y \circ \pi_Y}(x, s) \). Thus, the \( \tilde{\tau}_{f_Y \circ \pi_Y} \)-prolongations in \( X \times \mathbb{R} \) are exactly the pre-images of the \( \tilde{\tau}_{f_Y} \)-prolongations under the mapping \( \pi_Y \times \text{id}_{\mathbb{R}} \). If a Fell neighbourhood of \( \mathcal{D}_{T,f_Y \circ \pi_Y}(x, s) \) is defined by the non-empty open sets \( U_1, \ldots, U_k \) and a compact set \( K \) in \( X \times \mathbb{R} \), then by the openness of \( \pi_Y \) the \( \pi_Y \times \text{id}_{\mathbb{R}} \)-images of these sets define a Fell neighbourhood of \( \mathcal{D}_{T,f_Y}(\pi_Y(x), s) \). Therefore, the mapping \((x, s) \mapsto \mathcal{D}_{T,f_Y \circ \pi_Y}(x, s) \) is Fell continuous with a compact range, and an isomorphism of the minimal right translation actions is given the mapping \( \mathcal{D}_{T,f_Y}(y, s) \mapsto (\pi_Y \times \text{id}_{\mathbb{R}})^{-1}(\mathcal{D}_{T,f_Y}(y, s)) \).

Let \( b : X \to \mathbb{R} \) be the transfer function in the structure theorem. The homeomorphism \((x, s) \mapsto (x, s + b(x)) \) on \( X \times \mathbb{R} \) defines a homeomorphism of the hyperspace equipped with the Fell topology, which commutes with the right translation. Since it maps \( \mathcal{D}_{T,f}(x, s) \) onto \( \mathcal{D}_{T,f \circ \pi_Y}(x, s + b(x)) \), it is an isomorphism of the right translation actions and all properties carry over.

\[ \square \]

Remark 6. Though the compact metric flow \((X, T)\) is not necessarily itself a Rokhlin extension, the existence of a real-valued recurrent non-coboundary cocycle \( f(\tau, x) \) with a non-transitive skew product extension \( \tilde{\tau}_f \) implies the existence of a Rokhlin extension factor \((Y, T) = \pi_Y(X, T)\) with a non-trivial flow \( \{\delta^t : t \in \mathbb{R}\} \) and a cocycle \( f_Y(\tau, y) \) of the flow \((Y, T)\) with \( f_Y \circ \pi_Y \) cohomologous to \( f \).

The surjectivity of the mapping \( t \mapsto (1 + g)(t, m) \) for every \( m \in M \) implies that the set \( \{f(\tau, x) : \tau \in T\} \) is dense in \( \mathbb{R} \) for all \( x \) in a dense \( G_\delta \) subset of \( X \).

For other well-known topologies on the hyperspace like the Vietoris topology and the Hausdorff topology with respect to the product metric on \( X \times \mathbb{R} \), the continuity of the mapping \((x, s) \mapsto \mathcal{D}_{T,f}(x, s) \) can be disproved by Example 2.

2. Proof of the structure theorem

Furstenberg’s structure theorem for distal minimal flows shall be our main tool for studying the structure of cocycles.

Definition 8. Let \( X \) and \( Y \) be compact metric spaces, let \( \pi \) be a continuous mapping from \( X \) onto \( Y \), and let \( M \) be a homogeneous compact metric space, that is, for arbitrary points \( m, m' \in M \) there is an isometry of \( M \) mapping \( m \) to \( m' \). Suppose that \( \rho(x, x') \) is a continuous real-valued function defined on the set

\[ R_\pi = \{(x, x') \in X^2 : \pi(x) = \pi(x')\} \]

so that for every \( y \in Y \) the function \( \rho \) is a metric on the fibre \( \pi^{-1}(y) \) with an isometric mapping between \( \pi^{-1}(y) \) and \( M \). Then \( X \) is called an \( M \)-bundle over \( Y \).

Now let \((X, T)\) and \((Y, T) = \pi(X, T)\) be compact metric flows with \( X \) an \( M \)-bundle over \( Y \). If the function \( \rho \) satisfies \( \rho(x, x') = \rho(\tau x, \tau x') \) for all \((x, x') \in R_\pi\) and \( \tau \in T \), then \((X, T)\) is called an isometric extension of \((Y, T)\).

Fact 2.1 (Furstenberg’s structure theorem [9]). Let \((X, T)\) be a distal minimal compact metric flow. Then there exist a countable ordinal \( \eta \) and factors \((X_\xi, T) = \pi_\xi(X, T)\) for each...
ordinal $0 \leq \xi \leq \eta$ with the following properties:

(i) $(X_\eta, T) = (X, T)$ and $(X_0, T)$ is the trivial flow;
(ii) $(X_\xi, T) = \pi_\xi^\eta(X_\zeta, T)$ is a factor of $(X_\zeta, T)$ for all ordinals $0 \leq \xi < \zeta \leq \eta$;
(iii) for every ordinal $0 \leq \xi < \eta$ the flow $(X_{\xi+1}, T)$ is an isometric extension of $(X_{\xi}, T)$;
(iv) for a limit ordinal $0 < \xi \leq \eta$ the flow $(X_\xi, T)$ is the inverse limit of the flows $\{(X_\zeta, T) : 0 \leq \zeta < \xi\}$.

A system $\{(X_\xi, T) : 0 \leq \xi \leq \eta\}$ with the properties above is called a quasi-isometric system or I-system.

**Definition 9.** An I-system $\{(X_\xi, T) : 0 \leq \xi \leq \eta\}$ is called normal if for each ordinal $0 \leq \xi < \eta$ the flow $(X_{\xi+1}, T)$ is the maximal isometric extension of $(X_\xi, T)$ in $(X_\eta, T)$. This I-system gives the minimal ordinal $\eta$ to represent the compact metric flow $(X, T) = (X_\eta, T)$ (cf. [9, Proposition 13.1, Definitions 13.2 and 13.3]).

It will be essential that the fibres of all the isometric extensions are connected, with a possible exception of extension from the trivial flow to the minimal isometric flow $(X_1, T)$. For a normal I-system this property will be ensured by results of the paper [22]. These results require the acting group to be generated by every open neighbourhood of a compact subset, and the group $T$ is even compactly generated.

**Proposition 2.2.** Let $\{(X_\xi, T) : 0 \leq \xi \leq \eta\}$ be a normal I-system with a compactly generated group $T$ acting minimally and distally. Then for all ordinals $1 \leq \xi < \zeta \leq \eta$ the extension from $(X_\xi, T)$ to $(X_\zeta, T)$ has connected fibres.

**Proof.** In the following arguments we shall refer to terminology and results provided in the paper [22]. At first suppose that $1 \leq \xi < \eta$ is not a limit ordinal. Let $S(\pi_{\xi-1}) \subset X^2$ denote the relativized equicontinuous structure relation of the homomorphism $\pi_{\xi-1} : (X, T) \to (X_{\xi-1}, T)$, hence the flow $(X, T)/S(\pi_{\xi-1})$ is the maximal isometric extension $(X_\xi, T)$ of the flow $(X_{\xi-1}, T)$ in $(X, T)$. By [22, Theorem 3.7] the homomorphism $\pi_{\xi} : (X, T) \to (X_\xi, T)$ has connected fibres. For a limit ordinal $1 < \xi < \eta$ and an ordinal $0 \leq \zeta < \xi$, the same argument shows the connectedness of the fibres of $\pi_{\zeta+1} : (X, T) \to (X_{\zeta+1}, T)$. Since

$$(\pi_{\xi})^{-1}(x_\xi) = \bigcap_{0 \leq \zeta < \xi} (\pi_{\zeta+1})^{-1}(\pi_{\xi}^\xi(x_\xi))$$

holds for every $x_\xi \in X_\xi$, the fibre $(\pi_{\xi})^{-1}(x_\xi)$ is connected as the limit of a sequence of connected sets in a compact metric space (cf. [18, p. 170, Theorem 14]). The hypothesis follows, since for all ordinals $1 \leq \xi < \zeta \leq \eta$ the fibres of the homomorphism $\pi_{\xi}^\zeta$ are the images under $\pi_{\xi}$ of the connected fibres of $\pi_{\xi}$.

We shall henceforth assume that $\{(X_\xi, T) : 0 \leq \xi \leq \eta\}$ is a normal I-system with $(X_\eta, T) = (X, T)$. For every ordinal $1 \leq \xi < \eta$, we can define a projection of the cocycles of $(X, T)$ to the cocycles of $(X_\xi, T)$ by families of probability measures, since every distal extension of compact metric flows is a so-called RIM-extension (relatively invariant measure, cf. [10]). For an isometric extension this RIM is unique (cf. [10]), and within the I-system the RIM’s obey to an integral decomposition formula.
FACT 2.3. Let $0 \leq \xi \leq \eta$ be an ordinal. Then there exists a family of probability measures \( \{\mu_{\xi,x}: x \in X_\xi\} \) on \( X \) so that
\[
\mu_{\xi,x}(\pi_\xi^{-1}(x_\xi)) = 1 \quad \text{and} \quad \mu_{\xi,x} \circ \tau^{-1} = \mu_{\xi,\tau x}
\]
for every \( x_\xi \in X_\xi \) and every \( \tau \in T \). The mapping \( x_\xi \mapsto \mu_{\xi,x_\xi} \) is continuous on \( X_\xi \) with respect to the weak-* topology on \( C(X)^* \). Given a continuous function \( \varphi \in C(X) \) and arbitrary ordinals \( 1 \leq \xi < \zeta \leq \eta \), for all \( x_\xi \in X_\xi \) holds
\[
\mu_{\xi,x_\xi}(\varphi) = \int_{X_\xi} \mu_{\zeta,\eta}(\varphi) \, d(\mu_{\xi,x_\xi} \circ \pi_\xi^{-1})(\eta_\zeta).
\] (2.1)

Proof. The proof follows the inductive construction of an invariant measure for \((X,T)\) out of the unique RIM's of the extensions \((X_\xi,T) = \pi_\xi^{-1}(X_{\xi+1},T)\) with \( 0 \leq \xi < \eta \) (cf. [9, Chapter 12]). The generalization to a relatively invariant measure is provided in [24, p. 494], where the induction process is initiated with the family of point measures on \( X_\xi \) instead of the point measure on the trivial space \( X_0 \). Since the \( I \)-system and the RIM's for the isometric extensions remain fixed, the decomposition formula follows from the inductive construction.

For a cocycle \( f(\tau, x) \) of the flow \((X,T)\) and an ordinal \( 1 \leq \xi < \eta \), the RIM \( \{\mu_{\xi,x_\xi}: x_\xi \in X_\xi\} \) defines a continuous function \( f_\xi: T \times X_\xi \to \mathbb{R} \) by
\[
f_\xi(\tau, x_\xi) = \mu_{\xi,x_\xi}(f(\tau, \cdot)) = \int_X f(\tau, x) \, d\mu_{\xi,x_\xi}(x).
\]
The properties of the RIM imply for all \( \tau, \tau' \in T \) and \( x_\xi \in X_\xi \) that
\[
f_\xi(\tau, \tau' x_\xi) + f_\xi(\tau', x_\xi) = \mu_{\xi,\tau' x_\xi}(f(\tau', \cdot)) + \mu_{\xi,x_\xi}(f(\tau', \cdot))
\]
\[
= \mu_{\xi,x_\xi}(f(\tau, \cdot) \circ \tau') + \mu_{\xi,x_\xi}(f(\tau', \cdot)) = f_\xi(\tau \tau', x_\xi),
\]
therefore, \( f_\xi \) is a cocycle of the flow \((X_\xi,T)\). Furthermore, for ordinals \( \xi \) and \( \zeta \) with \( 1 \leq \xi < \zeta < \eta \) and every \( x_\xi \in X_\xi \), the integral of the cocycle \((f_\xi - f_\xi \circ \pi_\xi^{-1})(\tau, x_\xi)\) by the measure \( \mu_{\xi,x_\xi} \circ \pi_\zeta^{-1} \) on \( X_\zeta \) vanishes for every \( \tau \in T \):
\[
\int_{X_\zeta} (f_\xi - f_\xi \circ \pi_\xi^{-1})(\tau, x_\xi) \, d(\mu_{\xi,x_\xi} \circ \pi_\zeta^{-1})(x_\xi)
\]
\[
= \int_{X_\xi} (\mu_{\xi,x_\xi}(f(\tau, \cdot))) \, d(\mu_{\xi,x_\xi} \circ \pi_\zeta^{-1})(x_\xi) - \mu_{\xi,x_\zeta}(f(\tau, \cdot)) = 0.
\]
Since the measure \( \mu_{\xi,x_\xi} \circ \pi_\zeta^{-1} \) is supported by the connected fibre \((\pi_\xi^{-1}(x_\xi)) \) in \( X_\xi \), for every \( \tau \in T \) and every \( x_\xi \in X_\xi \) the function \( x_\xi \mapsto (f_\xi - f_\xi \circ \pi_\xi^{-1})(\tau, x_\xi) \) takes zero value on the fibre \((\pi_\xi^{-1}(x_\xi)) \). This property will be essential, as well as the representation of extensions of distal flows by so-called regular extensions.

FACT 2.4. Let \((X,T)\) be a distal minimal compact metric flow with a factor \((Y,T) = \sigma(X,T)\). Then there exist a distal minimal compact Hausdorff flow \((\tilde{X},T)\) with \((X,T) = \pi(\tilde{X},T)\) as a factor and a Hausdorff topological group \( G \subset \text{Aut}(X,T) \) acting freely on \( \tilde{X} \) (that is, \( g(\tilde{x}) = \tilde{x} \) for some \( \tilde{x} \in \tilde{X} \) implies that \( g = 1_G \)). The group \( G \) acts strictly transitive on the fibres \( \tilde{\sigma}^{-1}(\tilde{\sigma}(\tilde{x})) = \{g(\tilde{x}) : g \in G\} \) of the homomorphism \( \tilde{\sigma} = \sigma \circ \pi \) for every \( \tilde{x} \in \tilde{X} \), and \((X,T)\) is the orbit space of a subgroup \( H \) of \( G \) in \( \tilde{X} \) so that \( \pi \) is the mapping of a point in \( \tilde{X} \) to its \( H \)-orbit (cf. [7, 12.12, 12.13, and 14.26], with a direct proof in [22, Proposition 1.1]).
For an isometric extension \((X,T)\) of \((Y,T)\), the flow \((\tilde{X},\tilde{T})\) is metric and an isometric extension of \((Y,T)\), with a compact metric group \(G\) and a compact subgroup \(H\), hence called a compact metric group extension.

**Remark 7.** The construction above is also called the regularizer of an extension. In [11], it is verified that a compact Hausdorff flow \((\tilde{X},\tilde{T})\) with these properties is metrizable if and only if the extension from \((Y,T)\) to \((X,T)\) is isometric.

Studying the skew product extensions \(\tilde{\tau}_f : X_\xi \times \mathbb{R} \to X_\xi \times \mathbb{R}\) for the ordinals \(0 \leq \xi \leq \eta\) will require the following technical lemma.

**Lemma 2.5.** Let the minimal compact metric flow \((Z,T)\) be an extension of the flow \((Y,T) = \sigma(Z,T)\) and let \(g(\tau,y)\) be a real-valued cocycle of \((Y,T)\). Let \(h(\tau,z)\) be a real-valued cocycle of \((Z,T)\) so that for every \(\tau \in T\) and \(\tau' \in Z\) the image of \(\sigma^{-1}(\sigma(\tau'))\) under the function \(z \mapsto h(\tau,z)\) is connected and includes zero. Suppose that there exist a compact symmetric neighbourhood \(K \subset \mathbb{R}\) of \(0\) and \(\varepsilon > 0\) so that for all \(z \in Z\) and \(\tau \in T\) with \(d_Z(z,\tau z) < \varepsilon\) holds \((g \circ \sigma + h)(\tau,z) \notin K \setminus K^0\). Suppose that there exists a \(\delta > 0\) so that for all \(\tau \in T\) and \(z \in Z\) with \(d_Z(z,\tau z) < \delta\) holds \(d_Z(\tau' \tau z, \tau' z') < \varepsilon\) for every \(\tau' \in \sigma^{-1}(\sigma(z))\). Given \(z \in Z\) and a sequence \(\{\tilde{\tau}_k\} \subset T\) so that \(\tilde{\tau}_k z\) is convergent and \((g \circ \sigma)(\tilde{\tau}_k, z) \to 0\) as \(k \to \infty\), the sequence \(\{(g \circ \sigma)(\tilde{\tau}_k, z)\}_{k \geq 1}\) is bounded. Similarly, for a sequence \(\{\tilde{\tau}_k\} \subset T\) so that \(\tilde{\tau}_k z\) is convergent and \((g \circ \sigma + h)(\tilde{\tau}_k, z) \to 0\) as \(k \to \infty\), the sequence \(\{(g \circ \sigma)(\tilde{\tau}_k, z)\}_{k \geq 1}\) is bounded.

**Proof.** There exists a \(k_0 \geq 1\) so that for all \(k,k' \geq k_0\) holds \(d_Z(\tilde{\tau}_k z, \tilde{\tau}_{k'} z) < \delta\) and

\[
(g \circ \sigma)(\tilde{\tau}_{k'}, z) - (g \circ \sigma)(\tilde{\tau}_k, z) = (g \circ \sigma)(\tilde{\tau}_{k'} \tilde{\tau}_k^{-1}, \tilde{\tau}_k z) \in K^0.
\]

By the choice of \(K\), \(\varepsilon\) and \(\delta\) it follows that \((g \circ \sigma + h)(\tilde{\tau}_{k'} \tilde{\tau}_k^{-1}, z) \notin K \setminus K^0\) for all \(z \in \sigma^{-1}(\sigma(\tilde{\tau}_k z))\). Since the range of \((g \circ \sigma + h)(\tilde{\tau}_{k'} \tilde{\tau}_k^{-1}, z)\) on the fibre \(\sigma^{-1}(\sigma(\tilde{\tau}_k z))\) is connected and intersects \(K^0\), we can conclude that \((g \circ \sigma + h)(\tilde{\tau}_{k'} \tilde{\tau}_k^{-1}, \tilde{\tau}_k z) \in K^0\) for all \(k,k' \geq k_0\). Therefore, the sequence \(\{(g \circ \sigma + h)(\tilde{\tau}_k, z)\}_{k \geq 1}\) is bounded.

Provided a sequence \(\{\tilde{\tau}_k\} \subset T\) with convergent \(\tilde{\tau}_k z\) and \((g \circ \sigma + h)(\tilde{\tau}_k, z) \to 0\), there exists an integer \(k_0 \geq 1\) so that for all \(k,k' \geq k_0\) holds \(d_Z(\tilde{\tau}_k z, \tilde{\tau}_{k'} z) < \delta\) and \((g \circ \sigma + h)(\tilde{\tau}_{k'} \tilde{\tau}_k^{-1}, \tilde{\tau}_k z) \in K^0\). We can conclude as above that \((g \circ \sigma + h)(\tilde{\tau}_{k'} \tilde{\tau}_k^{-1}, z) \in K^0\) for all \(k,k' \geq k_0\) and \(z \in \sigma^{-1}(\sigma(\tilde{\tau}_k z))\). Since \(h(\tilde{\tau}_{k'} \tilde{\tau}_k^{-1}, z) = 0\) for some \(z \in \sigma^{-1}(\sigma(\tilde{\tau}_k z))\), the sequence \(\{(g \circ \sigma)(\tilde{\tau}_k, z)\}_{k \geq 1}\) is bounded.

At first, the step from an ordinal to its successor by an isometric extension shall be considered. The ‘local’ behaviour within the fibres of a compact group extension is similar to a skew product extension by a compact metric group, even if the global structure might be different since it does not necessarily split into a product.

**Lemma 2.6.** Let \(\gamma\) be an ordinal with \(1 \leq \gamma < \eta\). If there exists a sequence \(\{(\tau_k,x_k)\}_{k \geq 1} \subset T \times X_{\gamma+1}\) with \(d_{\gamma+1}(x_k,\tau_k x_k) \to 0\) so that \((f_\gamma \circ \pi_{\gamma+1})(\tau_k, x_k) \to 0\) and \(f_{\gamma+1}(\tau_k, x_k) \to 0\) as \(k \to \infty\) (or equivalently \((f_{\gamma+1} \circ \pi_{\gamma+1})(\tau_k, x_k) \to 0\) and \(f_{\gamma}(\tau_k, x_k) \to 0\)), then the skew product \(\tilde{\tau}_{f_{\gamma+1}}\) is necessarily point transitive. Therefore, if \(f_\gamma(\tau,x_\gamma)\) is transient, then \(f_{\gamma+1}(\tau,x_{\gamma+1})\) is either transient or the skew product \(\tilde{\tau}_{f_{\gamma+1}}\) is point transitive.

**Proof.** Suppose that \(\tilde{\tau}_{f_{\gamma+1}}\) is not point transitive and let \(G \subset \text{Aut}(Z,T)\) define a compact metric group extension of \((X_\gamma,T)\) with \((X_{\gamma+1},T) = \pi(Z,T)\). Then the skew product extension
of the flow \((Z,T)\) is also not point transitive, and Lemma 1.5 provides \(K \subset \mathbb{R}\) and \(\varepsilon > 0\). Since \(G\) acts uniformly equicontinuous, there exists a \(\delta > 0\) so that for all \(\tau \in T\) and \(z \in Z\) with \(d_Z(z, \tau z) < \delta\) holds \(d_Z(k(z), k(\tau z)) = d_Z(\varepsilon \delta, z, \tau) < \varepsilon\) for all \(k \in K\). For every \(z \in Z\), the \(G\)-orbit of \(z\) is all of the fibre \((\pi_\gamma^+ \circ \pi)^{-1}((\pi_\gamma^+ \circ \pi)(z))\). Since the \(\pi_\gamma^+\)-fibres are connected, for every \(\tau \in T\) and \(z' \in Z\) the range of \((f_{\tau + 1} - f_\gamma \circ \pi_\gamma^+)(\tau, \pi(z))\) on the fibre \((\pi_\gamma^+ \circ \pi)^{-1}((\pi_\gamma^+ \circ \pi)(z'))\) is connected and contains zero. Hence, Lemma 2.5 applies with \((Y, T) = (X, T), \sigma = \pi_\gamma^+ \circ \pi, g = f_\gamma\) and \(h(\tau, z) = (f_{\tau + 1} - f_\gamma \circ \pi_\gamma^+)(\tau, \pi(z))\).

However, given the sequence \((\tau_k, x_k)\) for every \(\tau \in T\), the hypothesis \(1.3\) provides a point \(\bar{x} \in X_{\gamma + 1}\). a point transitive skew product extension \(\tilde{\tau}_{\gamma + 1}\) recurrent. Let \(x' \in X_{\gamma + 1}\) be so that \((x', 0)\) is \(\tilde{\tau}_{\gamma + 1}\)-recurrent. Since \((x', 0)\) cannot be \(\tilde{\tau}_{\gamma + 1}\)-recurrent, there exist a neighbourhood \(V \subset X_{\gamma + 1} \times \mathbb{R}\) of \((x', 0)\) and a replete semigroup \(P \subset T\) such that \(\tilde{\tau}_{\gamma + 1}(x', 0) \notin V\) for every \(\tau \in P\). Given an arbitrary compact set \(C \subset T\), by \([13, Theorem 6.32]\) there exists a replete semigroup \(Q \subset P \setminus C\). Since \((x', 0)\) is \(\tilde{\tau}_{\gamma + 1}\)-recurrent, we can inductively construct a sequence \((\tau_k)_{k \geq 1} \subset P\) with \(\tau_k x' \rightarrow x', f_{\tau + 1}(\tau_k, x') \rightarrow 0\), and \((f_\gamma \circ \pi_\gamma^+)(\tau_k, x') \rightarrow 0\) as \(k \rightarrow \infty\). The point transitivity of \(\tilde{\tau}_{\gamma + 1}\) follows from the preceding statement.

Furthermore, we shall study the case of transfinite induction to a limit ordinal. The arguments are quite similar; however, with an approximation of a limit ordinal instead of an isometric group extension.

**Lemma 2.7.** Suppose that \(\gamma\) is a limit ordinal with \(1 < \gamma \leq \eta\).

(i) If for every ordinal \(1 < \alpha < \gamma\) there exists an ordinal \(\alpha \leq \xi < \gamma\) so that \(f_\xi(\tau, x_\xi)\) has a point transitive skew product extension, then \(f_\gamma(\tau, x_\gamma)\) has a point transitive skew product extension.

(ii) If there exist an ordinal \(1 \leq \alpha < \gamma\) and a sequence \((\tau_k, x_k)\) with \(d_\gamma(\tau_k, x_k) \rightarrow 0\) so that \((f_\xi \circ \pi_\xi^+)(\tau_k, x_k) \rightarrow 0\) for every \(\alpha \leq \xi < \gamma\) and \(f_\gamma(\tau_k, x_k) \rightarrow 0\) as \(k \rightarrow \infty\) (or equivalently \((f_\xi \circ \pi_\xi^+)(\tau_k, x_k) \rightarrow 0\) for every \(\alpha \leq \xi < \gamma\) and \(f_\gamma(\tau_k, x_k) \rightarrow 0\)), then \(\tau_\gamma\) is necessarily point transitive.

(iii) If there exists an ordinal \(1 \leq \alpha < \gamma\) so that for all \(\alpha \leq \xi < \gamma\) the cocycle \(f_\xi(\tau, x_\xi)\) is transient, then \(f_\gamma(\tau, x_\gamma)\) is either transient or its skew product extension is point transitive.

**Proof.** Suppose that the skew product of \(\tau_\gamma\) on \(X_\gamma \times \mathbb{R}\) is not point transitive, and let \(K \subset \mathbb{R}\) and \(\varepsilon > 0\) be provided by Lemma 1.5. Since \(\gamma\) is a limit ordinal and \((X_\gamma, T)\) is the inverse limit of the flows \((\{X_\xi, T\} : 0 \leq \xi < \gamma)\), we can choose an ordinal \(\zeta < \gamma\) so that for all \(x, x' \in X_\gamma\) with \(\pi_\gamma^+ (x) = \pi_\gamma^+ (x')\) it holds that \(d_\gamma(x, x') < \varepsilon/3\). If we put \(\delta = \varepsilon/3\), then \(d_\gamma(x', x') < \delta\) for \(x' \in X_\gamma\) and \(\tau \in T\) implies that \(d_\gamma(x, \tau x') < \varepsilon\) for all \(x \in (\pi_\gamma^+)^{-1}(\pi_\gamma^+ (x'))\). These conditions remain valid even if the ordinal \(\zeta\) will be increased later. Since the \(\pi_\gamma^+\)-fibres are connected, for every \(\tau \in T\) and \(x'_\gamma \in X_\gamma\), the range of \((f_\gamma - f_\xi \circ \pi_\xi^+)(\tau, x_\gamma)\) on the \(\pi_\gamma^+\)-fibre of \(x'_\gamma\) is connected and contains 0.

Under the hypothesis (i), we can choose \((\tau, x_\xi)\) so that \(f_\xi(\tau, x_\xi) \in K \setminus K^0\) and \(d_\gamma(x'_\gamma, \tau x'_\gamma) < \delta\) for some \(x'_\gamma \in (\pi_\gamma^+)^{-1}(x_\xi)\). Thus, \(d_\gamma(x_\gamma, \tau x_\gamma) < \varepsilon\) holds for all \(x_\gamma \in (\pi_\gamma^+)^{-1}(x_\xi)\), and for \(x_\gamma\) with \((f_\gamma - f_\xi \circ \pi_\xi^+)(\tau, x_\gamma) = 0\) this contradicts \(f_\gamma(\tau, x_\gamma) \notin K \setminus K^0\). Thus, assertion (i) is verified.
We apply Lemma 2.5 with \((Z, T) = (X_\alpha, T), (Y, T) = (X_\gamma, T), \sigma = \pi_\gamma^\beta, h = (f_\gamma - f_\xi \circ \pi_\gamma^\beta),\) and \(g = f_\xi.\) However, given the sequence \(\{\tau_k(x_\xi)\}_{k \geq 1} \subset T \times X_\gamma\) in hypothesis (ii), Lemma 1.3 provides a point \(\bar{x} \in X_\gamma\) and a sequence \(\{\bar{\tau}_k\} \subset T\) so that
\[
(f_\xi \circ \pi_\gamma^\beta, f_\gamma - f_\xi \circ \pi_\gamma^\beta)(\bar{\tau}_k, \bar{x}) = (g \circ \sigma, h)(\bar{\tau}_k, \bar{x}) \longrightarrow (0, \infty) \quad \text{(or } \infty, 0)\]
and \(\bar{\tau}_k \bar{x} \longrightarrow \bar{z}\) as \(k \rightarrow \infty.\) This is a contradiction to Lemma 2.5 and verifies (ii).

Now suppose that \(f_\xi(\tau, x_\xi)\) is transient and \(f_\gamma(\tau, x_\gamma)\) is recurrent, and choose \(x' \in X_\gamma\) so that \((x', 0)\) is \(\tilde{\tau}_{f_\gamma}\)-recurrent. Since \((x', 0)\) is not \(\tilde{\tau}_{f_\gamma \circ \pi_\gamma^\beta}\)-recurrent, there exist a neighbourhood \(V \subset X_\gamma \times \mathbb{R}\) of \((x', 0)\) and a replete semigroup \(P \subset T\) with \(\tilde{\tau}_{f_\gamma \circ \pi_\gamma^\beta}(x', 0) \notin V\) for every \(\tau \in P.\)

By induction, there exists a sequence \(\{\bar{\tau}_k\}_{k \geq 1} \subset P\) with \((\bar{\tau}_k)_{f_\gamma}(x', 0) \rightarrow (x', 0)\) as \(k \rightarrow \infty,\) and by Lemma 1.3 there exist a point \(\bar{x} \in X_\gamma\) and a sequence \(\{\bar{\tau}_k\} \subset T\) so that
\[
(f_\gamma, f_\xi \circ \pi_\gamma^\beta)(\bar{\tau}_k, \bar{x}) = (g \circ \sigma + h, g \circ \sigma)(\bar{\tau}_k, \bar{x}) \longrightarrow (0, \infty)
\]
and \(\bar{\tau}_k \bar{x} \longrightarrow \bar{z}\). This contradiction to Lemma 2.5 verifies the statement (iii).

**Proposition 2.8.** If the real-valued cocycle \(f(\tau, x)\) is topologically recurrent apart from a coboundary, then there exists a maximal ordinal \(1 \leq \alpha \leq \eta\) so that the skew product extension \(\tilde{\tau}_{f_\alpha}\) is point transitive on \(X_\alpha \times \mathbb{R}\). The cocycle \((f - f_\alpha \circ \pi_\alpha)(\tau, x)\) is relatively trivial with respect to \((f_\alpha \circ \pi_\alpha)(\tau, x)\).

**Proof.** Let us first suppose that the cocycle \(f_\xi(\tau, x_\xi)\) is recurrent for every ordinal \(1 \leq \xi < \eta,\) and let \(O = \{1 \leq \xi \leq \eta : f_\xi(\tau, x_\xi)\) is not a coboundary\}. This set is non-empty since \(f_\eta(\tau, x)\) is not a coboundary, and let \(\beta\) be its minimal element. If \(\beta = 1,\) then by Proposition 1.6 the recurrent skew product extension \(\tilde{\tau}_{f_\beta}\) of the isometric flow \((X_1, T)\) is point transitive. If \(\beta > 1,\) then Fact 1.4 provides a sequence \(\{(\tau_k, x_k)\}_{k \geq 1} \subset T \times X_\beta\) with \(d_\beta(x_k, \tau_k x_k) \rightarrow 0\) and \(f_\beta(\tau_k, x_k) \rightarrow \infty.\) For all \(1 \leq \zeta < \beta\) it holds that \((f_\beta \circ \pi_\zeta^\beta)(\tau_k, x_k) \rightarrow 0,\) and by Lemmas 2.6 and 2.7(ii) \(\tilde{\tau}_{f_\beta}\) is point transitive.

If \(f_\xi(\tau, x_\xi)\) is transient for an ordinal \(1 \leq \xi < \eta,\) then let \(\beta\) be the minimal element of the set \(O = \{\xi < \zeta \leq \eta : f_\xi(\tau, x_\xi)\) is topologically recurrent\}. This set is non-empty since \(f_\eta(\tau, x_\eta)\) is topologically recurrent, and it follows from Lemmas 2.6 and 2.7(iii) that \(\tilde{\tau}_{f_\beta}\) is even point transitive.

Now let \(O = \{1 \leq \xi \leq \eta : \tilde{\tau}_{f_\xi}\) is not point transitive for all \(\xi \leq \zeta \leq \eta\}\). If \(O\) is empty, then \(\tilde{\tau}_{f_\xi}\) is point transitive and \(\alpha = \eta.\) Otherwise, the set \(O\) has a minimal element \(\gamma > 1\) since \(\tilde{\tau}_{f_\gamma}\) is point transitive for some \(1 \leq \beta \leq \eta.\) Since \(\gamma\) cannot be a limit ordinal by Lemma 2.7(i), there exists a maximal ordinal \(\alpha \geq 1\) with point transitive \(\tilde{\tau}_{f_\alpha}.\) Let \(\{(\tau_k, x_k)\}_{k \geq 1} \subset T \times X\) be a sequence with \(d(x_k, \tau_k x_k) \rightarrow 0\) and \((f_\alpha \circ \pi_\alpha)(\tau_k, x_k) \rightarrow 0.\) Since the ordinal \(\alpha\) is maximal, it follows by transfinite induction with Lemmas 2.6 and 2.7(ii) that \((f_\alpha \circ \pi_\xi)(\tau_k, x_k) \rightarrow 0\) for every \(\alpha \leq \xi \leq \eta.\)

After the flow \((X_\alpha, T)\) with a point transitive skew product extension \(\tilde{\tau}_{f_\alpha}\) has been identified, we shall study the extension from \((X_\alpha, T)\) to \((X, T).\) There might be infinitely many isometric extensions in between, and therefore this extension is in general a distal extension. Since our construction will use the regularizer of this extension, it is necessary to leave the category of compact metric flows for the category of compact Hausdorff flows during the following construction (cf. Remark 7). However, the flow which will be constructed by means of the regularizer will be metric as a factor of the compact metric flow \((X, T).\)

**Proposition 2.9.** There exists a factor \((Y, T) = (X_\alpha \times M, T) = \pi_\gamma(X, T)\) which is the Rokhlin extension of \((X_\alpha, T) = \rho_\alpha(Y, T)\) by a distal minimal compact metric \(\mathbb{R}\)-flow.
(\mathcal{M}, \{\phi^t : t \in \mathbb{R}\}) and the cocycle \(f_\alpha(\tau, x_\alpha)\) so that for every \(x \in X\) holds
\[
\pi_Y^{-1}(\pi_Y(x)) \times \{0\} = \mathcal{D}_{T,f_\alpha \circ \pi_\alpha}(x, 0) \cap (\pi_\alpha^{-1}(\pi_\alpha(x)) \times \{0\}).
\] (2.2)

The \(\mathbb{R}\)-flow \(\{\phi^t : t \in \mathbb{R}\} \subset \text{Aut}(Y, T)\) defined by \(\psi^t(x_\alpha, m) = (x_\alpha, \phi^t(m))\) for \((x_\alpha, m) \in Y = X_\alpha \times M\) fulfils for every \(y \in Y\) and every \(t \in \mathbb{R}\) that
\[
\mathcal{O}_{T,f_\alpha \circ \pi_\alpha}(y, 0) \cap (\rho_\alpha^{-1}(\rho_\alpha(y)) \times \{t\}) \subset (\psi^t(y, t)) \cap (\psi^t(y, t)) \tag{2.3}
\]
with coincidence of these sets if \((\rho_\alpha(y), 0) \in X_\alpha \times \mathbb{R}\) is transitive for \(\tau_{f_\alpha}\).

**Proof.** We shall construct a factor \((Y, T)\) of \((X, T)\) and a flow \(\{\phi^t : t \in \mathbb{R}\} \subset \text{Aut}(Y, T)\), and then we shall represent \((Y, T)\) as a Rokhlin extension of \((X_\alpha, T)\). Let \((\tilde{X}, T)\) be a distal minimal compact Hausdorff flow with \((X, T) = \pi(\tilde{X}, T)\) and a Hausdorff topological group \(G \subset \text{Aut}(\tilde{X}, T)\) acting freely on the fibres of \(\pi_\alpha \circ \pi\) so that \((X, T)\) is the \(H\)-orbit space of a subgroup \(H \subset G\) (cf. Fact 2.4). For an arbitrary point \(\tilde{z} \in \tilde{X}\) and \(t \in \mathbb{R}\), we define a closed subset of \(G\) by
\[
G_{\tilde{z},t} = \{g \in G : (\pi(g(\tilde{z})), t) \in \mathcal{D}_{T,f_\alpha \circ \pi_\alpha}(\pi(\tilde{z}), 0)\}.
\] (2.4)

The mapping \(\pi\) is open as a homomorphism of distal minimal compact flows, and hence for every \(g \in G_{\tilde{z},t}\) there exist nets \(\{\tilde{z}_i\}_{i \in I} \subset \tilde{X}\) and \(\{\tau_i\}_{i \in I} \subset T\) with \(\tilde{z}_i \to \tilde{z}, \tau_i \pi(\tilde{z}_i) \to \pi(g(\tilde{z}))\), and \(f_\alpha(\tau_i, \pi_\alpha(\pi(\tilde{z}_i))) \to t\). Since the cocycle \((f_\alpha \circ \pi_\alpha)(\tau, x_\alpha)\) is constant on the fibres of \(\pi_\alpha\) and \(T\) is Abelian, it follows for every fixed \(\tau \in T\) that
\[
\tau_i \pi(\tau \tilde{z}_i) = \tau \tau_i \pi(\tilde{z}_i) \longrightarrow \pi(\tau g(\tilde{z})) = \pi(\tau g(\tilde{z}))
\]
and by the cocycle identity
\[
f_\alpha(\tau_i, \pi_\alpha(\tau \pi(\tilde{z}_i))) = f_\alpha(\tau_i, \pi_\alpha(\pi(\tilde{z}_i))) - f_\alpha(\tau, \pi_\alpha(\pi(\tilde{z}_i))) + f_\alpha(\tau, \pi_\alpha(\pi(\tau \tilde{z}_i))) \longrightarrow t.
\]
By the density of the \(T\)-orbit of \(\tilde{z}\) and a diagonalization of nets, for every \(\tilde{x} \in \tilde{X}\) there exist nets \(\{\tilde{x}_i\}_{i \in I} \subset \tilde{X}\) and \(\{\tau_i'\}_{i \in I} \subset T\) with \(\tilde{x}_i \to \tilde{x}, \tau_i' \pi(\tilde{x}_i) \to \pi(g(\tilde{x}))\), and \(f_\alpha(\tau_i, \pi_\alpha(\pi(\tilde{x}_i))) \to t\).

Therefore,
\[
(\pi(g(\tilde{x})), t) \in \mathcal{D}_{T,f_\alpha \circ \pi_\alpha}(\pi(\tilde{x}), 0)
\]
so that \(g \in G_{\tilde{x},t} = G_{\tilde{z},t} = G_t\). By symmetry it now follows that \(G_{-t} = (G_t)^{-1}\).

Then we fix a point \(x' \in X\) according to Fact 1.1 so that
\[
\mathcal{O}_{T,f_\alpha}(\pi_\alpha(x'), 0) = X_\alpha \times \mathbb{R}
\]
and \(\mathcal{D}_{T,f_\alpha \circ \pi_\alpha}(x', 0) = \mathcal{O}_{T,f_\alpha \circ \pi_\alpha}(x', 0)\).

The set \(G_t\) is non-empty for every \(t \in \mathbb{R}\), since \(\mathcal{O}_{T,f_\alpha}(\pi_\alpha(x'), 0) = X_\alpha \times \mathbb{R}\) and the compactness of \(X\) ensure that
\[
\mathcal{O}_{T,f_\alpha \circ \pi_\alpha}(x', 0) \cap (\pi_\alpha^{-1}(\pi_\alpha(x')) \times \{t\}) \neq \emptyset.
\]
For arbitrary \(t, t' \in \mathbb{R}\) and \(g \in G_t, g' \in G_{t'}\), we select \(\hat{x}, \hat{z} \in \tilde{X}\) so that \(\pi(\hat{x}) = x'\) and \(\hat{x} = g'(\tilde{z})\).

Then we have
\[
(x', t') = (\pi(g'(\tilde{z})), t') \in \mathcal{D}_{T,f_\alpha \circ \pi_\alpha}(\pi(\tilde{x}), 0),
\]
and for \(\tilde{y} = g(\tilde{x}) = gg'(\tilde{z})\) it holds that \((\pi(\tilde{y}), t) \in \mathcal{O}_{T,f_\alpha \circ \pi_\alpha}(x', 0) = \mathcal{D}_{T,f_\alpha \circ \pi_\alpha}(x', 0)\). Remark 1 implies that \((\pi(\tilde{y}), t + t') \in \mathcal{D}_{T,f_\alpha \circ \pi_\alpha}(\pi(\tilde{z}), 0)\) and thus \(gg' \in G_{t+t'}\). Hence \(G_t G_{t'} \subset G_{t+t'}\) holds for all \(t, t' \in \mathbb{R}\), and from \(G_{-t} = (G_t)^{-1}\) it follows that \((G_t)^{-1} G_{t+t'} = G_{-t} G_{t+t'} \subset G_{t'}\) so that \(G_t G_{t'} = G_{t+t'}\). Thus, the Hausdorff topological group
\[
\tilde{G} = \bigcup_{t \in \mathbb{R}} G_t
\]
has the closed set \(G_0 \supset H\) as a normal subgroup so that \(G_t\) is a \(G_0\)-coset in \(\tilde{G}\) for every \(t \in \mathbb{R}\). Moreover, the mapping \(t \mapsto G_t\) is a group homomorphism from \(\mathbb{R}\) into \(\tilde{G}/G_0\). The group
$G_0$ is not necessarily compact, however, its orbit space on $\tilde{X}$ defines a partition into sets invariant under $H \subset G_0$. Hence, this is also a partition of $X$, and the equivalence relation $R_Y$ of this partition of $X$ is $T$-invariant since $G_0 \subset \text{Aut}(X,T)$. Moreover, $R_Y$ is closed in $X$ squared, since definition (2.4) implies that $(x,x') \in R_Y$ if and only if

$$(x',0) \in \mathcal{D}_{T,f_0,\alpha}(x,0) \cap (\pi^{-1}_\alpha(\pi(x)) \times \{0\}).$$

The factor $(Y,T) = \pi_Y(X,T)$ defined by the $T$-invariant closed equivalence relation $R_Y$ is an extension of $(X_\alpha,T) = \rho_\alpha(Y,T)$, and equality (2.2) follows. The $\mathbb{R}$-action $\{\varphi^t : t \in \mathbb{R}\} \subset \text{Aut}(Y,T)$ is well defined for every $y \in Y$ and $t \in \mathbb{R}$ by

$$\varphi^t(y) = G_t((\pi_Y \circ \pi^{-1})(y)) = G_t(\{\tilde{x} \in \tilde{X} : G_0(\tilde{x}) = y\}).$$

Let $\{(t_k, y_k)\}_{k \geq 1} \subset \mathbb{R} \times Y$ be a sequence with $(t_k, y_k) \to (t,y)$, then $\varphi^{t_k}(y_k) = G_0 g_k(\tilde{x}_k)$ for a sequence $\{\tilde{x}_k\}_{k \geq 1} \subset \tilde{X}$ with $\pi_Y \circ \pi(\tilde{x}_k) = y_k$ and $g_k \in G_k$. We can assume that $\tilde{x}_k \to \tilde{x}$ and $g_k(\tilde{x}_k) \to z$ so that $(\pi(\tilde{z}), t) \in \mathcal{D}_{T,f_0,\alpha}(\pi(\tilde{x}), 0)$ and $z = g_t(\tilde{x})$ for some $g_t \in G_t$. From $\pi_Y \circ \pi(\tilde{x}) = y$ and $\varphi^{t_k}(y_k) = \pi_Y \circ \pi(g_k(\tilde{x}_k)) \to \pi_Y \circ \pi(\tilde{x}) = \varphi^t(y)$ follows the continuity of the action $\{\varphi^t : t \in \mathbb{R}\}$ on $Y$.

We turn to the inclusion (2.3). Suppose that $(y_t, t) \in \tilde{\mathcal{O}}_{T,f_0,\alpha}(y,0) \cap \rho^{-1}_\alpha(x,0) \times \{t\}$ for some $x \in X_\alpha$ and $t \in \pi^{-1}(y)$. By the compactness of $X$ there exist points $x_t \in \tilde{\pi}^{-1}(y_t) \subset \rho^{-1}_\alpha(x_t)$ so that $(x_t, t) \in \tilde{\mathcal{O}}_{T,f_0,\alpha}(x,0)$, and therefore $(x_t, 0) \in \mathcal{D}_{T,f_0,\alpha}(x,0)$. The equality (2.2) implies that $y_t = \pi_Y(x_t) = \pi_Y(x_2) = y_2$, and thus for every $y \in Y$ and $t \in \mathbb{R}$ holds

$$\text{card}\{\tilde{\mathcal{O}}_{T,f_0,\alpha}(y,0) \cap \rho^{-1}_\alpha(\rho_\alpha(y)) \times \{t\}\} \leq 1.$$ (2.5)

Moreover, for $x = \rho_\alpha(y)$ we can conclude that $x_t = \pi(g_t(\tilde{x}))$ with $g_t \in G_t$ and $\tilde{x} \in \pi^{-1}(x) \subset \tilde{X}$. Hence, $y_t = \pi_Y(x_t) = \varphi^t(y)$ and inclusion (2.3) is verified. If the point $(\rho_\alpha(y), 0) \in X_\alpha \times \mathbb{R}$ is $\tilde{\tau}_{f_0}$-transitive, then the cardinality in (2.5) is equal to 1 for every $y \in Y$ and $t \in \mathbb{R}$, and for $y' \in \rho^{-1}_\alpha(\rho_\alpha(y))$ and $x \in X_\alpha$ it holds that $\tilde{\mathcal{O}}_{T,f_0,\alpha}(y',0) \cap \rho^{-1}_\alpha(x,0) \times \{0\} = \{(y_0,0)\}$. We fix a point $\tilde{x}_0 \in X_\alpha$ with $\tilde{\tau}_{f_0}$-transitive $(\tilde{x}_0,0)$. If $(y_2, 0) \in \mathcal{D}_{T,f_0,\alpha}(y',0) \cap \rho^{-1}_\alpha(x,0) \times \{0\}$, then Remark 1 implies that $(y_2, 0) \in \mathcal{O}_{T,f_0,\alpha}(y,0)$, and as above it follows that $y_2 = y_1$. Hence,

$$\tilde{\mathcal{O}}_{T,f_0,\alpha}(y',0) \cap \rho^{-1}_\alpha(x,0) \times \{0\} = \mathcal{D}_{T,f_0,\alpha}(y',0) \cap \rho^{-1}_\alpha(x,0) \times \{0\}.$$ (2.6)

holds for every $y' \in \rho^{-1}_\alpha(\tilde{x}_0)$ and $x \in X_\alpha$. For distinct points $y', y'' \in \rho^{-1}_\alpha(\tilde{x}_0)$, we can verify that

$$\tilde{\mathcal{O}}_{T,f_0,\alpha}(y',0) \cap \tilde{\mathcal{O}}_{T,f_0,\alpha}(y'',0) \cap \rho^{-1}_\alpha(x,0) \times \{0\} = \emptyset.$$

Indeed, given a point $\tilde{y}$ in this intersection and an arbitrary sequence $\{\tau_k\}_{k \geq 1} \subset T$ with $\tau_k \tilde{x}_0 \to \rho_\alpha(\tilde{y})$ and $f_\alpha(\tau_k, \tilde{x}_0) \to 0$, it follows from equality (2.5) that $d_\mathcal{Y}(\tau_k y', \tau_k y'') \to 0$, in contradiction to the distality of $(Y,T)$. Hence the mapping $\iota : X_\alpha \times \rho^{-1}_\alpha(\tilde{x}_0) \to \mathcal{Y}$

$$(x, y') \mapsto \rho^{-1}_\alpha(x) \cap \{y \in Y : (y,0) \in \tilde{\mathcal{O}}_{T,f_0,\alpha}(y',0)\}$$

is well-defined, one-to-one and by equality (2.6) it is also continuous. For a dense set of points $\tilde{y} \in \mathcal{Y}$ holds the $\tilde{\tau}_{f_0}$-transitivity of $(\rho_\alpha(\tilde{y}), 0)$, since $\rho_\alpha$ is open. We can conclude for every $y \in Y$ that $\mathcal{D}_{T,f_0,\alpha}(y,0) \cap \rho^{-1}_\alpha(\tilde{x}_0) \times \{0\} \neq \emptyset$, and thus for some $y' \in \rho^{-1}_\alpha(\tilde{x}_0)$

$$\mathcal{D}_{T,f_0,\alpha}(y',0) \cap \{y\} \times \{0\} \neq \emptyset = \tilde{\mathcal{O}}_{T,f_0,\alpha}(y',0) \cap \{y\} \times \{0\} \neq \emptyset,$$

Hence $\iota$ is onto and by compactness $Y$ and $X_\alpha \times \rho^{-1}_\alpha(\tilde{x}_0)$ are homeomorphic.

Let $\{\varphi^t : t \in \mathbb{R}\}$ be the restriction of $\{\varphi^t : t \in \mathbb{R}\}$-invariant compact metric space $M = \rho^{-1}_\alpha(\tilde{x}_0)$. For every $y' \in M$ and $\tau \in T$ holds

$$\mathcal{O}_{T,f_0,\alpha}(y',0) \cap \rho^{-1}_\alpha(\tilde{x}_0) \times \{0\} = \mathcal{O}_{T,f_0,\alpha}(y',0) \cap \{y\} \times \{0\} \neq \emptyset,$$

and

$$\tilde{\tau}_{f_0,\alpha}(\varphi^{-1}_f(\tau, \tilde{x}_0))(y') \in \rho^{-1}_\alpha(\tau \tilde{x}_0) \cap \{y \in Y : (y,0) \in \tilde{\mathcal{O}}_{T,f_0,\alpha}(y',0)\}.$$

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Therefore $\tau \phi^\tau Z(x, \tau x) (y') = \iota (\tau x, y')$ and $\tau y = \iota (\tau \phi, \tau x)$ for every $y \in M$ and $\tau \in T$. The minimality of $(Y, T)$ implies that $(X, M, T)$ and $(Y, T)$ are isomorphic via $\iota$. Moreover, for the mapping $\psi^\tau (x, \tau x, m) = (x, \psi^\tau (m))$ with $\psi \in \text{Aut}(X, M, T)$ and every $m \in M = \rho^{-1}(\bar{x})$ and $\tau \in \mathbb{R}$ holds

$$\varphi^\tau (\iota (\tau \phi, \tau x, m)) = \varphi^\tau (\tau x) = \iota (\tau \phi, \tau x, m))$$

By the minimality of $(X, M, T)$ it follows that $\psi^\tau = \rho^{-1} \circ \varphi^\tau \circ \iota$ for every $\tau \in \mathbb{R}$. The flow $(M, \{\phi^\tau : \tau \in \mathbb{R}\})$ is minimal and distal, since a non-transitive point $m' \in M$ and a proximal pair $(m', m'') \in M^2$, respectively, give rise to a non-transitive point $(x, \tau x, m') \in Y$ and a proximal pair $((x, \tau x, m'), (x, \tau x, m'')) \in Y^2$, respectively. \hfill \Box

It should be mentioned that an ordinal $\xi = \eta$ with $(Y, T) = (X, \xi, T)$ does not necessarily exist. Therefore, we shall define a cocycle $f_Y : T \times Y \to \mathbb{R}$ independently of the cocycles $f_\xi (\tau, x)$, and it will turn out that $(f_Y \circ \pi_Y) (\tau, x)$ can be chosen topologically cohomologous to $f$.

**Proposition 2.10.** There exists a topological cocycle $f_Y (\tau, y)$ of the flow $(Y, T)$ so that $(f_Y \circ \pi_Y) (\tau, x)$ is topologically cohomologous to $f(\tau, x)$ and $f_Y (\tau, y)$ is relatively trivial with respect to $(f, \rho)(\tau, y)$.

We shall prove another technical lemma first.

**Lemma 2.11.** Let $(Z, T)$ be a distal minimal compact metric flow which extends $(X, T) = \sigma_{\alpha}(Z, T)$, and let $G \subset \text{Aut}(Z, T)$ be a Hausdorff topological group preserving the fibres of $\sigma_{\alpha}$. Suppose that there exists a continuous group homomorphism $\varphi : G \to \mathbb{R}$ so that for every $g \in G$ and every $z \in Z$ it holds that

$$(g(z), \varphi(g)) \in D_{T, f_{\alpha}, \sigma_{\alpha}(z), 0).$$

Furthermore, suppose that $h(\tau, z)$ is a real-valued cocycle of $(Z, T)$ which is relatively trivial with respect to $(f_{\alpha} \circ \sigma_{\alpha})(\tau, z)$. Then there exists a continuous cocycle $h(\tau, g(z), z)$ of the flow $(Z, T \times G)$ with the action $\{g \circ \tau : (\tau, g) \in T \times G\}$ so that $h(\tau, z) = h(\tau, 1_G, z)$ holds for every $(\tau, z) \in T \times Z$ and the mapping $\tau \to h$ is continuous. For $z \in Z$, $y \in G$, and a sequence $\{(\tau_k, z_k)\}_{k \geq 1} \subset T \times Z$ with $z_k \to z$, $\tau_k z_k \to g(z)$, and $(f_{\alpha} \circ \rho_{\alpha})(\tau_k, z_k) \to \varphi(g)$ holds

$$h(\tau_k, z_k) \to h((1_G, g)(z), z) \quad \text{as} \quad k \to \infty. \quad (2.7)$$

**Proof.** We put $F = (f_{\alpha} \circ \sigma_{\alpha}, h) : T \times Z \to \mathbb{R}^2$ and fix a point $\bar{z} \in Z$ so that $\bar{h} = F(\bar{z}, 0, 0)$ and $D_{T,F}(\bar{z}, 0, 0)$ coincide in $Z \times (\mathbb{R}^\infty)^2$ (cf. Fact 1.1). For every $g \in G$, we fix a sequence $\{\tau_k^g\}_{k \geq 1} \subset T$ with $(\tau_k^g)_{f_{\alpha} \circ \sigma_{\alpha}(z), 0} \to (g(z), \varphi(g))$ as $k \to \infty$, with $\{\tau_k^1_G = 1_T\}_{k \geq 1}$. Since $g \in \text{Aut}(Z, T)$ and $f_{\alpha} \circ \sigma_{\alpha} \circ g = f_{\alpha} \circ \sigma_{\alpha}$, we can conclude for every $t \in T$ that $\tau_k^g t \bar{z} \to t g(z)$ as $k \to \infty$ as well as

$$(f_{\alpha} \circ \sigma_{\alpha})(\tau_k^g, t \bar{z}) = (f_{\alpha} \circ \sigma_{\alpha})(t, \tau_k^g \bar{z}) + (f_{\alpha} \circ \sigma_{\alpha})(\tau_k^g, \bar{z}) - (f_{\alpha} \circ \sigma_{\alpha})(t, \bar{z}) \to \varphi(g). \quad (2.8)$$

By the relative triviality of $h(\tau, z)$ with respect to $(f_{\alpha} \circ \sigma_{\alpha})(\tau, z)$, the sequence $\{h(\tau, \tau_k^g, t \bar{z})\}_{k \geq 1}$ converges for all $\tau, t \in T$. Thus, we can put

$$\bar{h}(\tau, t \bar{z}) = \lim_{k \to \infty} h(\tau, \tau_k^g, t \bar{z}) = h(\tau, g(t \bar{z})) = \lim_{k \to \infty} h(\tau_k^g, t \bar{z}) \quad (2.9)$$

for every $(\tau, g(t \bar{z})) \in T \times G \times Z$. Suppose that there exist sequences $\{(\tau_k^i, z_k^i)\}_{k \geq 1} \subset T \times Z$ for $i = 1, 2$ so that $z_k^i \to z$, $\tau_k^i z_k^i \to g(z)$, and $(f_{\alpha} \circ \sigma_{\alpha})(\tau_k^i, z_k^i) \to \varphi(g)$ as $k \to \infty$,
while for $i = 1, 2$ the limit points $\bar{h}_i = \lim_{k \to \infty} h(\tau_k, z_k) \in \mathbb{R}_\infty$ are either distinct or both equal to $\infty$. Then $(g(z), \varphi(g), \bar{h}_i) \in D_{T,F}(z, 0, 0)$ for $i = 1, 2$, and for every $\tau' \in T$ we can conclude from $g \in \text{Aut}(Z,T)$ and the cocycle identity that
\[
(g(\tau'z), \varphi(g) + (f_0 \circ \sigma_0)(\tau', g(z)) - (f_0 \circ \sigma_0)(\tau', z)h(\tau', g(z)) + \bar{h}_i - h(\tau', z))
= (g(\tau'z), \varphi(g), h(\tau', g(z)) + \bar{h}_i - h(\tau', z)) \in D_{T,F}(\tau'z, 0, 0).
\]
Since $\bar{O}_T(z) = Z$, either there are two distinct points $a_1, a_2 \in \mathbb{R}_\infty$ with $(g(\bar{z}), \varphi(g), a_i) \in D_{T,F}(\bar{z}, 0, 0)$ or $(g(\bar{z}), \varphi(g), \infty) \in D_{T,F}(\bar{z}, 0, 0)$. In either case, since $\bar{O}_T,F(\bar{z}, 0, 0) = D_{T,F}(\bar{z}, 0, 0)$ in $Z \times (\mathbb{R}_\infty)^2$, this contradicts the relative triviality of $h(\tau, z)$ with respect to $(f_0 \circ \sigma_0)(\tau, z)$. Therefore equality (2.7) holds true, and the definition (2.9) extends uniquely from the $T$-orbit of $\bar{z}$ to a continuous mapping $\bar{h} : T \times G \rightarrow \mathbb{R}$ since the action of $T \times G$ on $X$ and $\varphi$ are continuous.

For the cocycle identity let $(\tau_1, g_1), (\tau_2, g_2) \in T \times G$ be arbitrary with according sequences $\{\tau_{k_1}\}_{k \geq 1}, \{\tau_{k_2}\}_{k \geq 1} \subseteq T$. By equality (2.8), we select a sequence $\{l_i\}_{i \geq 1} \subseteq \mathbb{N}$ with
\[
\tau_{k_1}^l \tau_{k_2}^l \bar{z} = \tau_{l}^{\tau_{k_1}} \tau_{l}^{\tau_{k_2}} \bar{z} = g_2(g_1(\bar{z})) \text{ and } (f_0 \circ \sigma_0)(\tau_{l}^{\tau_{k_1}}, \tau_{l}^{\tau_{k_2}}, \bar{z}) = \varphi(g_2) + \varphi(g_1) = \varphi(g_2g_1) \quad \text{as } l \rightarrow \infty.
\]
Thus, we can put $\{\tau_{k_i}^{l_i}\}_{i \geq 1} = \{\tau_{l}^{\tau_{k_1}}, \tau_{l}^{\tau_{k_2}}\}_{i \geq 1}$, and for every $t \in T$ equality (2.8) implies that $\tau_{l_i}^{-\tau_{k_1}} \tau_{l_i}^{-\tau_{k_2}} t \bar{z} = g_2(t \bar{z})$, $\tau_{l_i}^{-\tau_{k_1}} \tau_{l_i}^{-\tau_{k_2}} t \bar{z} = (g_2g_1)(t \bar{z})$, and
\[
(f_0 \circ \sigma_0)(\tau_{l_i}^{-\tau_{k_1}}, \tau_{l_i}^{-\tau_{k_2}} t \bar{z}) = (f_0 \circ \sigma_0)(\tau_{l_i}^{-\tau_{k_1}}, \tau_{l_i}^{-\tau_{k_2}} t \bar{z}) + (f_0 \circ \sigma_0)(\tau_{l_i}^{-\tau_{k_1}}, \tau_{l_i}^{-\tau_{k_2}} t \bar{z}) - (f_0 \circ \sigma_0)(\tau_{l_i}^{-\tau_{k_1}}, \tau_{l_i}^{-\tau_{k_2}} t \bar{z})
\]
\[
\text{as } l \rightarrow \infty.
\]
However, the uniqueness according to equality (2.7) verifies that $\bar{h}((\tau_1, g_2g_1g_2^{-1}), g_2(t \bar{z})) = \lim_{l \rightarrow \infty} h(\tau_1, \tau_2, \tau_2t \bar{z})$, and therefore
\[
\bar{h}((\tau_1, g_2g_1g_2^{-1}), g_2(t \bar{z})) + \bar{h}((\tau_2, g_2), t \bar{z}) = \lim_{l \rightarrow \infty} h(\tau_1, \tau_2, \tau_2t \bar{z}) + \lim_{l \rightarrow \infty} h(\tau_2, \tau^t \bar{z}, t \bar{z})
= \lim_{l \rightarrow \infty} h(\tau_1, \tau_2, \tau^t \bar{z}, t \bar{z}) = \bar{h}((\tau_1, \tau_2, g_2g_1), t \bar{z}).
\]
We substitute $g_2^{-1}g_1g_2$ for $g_1$ and obtain from $\bar{O}_T(z) = Z$ and the continuity of $\bar{h}$ that cocycle identity is valid.

**Proof of Proposition 2.10.** Let $(Y_c, T) = \pi_c(X, T)$ be the flow defined by the connected components of the fibres of $\pi_Y$ (cf. [22, Definition 2.2]), and let $\rho$ be the distal homomorphism from $(Y_c, T)$ onto $(Y, T) = \rho(Y_c, T)$. With a Rim $\{y, y' \in Y_c\}$ for the distal extension $(Y_c, T) = \pi_c(X, T)$ we define a cocycle $f_3(x, y) = \mu_{c,y}(f(\cdot, \cdot))$ for every $(\tau, y) \in T \times Y_c$. We fix a point $\bar{x} \in X$ with $D_{T,f_0 \circ \pi_0}(\tau \bar{x}, 0) = \bar{O}_{T,f_0 \circ \pi_0}(\bar{x}, 0)$ for all $\tau \in T$. From equality (2.2) and $\pi_{c^{-1}}(\pi_c(\tau \bar{x})) \subseteq \pi_{c^{-1}}(\pi_Y(\tau \bar{x}))$ it follows for all $\tau \in T$ that
\[
\bar{O}_{T,f_0 \circ \pi_0}(\bar{x}, 0) \cap (\pi_{c^{-1}}(\pi_c(\tau \bar{x})) \times \mathbb{R}) = \pi_{c^{-1}}(\pi_c(\tau \bar{x})) \times (f_0 \circ \pi_0(\tau \bar{x}), x) \times \mathbb{R}
\]
\[
= \{(x, (f_0 \circ \pi_0)(\tau \bar{x}), b_\tau(x)) : x \in \pi_{c^{-1}}(\pi_c(\tau \bar{x}))\}
\]
(2.10)
for every $\tau \in T$, in which $b_\tau : \pi_{c^{-1}}(\pi_c(\tau \bar{x})) \rightarrow \mathbb{R}$ is a continuous function. Indeed, given a sequence $\{l_k\}_{k \geq 1} \subseteq T$ with $l_k \tau \bar{x} \rightarrow x \in \pi_{c^{-1}}(\pi_c(\tau \bar{x}))$ and $(f_0 \circ \pi_0)(l_k, \tau \bar{x}) \rightarrow 0$, we can conclude from the relative triviality of $(f_0 \circ \pi_0)(\tau, x)$ the existence and uniqueness of the limit $b_\tau(x)$ of $f(\tau l_k, x)$. Moreover, for every $\varepsilon > 0$, there exists a $\delta > 0$ so that for all $\tau \in T$ and $x, x' \in \pi_{c^{-1}}(\pi_c(\tau \bar{x}))$ with $d(x, x') < \delta$ holds $|b_\tau(x) - b_\tau(x')| < \varepsilon$. Since the fibres of $\pi_c$ are connected, a covering of $X$ by $\delta$-neighbourhoods provides a constant $D > 0$ with $|b_\tau(x) - b_\tau(x')| < D$ for all $\tau \in T$ and $x, x' \in \pi_{c^{-1}}(\pi_c(\tau \bar{x}))$. Equality (2.10) shows for
$x \in \pi^{-1}_c(\pi_c(x))$ and $\tau \in T$ that $b(x) + f(\tau, x) = b(\tau) x$ and hence $|f(\tau, x) - f(\tau, x)| \leq 2D$. Since $(f - f_c \circ \pi_c)(\tau, x)$ takes zero value on each $\pi_c$-fibre, for all $\tau \in T$ holds the upper bound $|f - f_c \circ \pi_c)(\tau, x)| < 2D$ so that the cocycle $(f - f_c \circ \pi_c)(\tau, x)$ is a coboundary.

The extension from $(Y, T)$ to $(Y_c, T)$ is an isometric extension due to [22, Theorem 3.7], and by Fact 2.4 there exists a compact group extension $(\bar{Y}, T) \rightarrow (Y, T)$ by $G \subset \text{Aut}(\bar{Y}, T)$ so that $(Y_c, T) = \sigma(Y, T)$ is the orbit space of a compact subgroup $H \subset G$. For every sequence $\{y_k, \bar{y}_k\}_{k \geq 1} \subset T \times \bar{Y}$ with $\phi(\bar{y}_k, \bar{y}_k) \rightarrow 0$ and $(\rho \circ \phi(\bar{y}_k, \bar{y}_k) \rightarrow 0$ holds $(\rho \circ \phi(\bar{y}_k, \bar{y}_k) \rightarrow 0$. Otherwise, by Lemma 1.3, there exists a sequence $\{Y, T\}_{k \geq 1} \subset T \times X\}$ so that $d(x_k, \bar{x}_k) \rightarrow 0$, $(\pi \circ \pi(\bar{x}_k, x_k) \rightarrow 0$, and $(\pi \circ \pi(\bar{x}_k, x_k) \rightarrow \infty$, which contradicts Proposition 2.8 and the boundedness of the transfer function between $f$ and $f_c \circ \pi_c$. We can apply Lemma 2.11 for the flow $(Y, T)$, the cocycle $h = f_c \circ \sigma$, the group $G \subset \text{Aut}(\bar{Y}, T)$, and the group homomorphism $\varphi \equiv 0$, and we obtain a real-valued cocycle $\bar{h}(\tau, h, \bar{y})$ with $\bar{h}(\tau, \bar{1_c}, \bar{y}) = (f_c \circ \sigma(\bar{y})$ for every $(\tau, \bar{y}) \in T \times \bar{Y}$. We define a topological cocycle of $(Y, T)$ by $f_Y(\tau, y) = \mu_{\tau, y}(\tau \circ \sigma(\tau, \bar{y})$, where $\mu_{\tau, y} : y \in Y$ is the RIM for the extension $(Y, T) = \rho \circ \pi_c(Y, T)$.

From the cocycle identity

$\bar{h}(\tau, g, y) = (f_c \circ \sigma)(\tau, g(y)) + \bar{h}(\tau, h, \bar{y}) = \bar{h}(\tau, h, \bar{y}) + \bar{h}(\tau, \bar{y})$ 

and from the boundedness of $\bar{h}(\tau, h, \bar{y})$ for $(\tau, \bar{y}) \in G \times Y$, we can conclude that the real-valued cocycle $(f_c \circ \sigma \circ \rho) \circ \sigma(\tau, y)$ is uniformly bounded. Therefore, also $(f_c - f \circ \rho)\circ \sigma(\tau, y)$ is uniformly bounded and a coboundary, and the relative triviality of $f_Y(\tau, y)$ with respect to $(f_c \circ \sigma(\tau, y)$ can be verified as above for the cocycle $(f_c \circ \sigma(\tau, y)$.

Proposition 2.12. The cocycle $(f_Y - f_c \circ \rho)(\tau, y)$ of the flow $(Y, T)$ can be extended to a cocycle $\bar{f}(\tau, t, y)$ of the $T \times \mathbb{R}$-flow $(Y, \{\psi^t \circ \tau : (\tau, t) \in T \times \mathbb{R}\})$ so that

$(f_Y - f_c \circ \rho)(\tau, y) = \bar{f}(\tau, 0, y)$

for every $(\tau, y) \in T \times Y$. We put $L_c(x, m) = (x, m) \in X_a \times M = Y$. Then for arbitrary $x, m \in X_a$ there exists a continuous function $b : Y \rightarrow \mathbb{R}$ with $b(x, m) = 0$ so that for every $(\tau, y) \in T \times Y$ holds

$\bar{f}(\tau, -f_c \circ \rho)(\tau, y), y) = b(\psi(\rho(\tau, y)) - b(y) = b(y) - b(L, y), y)$

(2.11)

and the mapping $(\tau, y) \mapsto \bar{f}(\tau, -f_c \circ \rho)(\tau, y), y)$ is a topological coboundary of the distal flow $(Y, \{L_c \circ \tau : (\tau) \in T\}$ with transfer function $b$. For every $(x, m) \in Y$ and $t \in \mathbb{R}$ holds

$\bar{f}(\tau, t, (x, m)) = \bar{f}(\tau, t, (x, m)) + b(x, m(0, m)) - b(x, m, m)$

(2.12)

Proof. Since Lemma 2.11 can be applied to the cocycle $h = f_Y - f_c \circ \rho$, the group $G = \{\phi^t : t \in \mathbb{R}\} \subset \text{Aut}(Y, T)$, and the group homomorphism $\varphi = \text{id}_{\mathbb{R}}$, it provides the cocycle $\bar{f}(\tau, t, y)$. The mapping $\bar{f}(\tau, t, y) = \bar{f}(\tau, t, (f_c \circ \rho(\tau, y), y)$ fulfills

$\bar{f}(\tau, t, \psi^t(L, y)) + \bar{f}(\tau, t, y) = \bar{f}(\tau, t, (f_c \circ \rho(\tau, y), y)$

and is thus a cocycle of the minimal flow $(Y, \phi^t \circ L_c : (\tau, t) \in T \times \mathbb{R})$. Now let $(\tau, y)$ be arbitrary. By equality (2.3) and the density of $\tilde{\pi}_c$-transitive points in $X_a$, there exists a sequence $\{y_k, \bar{y}_k\}_{k \geq 1} \subset T \times \bar{Y}$ so that $y_k \rightarrow \tau y, \bar{y}_k \rightarrow \psi^t(\rho(\tau, y))$ and $(f_c \circ \rho(\bar{y}_k, \bar{y}_k) \rightarrow -f_c \circ \rho(\bar{y}_k, \bar{y}_k)$ as $k \rightarrow \infty$, and from equality (2.7) it follows that $(f_Y - f_c \circ \rho)(\tau, y) \rightarrow (f_Y - f_c \circ \rho)(\tau, y)$, $(f_Y - f_c \circ \rho)(\tau, y)$.
as \( k \to \infty \), and this limit coincides with \( f((\tau, -(f_\alpha \circ \rho_\alpha)(\tau, y)), y) = f'(((\tau, 0), y) \) due to the cocycle identity for \( f((\tau, t), y) \). Since \( f_y((\tau, y) \) is relatively trivial with respect to \((f_\alpha \circ \rho_\alpha)(\tau, y) \) for every \( \varepsilon > 0 \) there exists a \( \delta > 0 \) so that for every \( (\tau', y') \in T \times Y \) with \( d_Y(y', \tau'y') < \delta \) and \(|(f_\alpha \circ \rho_\alpha)(\tau', y')| < \delta \) holds \(|(f_\alpha - f_\alpha \circ \rho_\alpha)(\tau', y')| < \varepsilon \). From \( \tau^{-1}y_k \to y \), \( \tau_ky_k \to L\gamma \), and \((f_\alpha \circ \rho_\alpha)(\tau_k, \tau^{-1}y_k) \to 0 \) we can conclude for every \((\tau, y) \in T \times Y \) with \( d_Y(y, L\gamma(y)) < \delta \) that \( f'(((\tau, 0), y) < \varepsilon \). However, Fact 1.4 implies that the cocycle \((\tau, y) \mapsto f'((\tau, y), y) \) of the distal flow \((Y, \{L\gamma : \tau \in T\}) \) is a coboundary on the \( \{L\gamma : \tau \in T\} \)-orbit closure \( X_\alpha \times \{m\} \) with transfer function \( b_m : X_\alpha \to \mathbb{R} \) for every \( m \in M \). Since \( \delta > 0 \) is valid for all \( \{L\gamma : \tau \in T\} \)-orbit closures, the transfer functions \( \{b_m : m \in M\} \) are uniformly equicontinuous. We fix a point \( \bar{x}_\alpha \in X_\alpha \) and obtain from the cocycle identity for all \((\tau, t) \in T \times \mathbb{R} \) and \((x, m) \in Y \) that
\[
\begin{align*}
f_{\bar{x}_\alpha}((\tau, t), (x, m)) &= f'((\tau, t), (x, m)) - f'((1T, t), (\bar{x}_\alpha, m)) \\
&= b_{\phi_t(m)}(\tau x, \bar{x}) - b_{\phi_t(m)}(\bar{x}, \bar{x}) - b_m(\tau x, \bar{x} + b_m(\bar{x}, \bar{x}).
\end{align*}
\]
The function \( f_{\bar{x}_\alpha}((\tau, t), (x, m)) \) is also a cocycle of \((Y, \{\psi^t \circ L\gamma : (\tau, t) \in T \times \mathbb{R}\}) \) and bounded on \( T \times \mathbb{R} \times Y \), hence a coboundary with a transfer function \( \bar{b} : \bar{Y} \to \mathbb{R} \) so that \( \bar{b}(\rho_\alpha^{-1}(\bar{x})) = \{0\} \). Now equality (2.11) follows, and equality (2.12) follows from \( f'((1T, t), (\bar{x}, m)) = \bar{b}((1T, t), (\bar{x}, m)) \) for all \( t \in \mathbb{R} \) and \((x, m) \in Y \).

With these prerequisites we can conclude the proof of our main result.

**Proof of the structure theorem.** We let all elements of the structure theorem and the flow \( \{\psi^t : t \in \mathbb{R}\} \subset \text{Aut}(Y, \{L\gamma : \tau \in T\}) \cap \text{Aut}(Y, T) \) be defined according to Propositions 2.8–12.12. We fix a point \( \bar{x}_\alpha \in X_\alpha \) so that \((\bar{x}_\alpha, 0) \) is transitive for \( f_\alpha \) and \( \bar{b}(\rho_\alpha^{-1}(\bar{x}_\alpha)) = \{0\} \). Then we define a cocycle \( g(t, m) \) of the distal minimal flow \((M, \{\phi^t : t \in \mathbb{R}\}) \) by
\[
g(t, m) = f'((1T, t), (\bar{x}, m)) \quad \text{for all} \quad (t, m) \in \mathbb{R} \times M.
\]
From equalities (2.11) and (2.12) it follows for all \( \tau \in T \) and \((x, m) \in Y \) that
\[
\begin{align*}
(f_\gamma - f_\alpha \circ \rho_\alpha)(\tau, (x, m)) &= f((\tau, 0), (x, m)) \\
&= f'((1T, t), (\bar{x}, m)) = b(\tau (x, m)) - b(x, m) \quad \text{for all} \quad \tau \in T \times \mathbb{R} \times Y,
\end{align*}
\]
and, therefore, \( f(\gamma, t, (x, m)) = \bar{b}(\tau (x, m)) - b(x, m) \) for all \( \tau \in T \times \mathbb{R} \times Y \). Hence, equality (1.5) holds for the cocycle \( f_\gamma(\tau, y) = \bar{b}(\tau y) + b(y) \) cohomologous to \( f_\gamma(\tau, y) \), and this cocycle will be substituted for \( f_\gamma(\tau, y) \) henceforth. For every sequence \( \{(\tau_k, x_k)\}_{k \geq 1} \subset T \times \mathbb{R} \) with \((f_\alpha \circ \rho_\alpha)(\tau_k, x_k) \to 0 \) it also holds that \( f_\gamma(\tau_k, x_k) \to 0 \) as \( k \to \infty \), and thus identity (2.2) implies identity (1.4).

For every ordinal \( \xi \) with \( \alpha \leq \xi \leq \eta \) we can apply Propositions 2.9, 2.10, and 2.12 to the distal minimal flow \((X_\xi, T) \) and the cocycle \( f_\xi(\tau, x_\xi) \). We obtain a factor \((Y_\xi, T) = (X_\alpha \times M_\xi, T) = \pi Y_\xi(X_\xi, T) \) with \((X_\alpha, T) = \rho_\alpha^\xi(Y_\xi, T) \), an \( \mathbb{R} \)-flow \( \{\psi^t : t \in \mathbb{R}\} \subset \text{Aut}(Y_\xi, T) \), a cocycle \( f_\xi(\tau, y_\xi) \) of \((X_\xi, T) \), and a cocycle \( f_\xi((\tau, t), y_\xi) \) of the flow \((Y_\xi, T) \times \mathbb{R} \) extending \( f_\xi(t) - f_\alpha \circ \rho_\alpha^\xi(\tau, y_\xi) \). Striving for a contradiction to the maximality of the ordinal \( \alpha \) (cf. Proposition 2.8), we assume the recurrence of the cocycle \( (1 + g)(t, m) \) of the minimal flow \((M, \{\phi^t : t \in \mathbb{R}\}) \) so that the cocycle \( f_{\bar{x}_\alpha}((1T, t), y_\alpha) + t \) of the minimal flow \((\{\rho_\alpha^{-1}(\bar{x}_\alpha)\}, \{\psi^t : t \in \mathbb{R}\}) \) is as well recurrent. We let \( \beta \) be the minimal element of the non-empty set of ordinals
\[
\{\alpha \leq \xi \leq \eta : f_\xi((1T, t), y_\xi) + t \text{ is a recurrent cocycle of } (\{\rho_\alpha^{-1}(\bar{x}_\alpha)\}, \{\psi^t : t \in \mathbb{R}\})\},
\]
with \( \beta > \alpha \) since \( f_\alpha = f_\alpha \) and \( f_\alpha((1T, t), y_\alpha) \equiv 0 \). We fix a point \( \bar{x}_\beta \in (\pi_\alpha^\beta)^{-1}(\bar{x}_\alpha) \) so that \((\pi_\alpha(\bar{x}_\beta), 0) \) is a recurrent point for the skew product of the flow \((\{\rho_\alpha^{-1}(\bar{x}_\alpha), \{\psi^t \circ \pi_\beta : t \in \mathbb{R}\}) \) and the cocycle \( f_\beta((1T, t), y_\beta) + t \). Then there exists a sequence \( \{\tau_k\}_{k \geq 1} \subset T \) with \( f_\alpha(\tau_k, \pi_\beta(\bar{x}_\beta)) \to \infty \) so that
\[
\begin{align*}
f_\beta((1T, f_\alpha(\tau_k, \pi_\beta(\bar{x}_\beta))), \pi_\beta(\bar{x}_\beta)) + f_\alpha(\tau_k, \pi_\beta(\bar{x}_\beta)) &= f_\beta(\tau_k, \pi_\beta(\bar{x}_\beta)) \to 0.
\end{align*}
\]
as \( k \to \infty \), and by the cohomology of the cocycles \( f_\beta(\tau, x_\beta) \) and \( (f_{\gamma} \circ \pi_{\gamma})(\tau, x_\beta) \) the sequence \( f_\beta(\tilde{\tau}_k, \tilde{x}_\beta) \) is bounded in \( \mathbb{R} \). Hence, there exists a sequence \( \{k_t\}_{t \geq 1} \subset \mathbb{N} \) so that \( f_\alpha(\tilde{\tau}_{k_t+1}, \tilde{x}_\alpha(\tilde{\tau}_{k_t})) \to \infty \), \( d_\beta(\tilde{\tau}_{k_t+1}, \tilde{x}_\beta, \tilde{\tau}_k, \tilde{x}_\beta) \to 0 \), and \( f_\beta(\tilde{\tau}_{k_t}, \tilde{x}_\beta) \) is convergent as \( l \to \infty \), and then \( \{(\tau_l, x_l) = (\tilde{\tau}_{k_t+1}^{-1}, \tilde{x}_\beta)\}_{t \geq 1} \subset T \times X_\alpha \) is a sequence with \( f_\alpha(\tau_l, \pi_{\alpha}(x_l)) \to \infty \), \( d_\beta(x_l, x_l) \to 0 \), and \( f_\beta(\tau_l, x_l) \to 0 \) as \( l \to \infty \). However, for every \( \alpha \leq \xi < \beta \) holds

\[
\bar{f}_\xi((1_l, \pi_{\beta}(\tau_l)), \pi_{\gamma} \circ \pi_{\beta}(x_l)) + f_\alpha(\tau_l, \pi_{\alpha}(x_l)) = f_\gamma(\tau_l, \pi_{\gamma} \circ \pi_{\beta}(x_l)) \to \infty
\]
as \( l \to \infty \). Otherwise, since \( f_\xi((1_l, t), (a_\alpha, m_\xi)) - f_\xi((1_l, t), (a_\alpha, m_\xi)) \) is uniformly bounded for all \( t \in \mathbb{R} \), \( x_\alpha \in X_\alpha \), and \( m_\xi \in M_\xi \) (cf. identity (2.12)), there exists a non-trivial prolongation in the skew product of the minimal flow \((\rho_{\alpha}^{-1}((\bar{x}_\alpha), \{\psi_t : t \in \mathbb{R}\}) \) and the cocycle \( f_\xi((1_l, t), y_\xi) + t \), which is sufficient for its recurrence (cf. Lemma 1.9). For every \( \alpha \leq \xi < \beta \) tends \( f_\xi(\tau_l, \pi_{\xi}(x_l)) \to \infty \) as \( l \to \infty \), depending on the type of the ordinal \( \beta \) either Lemma 2.6 or Lemma 2.7 implies that \( \bar{f}_\beta \) is point transitive, in contradiction to the maximality of \( \alpha \). □

**Proof of the structure of the topological Mackey action.** In the proof of the decomposition theorem it is verified that the topological Mackey actions for the cocycle \( f(\tau, x) \) of \((X, T) \) and the transient cocycle \((1 + g)(t, m) \) of \((M, \{\varphi^t : t \in \mathbb{R}\}) \) are isomorphic. Let \( \{(M_{\xi}, \varphi_{\xi}^t : t \in \mathbb{R}) : 0 \leq \xi \leq \theta \} \) be the normal \( L \)-system for the distal minimal compact metric flow \((M, \{\varphi^t : t \in \mathbb{R}\}) \) with the homomorphisms \( \sigma_\xi : M \to M_\xi \). For every ordinal \( 0 \leq \xi \leq \theta \) a cocycle \( g_\xi(t, m_\xi) \) of \((M_{\xi}, \varphi_{\xi}^t : t \in \mathbb{R}) \) can be defined by a \( RIM \). Let \( \beta \) be the minimal element of the set of ordinals

\[
\theta \in \{0 \leq \xi \leq \theta : (g - g_\xi \circ \sigma_\xi)(t, m) \text{ is a coboundary of } (M, \{\varphi^t : t \in \mathbb{R}\})\}.
\]
The cocycle \((1 + g_\beta)(t, m_\beta) \) is transient, since the cocycle \((1 + g)(t, m) \) cohomologous to \((1 + g_\beta)(t, m) \) is transient. By Lemma 1.9, the mapping \( \chi : M_\beta \to D \) defined by \( m_\beta \mapsto \mathcal{O}_{\varphi_{\beta}^t(1 + g_\beta)}(m_\beta, 0) \) is Fell continuous onto the Fell compact space \( D \) of orbits in \( M_\beta \times \mathbb{R} \). The right translation action \( \{R_b : b \in \mathbb{R}\} \) on \( M_\beta \times \mathbb{R} \) acts minimally on \( D \), and for every \( t \in \mathbb{R} \) and \( m_\beta \in M_\beta \)

\[
\chi \circ \varphi_{\beta}^t(m_\beta) = R_t(g_\beta)(m_\beta) \circ \chi(m_\beta).
\]
For \( \beta = 0 \) the flow \((D, \{R_b : b \in \mathbb{R}\}) \) is trivial and thus weakly mixing. If \( \beta \geq 1 \) and the non-trivial minimal compact metric flow \((D, \{R_b : b \in \mathbb{R}\}) \) is not weakly mixing, then there exists a non-trivial equicontinuous minimal flow \((D_1, \{\varphi^t : t \in \mathbb{R}\}) = \nu(D, \{R_b : b \in \mathbb{R}\}) \) with homomorphism \( \nu \) (cf. [17]). By a generalized and relativized version of [6, Theorem 1], we shall obtain a contradiction to the minimality of \( \beta \). Since the flow \((D_1, \{\varphi^t : t \in \mathbb{R}\}) \) is minimal and non-trivial, for each small enough \( \varepsilon > 0 \) holds \( \varphi^t(d_1) \neq d_1 \) for all \( d_1 \in D_1 \). We shall verify as a sub-lemma that there are no sequences \( \{t_k\}_{k \geq 1} \subset \mathbb{R} \) and \( \{m_k\}_{k \geq 1} \subset M_\beta \) for \( i = 1, 2 \) so that \( m_k \to m_\beta , \varphi_{\beta}^t(m_k) \to m_\beta' \), and \( g_\beta(t_k, m_k) - g_\beta(t_k, m_k') \to \varepsilon \), as \( k \to \infty \). Indeed, for \( i = 1, 2 \) we can conclude that

\[
\varphi(1 + g_\beta)(t_k, m_k) \circ \nu \circ \chi(m_k) = \nu \circ \varphi_{\beta}^t(m_k) \to \nu \circ \chi(m_\beta'),
\]
and since the flow \((D_1, \{\varphi^t : t \in \mathbb{R}\}) \) is equicontinuous \( \varphi(1 + g_\beta)(t_k, m_k) \circ \nu \circ \chi(m_k') \to \nu \circ \chi(m_\beta') \) as \( k \to \infty \). From

\[
\varphi(1 + g_\beta)(t_k, m_k) \circ \nu \circ \chi(m_k) \to \varphi(\lim \varphi(1 + g_\beta)(t_k, m_k)) \circ \nu \circ \chi(m_\beta') \text{ as } k \to \infty,
\]

it follows that \( \nu \circ \chi(m_\beta') = \varphi^\varepsilon \circ \nu \circ \chi(m_\beta) \), in contradiction to the choice of \( \varepsilon \).

If \( \beta = \gamma + 1 \) for some ordinal \( 0 \leq \gamma < \theta \) and the extension is isometric, then the sub-lemma applies to all sequences with \( m_k \to m_\beta \) and \( \sigma_{\beta}^t(m_k) = \sigma_{\beta}^t(m_k') \) for all \( k \geq 1 \), and the condition on sufficiently small \( \varepsilon \) can be fulfilled by the connectedness of the \( \sigma_{\beta}^t \)-fibres. Therefore, the mapping \( m_\beta \mapsto g_\beta(t, m_\beta) \) is uniformly equicontinuous for all \( t \in \mathbb{R} \) on each \( \sigma_{\beta}^t \)-fibre. Since the cocycle \((g_\beta - g_\beta \circ \sigma_{\beta}^t)(t, m_\beta) \) is uniformly equicontinuous for all \( t \in \mathbb{R} \) on each connected \( \sigma_{\beta}^t \)-fibre and takes zero value, by Fact 1.4 it is a coboundary in contradiction to the minimality of \( \beta \).
For a limit ordinal $\beta$, the sub-lemma applies to sequences with $m^1_\xi \to \bar{m}$ so that for every ordinal $0 \leq \xi < \beta$ there exists an integer $k_\xi \geq 1$ with $\sigma^\beta_{\xi}(m^1_k) = \sigma^\beta_{\xi}(m^2_k)$ for all $k \geq k_\xi$. Indeed, as $\xi$ tends to $\beta$, the diameter of the $\sigma^\beta_{\xi}$ -fibres tends to 0 uniformly. Hence, there exists an ordinal $0 \leq \xi < \beta$ so that $|g_\xi(t, m_\beta) - g_\xi(t, m'_\beta)| < \varepsilon$ for all $t \in \mathbb{R}$ and $m_\beta, m'_\beta \in M_\beta$ with $\sigma^\beta_{\xi}(m_\beta) = \sigma^\beta_{\xi}(m'_\beta)$. Therefore, the cocycle $(g_\xi - g_{\xi', \sigma^\beta_{\xi}})(t, m_\beta)$, which takes zero value on every connected $\sigma^\beta_{\xi}$ -fibre, is in a contradiction to the minimality of $\beta$.

The topological Mackey actions of the transient cocycle $(1 + g)(t, m)$ and the cohomologous cocycle $(1 + g_\beta \circ \sigma_\beta)(t, m)$ are isomorphic (cf. the proof of the decomposition theorem).

The weakly mixing flow $(D, \{R_h : b \in \mathbb{R}\})$ is a factor of the latter, since for every $(m, s) \in M \times \mathbb{R}$ the mapping $\sigma_\beta \times \text{id}_\mathbb{R}$ maps the orbit $O_{\phi, (1 + g_\beta \circ \sigma_\beta)}(m, s)$ in $M \times \mathbb{R}$ to the orbit $O_{\phi, (1 + g_\beta)}(\gamma_\beta(m), s)$ in $M_\beta \times \mathbb{R}$ continuously with respect to the Fell topologies. Suppose that there exist two distinct $\phi^i_{(1 + g_\beta \circ \sigma_\beta)}$-orbits $O_1, O_2$ in $M \times \mathbb{R}$ within the same $\sigma_\beta \times \text{id}_\mathbb{R}$-fibre and $\{t_k\}_{k \geq 1} \subset \mathbb{R}$ is a sequence with $R_{t_k}O_i \to O'$ for $i = 1, 2$. Since for every $m_\beta \in M_\beta$ the mapping $t \mapsto (1 + g_\beta)(t, m_\beta)$ is onto $\mathbb{R}$ (cf. Lemma 1.9), there exists a point $\bar{m} \in M_\beta$ and distinct $m_i \in \sigma_\beta^{-1}(\bar{m})$ for $i = 1, 2$ so that $(m_i, 0) \in O_i$. Moreover, for every integer $k \geq 1$ we can select a real number $t'_k$ so that $(1 + g_\beta)(t'_k, \bar{m}) = t_k$, and therefore $(\phi^i_{k}(m_i), 0) \in R_{t_k}O_i$ for $i = 1, 2$. By passing to a subsequence we can assume that $\phi^k(m_i) \to m'_k$ as $k \to \infty$ with $m'_k \neq m'_2$, since the flow $(M, \{\phi^i : t \in \mathbb{R}\})$ is distal. However, since $(m'_1, 0), (m'_2, 0) \in O'$, the point $(\bar{m}, 0)$ is a periodic point in $(D, \{R_{t} : b \in \mathbb{R}\})$, in contradiction to the transience of the cocycle $(1 + g_\beta)(t, m_\beta)$. Thus, $\sigma_\beta \times \text{id}_\mathbb{R}$ is a distal homomorphism of the topological Mackey action of the cocycle $(1 + g_\beta \circ \sigma_\beta)(t, m)$ onto the weakly mixing flow $(D, \{R_h : b \in \mathbb{R}\})$.

The topological Mackey action of the cocycle $(1 + g)(t, m)$ is distal if and only if $\beta = 0$. Equivalently, the cocycle $g(t, m)$ is cohomologous to $(g_\beta \circ \sigma_\beta)(t, m)$ for a cocycle $g_\beta(t, m_\beta)$ of the trivial $\mathbb{R}$-flow, however, all topological cocycles of the trivial $\mathbb{R}$-flow are of the form $c\mathbb{I}(t, m_0) = ct$ for all $(t, m_0) \in \mathbb{R} \times M_\beta$ with $c \in \mathbb{R}$.

Proof of the uniqueness of the representation. Let two representations for one and the same cocycle $f(\tau, x)$ according to the structure theorem be given, with cocycles $(fY_i \circ \pi_{Y_i})(\tau, x)$ cohomologous to $f(\tau, x)$, distal minimal flows $(X^i, T) = \pi_{X^i}(X, T), (M_i, \{\phi^i : t \in \mathbb{R}\}),$ and $\mathbb{R}$-valued cocycles $f^i_{\alpha}(\tau, x^i_{\alpha})$, $(1 + g_i)(t, m_i)$ for $i \in \{1, 2\}$. We shall verify at first that the cocycle $(f^2_{\alpha} \circ \pi^2_{\alpha})(\tau, x)$ of $(X, T)$ is relatively trivial with respect to $(f^1_{\alpha} \circ \pi^1_{\alpha})(\tau, x)$. If this is not the case, then by Lemma 1.3 there exist a sequence $\{\tau_k\}_{k \geq 1} \subset T$ and a point $\bar{x} \in X$ so that $d(\tau_k \bar{x}, \bar{x}) \to 0$ and $(f^2_{\alpha} \circ \pi^2_{\alpha})(\tau_k, \bar{x}) \to 0$ as $k \to \infty$, however $(f^2_{\alpha} \circ \pi^2_{\alpha})(\tau_k, \bar{x}) \to \infty$. We represent the cocycles $(fY_i \circ \pi_{Y_i})(\tau, x)$ for $i \in \{1, 2\}$ according to the structure theorem by

$$(fY_1 \circ \pi_{Y_1})(\tau, x) = (1 + g_1)((f^1_{\alpha} \circ \pi^1_{\alpha})(\tau, x), \pi_{M_1}(x)),$$ 

in which $\pi_{M_1}$ denotes the composition of $\pi_{Y_1}$ and the second projection of $Y_1 = X^1_{\alpha} \times M_1$. On the one hand, it follows that $(fY_1 \circ \pi_{Y_1})(\tau, x) \to 0$, while on the other hand $(fY_2 \circ \pi_{Y_2})(\tau, x) \to \infty$ as $k \to \infty$ by the transience of the cocycle $(1 + g_2)(t, m_2)$ and Lemma 1.9. This is clearly a contradiction to the continuous cohomology of the cocycles $(fY_1 \circ \pi_{Y_1})(\tau, x)$ and $(fY_2 \circ \pi_{Y_2})(\tau, x)$, both of which are cohomologous to $f(\tau, x)$.

Since $(f^2_{\alpha} \circ \pi^2_{\alpha})(\tau, x)$ is relatively trivial with respect to $(f^1_{\alpha} \circ \pi^1_{\alpha})(\tau, x)$, we can proceed with Propositions 2.10 and 2.12 as in the proof of the structure theorem, however with $(f^2_{\alpha} \circ \pi^2_{\alpha})(\tau, x)$ substituted for $f(\tau, x)$. Indeed, the only property exploited to obtain the cocycle $g(t, m)$ in the proof of the structure theorem was the relative triviality of $(f - f_\alpha \circ \pi_{\alpha})(\tau, x)$ with respect to $(f_\alpha \circ \pi_{\alpha})(\tau, x)$. Hence, there exists a continuous function $b' : X \to \mathbb{R}$ and a real-valued cocycle $h(t, m_1)$ of the flow $(M_{\alpha}, \{\phi^1_{\alpha} : t \in \mathbb{R}\})$ so that

$$(f^2_{\alpha} \circ \pi^2_{\alpha})(\tau, x) = h((f^1_{\alpha} \circ \pi^1_{\alpha})(\tau, x), \pi_{M_1}(x)) + b'(\tau x) - b'(x).$$
for all \((\tau, x) \in T \times X\). We shall verify that the cocycle \(h(t, m_1)\) is transient. Suppose that \((\bar{m}_1, 0)\) is a recurrent point for the skew product \((\phi^1_t)_{\bar{m}_1}\) and \((\bar{x}^1_0, 0) \in X^1_{\bar{m}_1} \times \mathbb{R}\) is a transitive point for \(\bar{\tau}_1\).

Then we can find a sequence \(\{\tau_k\}_{k \geq 1} \subset T\) with \(f^1_{\alpha}(\tau_k, x^1_\alpha) \to \infty\) and \(h(f^1_{\alpha}(\tau_k, x^1_\alpha), \bar{m}_1) \to 0\) as \(k \to \infty\). For an arbitrary point \(x \in \pi^{-1}_1(\bar{x}^1_\alpha, \bar{m}_1)\), we put \((\bar{x}^2_\alpha, \bar{m}_2) = \pi_2(\bar{x})\). Then the transience of the cocycle \((1 + g_1)(t, \bar{m}_1)\) implies that \((f_{\alpha} \circ \bar{\tau}_1)(\tau_k, x) \to \infty\) as \(k \to \infty\), however,

\[
(f_{\alpha} \circ \bar{\tau}_1)(\tau_k, \bar{x}) = (1 + g_2)(f_{\alpha}(\tau_k, x^1_\alpha), \bar{m}_2) = (1 + g_2)(h(f_{\alpha}(\tau_k, x^1_\alpha), \bar{m}_1) + b'(\bar{x}) - b'(\bar{x}), \bar{m}_2)
\]

is a bounded sequence since \(h(f^1_{\alpha}(\tau_k, x^1_\alpha), \bar{m}_1) \to 0\) as \(k \to \infty\). This is again a contradiction to the continuous cohomology of the cocycles \((f_{\alpha} \circ \bar{\tau}_1)(\tau, x)\) and \((f_{\alpha} \circ \bar{\tau}_2)(\tau, x)\).

By Lemma 1.9 the mapping \(\chi_2 : M_2 \to D_2\) defined by \(m_2 \mapsto O_{\phi_{2,1}}(m_2, 0)\) is continuous onto the compact space \(D_2\) of orbits in \(M_2 \times \mathbb{R}\), and it is even one-to-one and hence a homeomorphism. The right translation action \(\{R_0 : b \in \mathbb{R}\} \to M_2 \times \mathbb{R}\) acts minimally on \(D_2\), and for every \(t \in \mathbb{R}\) holds \(\chi_2 \circ \phi^1_{t} = R_t \circ \chi_2\) so that \(\chi_2\) is an isomorphism and \((D_2, \{R_0 : b \in \mathbb{R}\})\) is a distal minimal flow. Since the cocycle \(h(t, m_1)\) is transient, the mapping \(\chi_1 : M_1 \to D_1\) defined by \(m_1 \mapsto O_{\phi_{1,h}}(m_1, 0)\) is continuous onto the compact space \(D_1\) of orbits in \(M_1 \times \mathbb{R}\) with \(\{R_0 : b \in \mathbb{R}\} \in \mathbb{R}\) acting minimally and \(\chi_1 \circ \phi^1_{t} = R_h(t, m_1) \circ \chi_1(m_1)\) for every \(t \in \mathbb{R}\) and \(m_1 \in M_1\). By the decomposition theorem, the flows \((D_1, \{R_0 : b \in \mathbb{R}\})\) and \((D_2, \{R_0 : b \in \mathbb{R}\})\) are both isomorphic to the topological Mackey action of \((\alpha^1_\tau, \alpha^2_\tau)\), and therefore the flow \((D_1, \{R_0 : b \in \mathbb{R}\})\) is distal by the distality of \((D_2, \{R_0 : b \in \mathbb{R}\})\).

According to the structure of the topological Mackey action, the transient cocycle \(h(t, m_1)\) is then cohomologous to

\[
c_1(t, m_1) = ct = h(t, m_1) + b'' \circ \phi^1_{t}(m_1) - b''(m_1)
\]

for some \(c \in \mathbb{C} \setminus \{0\}\) and a continuous function \(b'' : M_1 \to \mathbb{R}\). We can conclude that the flows \((D'_1, \{R_0 : b \in \mathbb{R}\})\) with \(D'_1 = \{O_{\phi_{1,c}}(m_1, 0) : m_1 \in M_1\}\) and \((D_1, \{R_0 : b \in \mathbb{R}\})\) are isomorphic, and the isomorphism from \(D'_1\) to \(D_1\) is induced by the homeomorphism \((m_1, t) \mapsto (m_1, t - b''(m_1))\) of \(M_1 \times \mathbb{R}\). Since the flows \((D'_1, \{R_0 : b \in \mathbb{R}\})\) and \((D_1, \{\phi^{t/c} : t \in \mathbb{R}\})\) are isomorphic, there exists a homeomorphism \(\iota : M_1 \to M_2\) so that \(\iota \circ \phi^1_{t} = \phi^2_{t \circ \iota}\) for all \(t \in \mathbb{R}\).

It remains to verify the correspondence between the cocycles \(g_1(t, m_1)\) and \(g_2(t, m_2)\). We let \(x \in X\) be arbitrary with \(m_1 = \pi_{M_1}(x)\). The isomorphism of the topological Mackey actions of \(h((f^1_{\alpha} \circ \phi^1_{t})(\tau, x), \pi_{M_1}(x))\) and \((f^2_{\alpha} \circ \phi^2_{t})(\tau, x)\) is induced by the homeomorphism \((x, t) \mapsto (x, t + b'(x))\) of \(X \times \mathbb{R}\). Therefore, we map \(m_1 \mapsto O_{\phi_{1,c}}(m_1, 0) \to O_{\phi_{1,h}}(m_1, -b''(m_1)) \to C_{m_1}\), in which the closed set

\[
C_{m_1} = \{(x', t') \in X \times \mathbb{R} : \pi_{M_1}(x') = \phi^1_t(m_1), t' = h(t, m_1) + b'(x') - b''(m_1), t \in \mathbb{R}\}
\]

is an element in the topological Mackey range of \((f^2_{\alpha} \circ \phi^2_{t})(\tau, x)\). The set \(C_{m_1}\) is then mapped to the unique element \(\iota(m_1) \in M_2\) with \(\pi_{M_2}(\iota(m_1)) \times \{0\} = C_{m_1} \cap (X \times \{0\})\). However, for \(x' = x\) and \(t = 0\), we obtain that \((x, b'(x) - b''(m_1)) \in C_{m_1}\), and thus for all \(x \in X\) holds

\[
\pi_{M_2}(x) = \phi^2_{b'(x) - b''(m_1)}(x) \circ \iota \circ \pi_{M_1}(x).
\]

The cocycles \((f_{\alpha} \circ \phi_{1})(\tau, x)\) and \((f_{\alpha} \circ \phi_{2})(\tau, x)\) are cohomologous as well as \((c \phi^1_{t} \circ \phi^1_{t})(\tau, x)\) and \((c \phi^2_{t} \circ \phi^2_{t})(\tau, x)\), hence we can conclude that \(|(1 + g_1)(t, \pi_{M_1}(x)) - (1 + g_2)(ct, \pi_{M_2}(x))|\) is uniformly bounded for all \((t, x) \in \mathbb{R} \times X\). Moreover, from equality (2.13) and the cocycle identity it follows that \(|(1 + g_2)(t, \iota \circ \pi_{M_1}(x)) - (1 + g_2)(ct, \pi_{M_2}(x))|\) is uniformly bounded for all \((t, x) \in \mathbb{R} \times X\). Hence the real-valued function \((1 + g_1)(t, m_1) - (1 + g_2)(ct, \pi_{M_1}(x))\) is uniformly bounded for all \((t, m_1) \in \mathbb{R} \times M_1\). However, since both of the mappings \((t, m_1) \mapsto (1 + g_1)(t, m_1)\) and \((t, m_1) \mapsto (1 + g_2)(ct, \pi_{M_1}(x))\) are cocycles of the minimal compact metric flow \((M_1, \phi^1_t : t \in \mathbb{R})\), these cocycles are cohomologous.

Acknowledgement. The author thanks Professor Jon Aaronson and Professor Eli Glasner for useful discussions and encouragement. The author also thanks an anonymous referee for comments which have led to improvements and extensions of this paper.
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