Pre-acceleration from Landau–Lifshitz series

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Received June 26, 2013; Revised October 19, 2013; Accepted October 21, 2013; Published December 1, 2013

The Landau–Lifshitz equation is considered as an approximation of the Abraham–Lorentz–Dirac equation. It is derived from the Abraham–Lorentz–Dirac equation by treating radiation reaction terms as a perturbation. However, while the Abraham–Lorentz–Dirac equation has pathological solutions of pre-acceleration and runaway, the Landau–Lifshitz equation and its finite higher-order extensions are free of these problems. So it seems mysterious that the properties of the solutions of these two equations are so different. In this paper we show that the problems of pre-acceleration and runaway appear when one considers a series of all-order perturbation which we call the Landau–Lifshitz series. We show that the Landau–Lifshitz series diverges in general. Hence a resummation is necessary to obtain a well-defined solution from the Landau–Lifshitz series. This resummation leads the pre-accelerating and the runaway solutions. The analysis focuses on the non-relativistic case, but we can extend the results obtained here to the relativistic case, at least in one dimension.

Subject Index A00, A01, G03, J25

1. Introduction

A charged particle emits radiation when it is accelerated. Since the radiation carries energy and momentum, so the conservation laws require that the equation of motion for the charged particle should be modified by a friction term. One such equation is the Abraham–Lorentz–Dirac (ALD) equation [1–3], which appears to be a third-order differential equation. The ALD equation itself is well established, but it has the infamous problem of runaway solutions which are solutions that describe the charged particle infinitely accelerated to the speed of light even when the external force vanishes. One can remove such runaway solutions by imposing a regular boundary condition at the future infinity. But then the solutions must be accelerated before the external force is applied, which is against causality. This is called pre-acceleration.

A programmatic solution to this problem is to treat the backreaction term as a perturbation. The Landau–Lifshitz equation (LL equation) is derived as the leading order of the perturbation series [4]. In contrast to the ALD equation, the LL equation is a second-order differential equation with the backreaction terms written in terms of the external force and its derivatives. When the external force vanishes, the backreaction also vanishes; neither the pre-acceleration nor the runaway occurs in the LL equation. This feature also holds for the higher orders of the perturbation. For practical purposes it is preferred that the LL equation is free of the problems of pre-acceleration and runaway. But this may also make the relation between the LL equation and the ALD equation appear to be mysterious, since the behaviors of their solutions are very different.
It is believed that the LL equation is valid for cases where the acceleration of the particle is sufficiently small. On the other hand, with the rapid progress of laser technologies, the intensities of lasers have reached the order of $10^{22}$ W cm$^{-2}$ [5–8]. In such a strong electromagnetic field, the radiation reaction is no longer negligible (for example, see [9]), and the acceleration of the charged particle becomes very large. Then the validity of the LL equation becomes important, and the differences between the solutions of the ALD and LL equations have also investigated numerically (see [10,11], for example).

In this paper, we investigate the analytical properties of an infinite series of all-order perturbation (the Landau–Lifshitz series). Our calculation is focused on the non-relativistic case, but the results are extended to cases of a relativistic charged particle moving in one dimension (see Appendix A). We find that the LL series diverges in general, and is an asymptotic series of the solutions of the ALD equation. Numerically, this means that to obtain the results of the best approximation, we have to stop the calculation at an appropriate order of the LL series; the higher-order terms don’t always make the results better. Theoretically, since the series diverges, a resummation is necessary to obtain well-defined solutions from the LL series. Although none of the terms of the LL series has the problems of pre-acceleration and runaway, the resummation can lead to these pathological solutions. Generally, a function may have different asymptotic expressions in different domains (the Stokes phenomenon), so the resummation of an asymptotic series is not unique. The pre-accelerating solution and the runaway solution appear from different domains of the asymptotic expression.

The paper is organized as follows. In Sect. 2, we derive the general properties of the LL series. After briefly reviewing the ALD and LL equations, we introduce the LL series and show that the convergence radius of the LL series is generally zero. We show that the Borel resummation will give us the pre-accelerating solution. In Sect. 3, we investigate the LL series for three examples of external force: a Gaussian function, a regularized step function, and a Fourier mode. In the first example, we perform the resummation explicitly and find the pre-accelerating solution and the runaway solution. In the second example, we show that the LL series also has the problems of pre-acceleration and runaway, in a way that is quite similar to the case of the ALD equation. In the third example, we find that the singular behavior of the LL series is caused by high-energy modes of the external force. Section 4 is devoted to conclusions and discussions.

**2. Landau–Lifshitz series**

The ALD equation is given by

$$\ddot{z}^\mu_{\text{ALD}} = \frac{e}{mc^2} F^{\mu\nu} z_{\text{ALD},\nu} + \frac{e^2}{6\pi mc^2} (\ddot{z}^\mu_{\text{ALD}} + \dot{z}^\mu_{\text{ALD}} \ddot{z}_{\text{ALD},\nu}) + \cdots.$$  \hspace{1cm} (1)

The first term on the right-hand side is the external force, while the second one represents the backreaction of the radiation. As we will show later, the ALD equation has problems of runaway and pre-acceleration. To avoid these pathological solutions, one possibility is to treat the backreaction term,

$$F^{\mu\nu}_{\text{ALD}} = \frac{e^2}{6\pi mc^2} (\dddot{z}^\mu_{\text{ALD}}^2 + \dddot{z}^\mu_{\text{ALD}}^2),$$  \hspace{1cm} (2)

as a perturbation. For the case that the backreaction is absent, the equation of motion is given by

$$\ddot{z}^\mu_{0} = \frac{e}{mc^2} F^{\mu\nu} z_{0,\nu}.$$
Substituting the above $\ddot{z}^\mu$ in the backreaction force (2), one obtains the equation

$$\ddot{z}^\mu_{LL} = \frac{e F^\mu\nu}{mc^2} \dot{z}_{LL,\nu} + \frac{e^2}{6\pi mc^2} \left( \frac{e \partial_\alpha F^\mu\nu}{mc^2} \dot{z}^\alpha_{LL,\nu} + \frac{e^2 F^\mu\nu F_{\nu\rho}}{m^2 c^4} \dot{z}^\rho_{LL} \right) ;$$

(3)

this is called the Landau–Lifshitz equation. Since the backreaction force of the LL equation is written in terms of the external force and its derivatives, the acceleration of the particle vanishes when $F^\mu\nu = 0$. So the LL equation is free from the problems of pre-acceleration and runaway. One can also continue to substitute the LL equation (3) in Eq. (2) and obtain higher-order correction terms for the LL equation. These correction terms are general very complicated, but all of them also vanish when $F^\mu\nu = 0$. The finite higher-order extensions of the LL equation are also free of the problems of pre-acceleration and runaway.

Note that though the treatment described above is covariant in form, the condition for the perturbation to be a good approximation is not covariant. The backreaction term (2) is a different 4-vector from the external force, so the relationship between their strengths depends on observers. Here we assume that the backreaction term (2) is much smaller than the external force in the comoving frame. In quantities of the comoving observer, the ALD equation is written in a simpler form which is quite similar to the non-relativistic case.

In the following, we consider non-relativistic motions of the charged particle. The non-relativistic equation of the backreaction is linearized and becomes much simpler than the relativistic case. Consider the path of the particle $z^\mu$ satisfying the condition

$$\left| \frac{d z^i}{dt} \right| \ll c .$$

(4)

Then the ALD equation (1) becomes a simple form

$$\dot{\vec{v}}_{ALD}(t) - g \vec{v}_{ALD}(t) = \vec{F}_{ext}(t),$$

(5)

with $\vec{v} = (dz^1/dt, dz^2/dt, dz^3/dt)$ and $g = \frac{e^2}{6\pi mc^2}$. We note that the above equation takes the same form as the equation of motion of a relativistic charged particle that moves in one dimension (Appendix A). Here, we consider the external force $\vec{F}_{ext}(t)$ to be a function which only depends on time and does not depend on the spatial coordinates or velocities of the charged particle. So, in our case, the external force is purely electric: $\vec{F}_{ext}(t) = e \vec{E}_{ext}(t)$. The general solution to (5) is given by

$$\dot{\vec{v}}_{ALD}(t) = -\frac{e^t/g}{g} \left( \int_0^t \frac{\vec{F}_{ext}(t')}{m} e^{-t'/g} dt' + \tilde{C} \right),$$

where $\tilde{C}$ is a constant vector. Generally, $\vec{v}_{ALD}$ diverges at the future infinity, $t \to \infty$, due to the factor $e^t/g$. This is called the problem of runaway. In order to obtain a realistic solution, we would like to eliminate the divergence at the future infinity. This can be done by choosing the constant $\tilde{C}$ to be

$$\tilde{C} = -\int_0^\infty \frac{\vec{F}_{ext}(t')}{m} e^{-t'/g} dt'.$$
Substituting the above equation in (5) and changing the integrating variable to \( s = \frac{t' - t}{g} \), one obtains the following solution:

\[
\dot{v}_{\text{pre}}(t) = \frac{1}{m} \int_0^\infty \bar{F}_{\text{ext}}(t + gs)e^{-s} ds.
\]  

(6)

This is called the pre-accelerating solution. The value of \( \dot{v}_{\text{pre}}(t) \) depends on the values of the external force in the future, which is against causality. To see the acausal feature more clearly, we consider the example

\[
\bar{F}_{\text{ext}} = \bar{F}_0 \theta(t),
\]  

(7)

where \( \bar{F}_0 \) is a constant vector and \( \theta(t) \) is the step function which takes the value 0 for negative \( t \) and \( \theta = 1 \) for positive \( t \). With the above external force, the pre-accelerating solution \( \dot{v}_{\text{pre}}(t) \) becomes

\[
\dot{v}_{\text{pre}}(t) = \frac{\bar{F}_0}{m} \left\{ \theta(-t)e^{-\frac{t}{g}} + \theta(t) \right\}.
\]  

(8)

which takes a finite value even before the external force is applied, \( t < 0 \).

Now we consider the LL equation. The \( n \)th-order extension of the LL equation is given by

\[
\dot{v}_{\text{LL}, n} = \left( 1 + g \frac{d}{dt} + g^2 \frac{d^2}{dt^2} + \cdots + g^n \frac{d^n}{dt^n} \right) \frac{\bar{F}_{\text{ext}}}{m}.
\]  

The Landau–Lifshitz series is given by

\[
\dot{v}_{\text{LL}}(t, g) = \sum_{n=0}^{\infty} g^n \frac{d^n}{dt^n} \frac{\bar{F}_{\text{ext}}(t)}{m}.
\]  

(9)

note that \( \dot{v}_{\text{LL}}(t, g) \) is a power series of \( g \) at each time \( t \). Formally, we may write the sum into the form \( \sum (g \frac{d}{dt})^n = (1 - g \frac{d}{dt})^{-1} \), and see that \( \dot{v}_{\text{LL}}(t, g) \) satisfies equation (5). Obviously, \( \dot{v}_{\text{LL}, n}(t) \) is linear in the external force, so it is free of the problems of pre-acceleration and runaway. However, this is only true for finite \( n \) of \( \dot{v}_{\text{LL}, n}(t) \), but not for the infinite series \( \dot{v}_{\text{LL}}(t) \).

The series (9) does not converge in general. To see this, consider the external force \( \bar{F}_{\text{ext}}(t) \) to be an analytic function. Then its Taylor series at \( t_0 \) is written as

\[
\bar{F}_{\text{ext}}(t) = \sum_{n=0}^{\infty} \bar{f}(n, t_0)(t - t_0)^n.
\]  

(10)

Generally, the above series is convergent in some region given by \( |t - t_0| < r \). The coefficients behave like \( \bar{f}(n, t_0) \sim \bar{a} r^{-n} \) at large \( n \). So the LL series becomes

\[
\dot{v}_{\text{LL}}(t_0) \sim \frac{\bar{a}}{m} \left( \frac{\bar{g}}{r} \right)^n n!,
\]  

(11)

which diverges for any finite value of \( \frac{\bar{g}}{r} \). Generally, the LL series is an asymptotic series of \( g \). An asymptotic series is a series that, even though it diverges everywhere, can still make a good approximation by stopping the summation at a finite order. One can improve the approximation by including the higher-order terms, but then the valid region of the approximation becomes shorter. When one includes all the terms of the series, the region of the approximation becomes zero and the summation itself diverges.
One of the solutions to obtaining a well-defined function from the asymptotic series is the Borel resummation. Consider a power series of \( g \),

\[
Z(g) = \sum_{n=0}^{\infty} Z_n g^n,
\]

where the coefficients \( Z_n \) don’t have to make the series \( Z(g) \) converge. The Borel transformation \( Z_B(t, \alpha) \) is defined as

\[
Z_B(s, \alpha) = \sum_{k=0}^{\infty} \frac{Z_k s^k}{\Gamma(k + \alpha)}.
\]

If it is possible to define \( Z_B(t, \alpha) \) in the region \( 0 \leq t \leq \infty \) by analytic continuation, then we can obtain a resummation \( Z_R(g) \):

\[
Z_R(g) = \int_{0}^{\infty} s^{\alpha - 1} Z_B(gs, \alpha) e^{-s}.
\]

Generally, \( Z_R(g) \) doesn’t depend on the values of \( \alpha \), so one can choose an \( \alpha \) to make the calculation easy.

By employing the Borel resummation, one can obtain the pre-accelerating solution from the Landau–Lifshitz series (9). The Borel transform \( \dot{\vec{v}}_B(t, s) \) is given by

\[
\dot{\vec{v}}_B(t, s) = \frac{1}{m} \sum_{k=0}^{\infty} \frac{s^k}{k!} \frac{d^k}{dt^k} \vec{F}_{\text{ext}}(t) = \frac{\vec{F}_{\text{ext}}(t + s)}{m};
\]

here we take \( \alpha = 1 \) in (13). The resummation \( \dot{\vec{v}}_R(t, g) \) is given by

\[
\dot{\vec{v}}_R(t, g) = \frac{1}{m} \int_{0}^{\infty} ds \; e^{-s} \vec{F}_{\text{ext}}(t + gs),
\]

which is exactly the same as the pre-accelerating solution (6). It is interesting that only the pre-accelerating solution appears from the Borel resummation of the LL series, while one has both the runaway solution and the pre-accelerating solution from the ALD equation (5). Generally, the issues of resummation of an asymptotic series are not unique. This corresponds to the fact that a function can have different asymptotic expressions, depending on the different domains. In the next section, we will see this explicitly and find that the runaway solution corresponds to the domains of \( g \rightarrow -0 \), while the pre-accelerating solution corresponds to \( g \rightarrow +0 \).

We note that, in the ALD equation (5), the backreaction term \( g \ddot{\vec{x}}_{\text{ALD}} \) is the highest derivative. The LL series is obtained by treating this term as a perturbation. It is known that such kinds of perturbation have singular behaviors in general (see [12], for example). This is because the number of initial values that one needs for specifying a solution is determined by the highest-order term. In our case, Eq. (5) is a third-order differential equation which needs the value of \( \dot{\vec{v}}(0) \) in additional to \( \vec{v}(0) \) and \( \vec{x}(0) \) to specify a solution \( \vec{x}(t) \). However, for the case that the perturbation term is absent, only \( \vec{v}(0) \) and \( \vec{x}(0) \) are required. This implies that, in the limit \( g \rightarrow 0 \), something nontrivial happens. So one can expect, from the general discussions, that the LL series (9) may diverge and may be an asymptotic series. But the behaviors of the pre-accelerating solution and the runaway solution depend on the details of the equation. For example, the solution (6) can be neither pre-accelerating nor runaway if the constant \( g \) takes a negative value. To understand the properties of the LL series better, we would like to investigate the resummation in detail for explicit examples.
3. Pre-acceleration and runaway solutions

In this section, we investigate the LL series in three cases. First, we investigate a case where the external force takes the form of a Gaussian function which is localized in time. We see that the runaway solution appears as an example of the Stokes phenomenon. After that, in order to investigate the pre-acceleration in detail, we consider the second example where the external force takes the form of a regularized step function. From different resummation solutions we obtain the pre-accelerating solutions, the runaway solutions, or solutions containing discontinuity. Then, in order to investigate the physical origins of the problems, we consider the third example where the external force takes the form of Fourier modes. We find that the problems are caused by the modes of high frequency which are outside the region of classical mechanics.

For simplicity, we consider the motions of the particle to be in one dimension. The extension to three dimensions in the non-relativistic case is straightforward, since each direction of the particle in the equation of motion (5) is independent of the others.

3.1. Gaussian function

Consider the external force \( F(t) \) taking the form

\[
F_{\text{ext}}(t) = f_0 e^{-\alpha t^2},
\]

where \( f_0 \) and \( \alpha \) is constants. We consider \( \alpha \) small enough so that \( \frac{1}{\sqrt{\alpha}} \) is a macroscopic timescale.

The Landau–Lifshitz series in this case is written in terms of Hermite polynomials:

\[
\dot{v}_{\text{LL}}(t) = \sum_{n=0}^{\infty} g^n \frac{d^n}{dt^n} F_{\text{ext}}(t) = e^{-\alpha t^2} \frac{f_0}{m} \sum_{n=0}^{\infty} (-\sqrt{\alpha}g)^n H_n(\sqrt{\alpha}t).
\]

By using the following equations

\[
H_{2s}(x) = \sum_{l=0}^{s} \frac{(-1)^{s-l} 2^{2l}(2s)!}{(2l)! (s-l)!} x^{2l}
\]

\[
H_{2s+1}(x) = \sum_{l=0}^{s} \frac{(-1)^{s-l} 2^{2l+1}(2s+1)!}{(2l+1)! (s-l)!} x^{2l+1},
\]

one can see that the convergence radius of (18) is 0 for any values of \( t \).

On the other hand, the pre-accelerating solution is given by

\[
\dot{v}_{\text{pre}}(t) = \frac{1}{m} \int_{0}^{\infty} F(t + gs) e^{-s} ds = \frac{f_0}{m} \int_{0}^{\infty} e^{-\alpha (t + gs)^2 - s} ds
\]

\[
= \frac{f_0}{m} \exp \left[ \frac{4\alpha gt + 1}{4\alpha g^2} \right] \sqrt{\pi} \ \text{erfc} \left( \frac{2\alpha gt + 1}{2\sqrt{\alpha} g} \right),
\]

where the error function \( \text{erfc}(x) \) is defined by

\[
\text{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_{x}^{\infty} e^{-s^2} ds,
\]

and \( \text{erfc}(0) = 1 \) while \( \text{erfc}(\infty) = 0 \). To relate the solution (21) and the LL serics (18), we need to find a power expansion of (21) around \( g = 0 \). This can be done by using the expansion of \( \text{erfc}(x) \) at
infinity \( x \to \infty \), which is given by
\[
\text{erfc}(x) \sim \sqrt{\frac{2}{\pi}} e^{-x^2} \sum_{k=0}^{\infty} \frac{(-1)^k (2k-1)!!}{(\sqrt{2} x)^{2k+1}},
\]
and is known as an asymptotic expansion. By using the above equation, we have
\[
\dot{v}_{\text{pre}}(t) \sim e^{-\alpha t^2} \frac{f_0}{m} \sum_{k=0}^{\infty} (-1)^k \frac{(2k-1)!!}{(2\alpha t + 1)^{2k+1}} (\sqrt{2} \alpha g)^{2k}.
\]
as \( \frac{2\alpha t + 1}{2\sqrt{\alpha g}} \to \infty \). And using the equation
\[
\frac{1}{(1 + x)^{2k+1}} = \sum_{i=0}^{\infty} \frac{(-1)^i (2k + i)!}{i! (2k)!} x^i,
\]
we obtain the power series
\[
\dot{v}_{\text{pre}}(t) \sim e^{-\alpha t^2} \frac{f_0}{m} \sum_{k=0}^{\infty} \sum_{i=0}^{\infty} (-1)^{k+i} \frac{(2k + i)!}{i! k!} \frac{(2\alpha g)^i}{(\sqrt{\alpha g})^{2k}}.
\]
With the replacement of \( 2k + i \sim n \) and \( i \sim l \), we see that the above series is the same as the LL series (18).

Here, we use the symbol “\( \sim \)” for the asymptotic expansion. This is because the correspondence between the asymptotic series and the original function is not one to one. One can obtain different solutions from the LL series. For example, \( \text{erfc}(x) \) can have another asymptotic expression as \( x \to -\infty \),
\[
\text{erfc}(x) \sim 2 + \sqrt{\frac{2}{\pi}} e^{-x^2} \sum_{k=0}^{\infty} \frac{(-1)^k (2k-1)!!}{(\sqrt{2} x)^{2k+1}},
\]
which is quite similar to (23), but has a different first term. So we can also have
\[
\dot{v}_{\text{LL}}(t) \sim \dot{v}_{\text{pre}}(t) - \frac{f_0}{m} \frac{\exp\left[-\frac{1}{4\alpha g^2}\right]}{g \sqrt{\alpha}} \sqrt{\pi} e^\frac{t^2}{2g}.
\]
as \( g \to -0 \). The second term in the above equation takes a form \( e^{t^2/g} \), which describes the runaway as \( t \to \infty \). In the real world, \( g \) is positive. In this sense, it is natural to choose the former resummation method which gives us the pre-accelerating solution from the LL series.

### 3.2. Regularized step function

The problem of the pre-acceleration is very clear in the case where the external force takes the form of a step function. Then, the solution (8) is against causality in an explicit form. However, the LL series is written in terms of derivatives of the external force. So we need a regularization to make the LL series well defined. We consider an external force \( F(t, a) \) taking the form
\[
F(t, a) = \frac{f_0}{1 + e^{-\alpha t}},
\]
with \( a \) a positive parameter. \( F(t, a) \) is a regularized step function and satisfies
\[
\lim_{a \to \infty} F(t, a) = f_0 \theta(t).
\]
The poles of \( F(t, a) \) are given by \( t = \frac{(2n+1)^2 \pi^2}{a^2} \). So the power expansion of \( F(t, a) \) around \( t_0 \) only converges at a finite region \( (t - t_0) < r \). According to the general discussions in Sect. 2, the LL series diverges.
The explicit form of the LL series can be written by

\[
\ddot{v}_{\text{LL}}(t) = \begin{cases} \\
\frac{f_0}{m} \sum_{n=0}^{\infty} \frac{(-1)^n}{1 - agn} \left(\sum_{l=0}^{\infty} (\pm ag)^l e^{ant} \right) & \text{(for } t < 0), \\
\frac{f_0}{m} \sum_{n=0}^{\infty} \frac{(-1)^n}{1 + agn} \left(\sum_{l=0}^{\infty} e^{-ant} \right) & \text{(for } t > 0), \end{cases}
\]  

(31)

where the summation of \(l\) diverges. One may make a naive resummation of the above series by simply exchanging the order of \(\sum_{n}\) and \(\sum_{l}\), then replacing \(\sum (\pm ag)^l\) with \(\frac{1}{1 \pm ag}\). The result is given by

\[
\ddot{v}_R(t) = \begin{cases} \\
\frac{f_0}{m} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{1 - agn} e^{ant} & \text{(for } t < 0), \\
\frac{f_0}{m} \sum_{n=0}^{\infty} \frac{(-1)^n}{1 + agn} e^{-ant} & \text{(for } t > 0). \end{cases}
\]  

(32)

It is easy to check that, at the limit \(a \to \infty\), the above series becomes

\[
\ddot{v}_R(t) = \frac{f_0}{m} \theta(t),
\]  

(33)

which might seem nice since it doesn’t have the behaviors of either the pre-acceleration or the runaway.

However, \(\ddot{v}_R(t)\) has a discontinuity at \(t = 0\) even for finite \(a\):

\[
\ddot{v}_R(+) - \ddot{v}_R(-) = \frac{f_0}{m} \left\{ 1 + 2 \sum_{n=1}^{\infty} \frac{(-1)^n}{a^2 n^2} \right\} = \frac{f_0}{m} \frac{\pi}{n} \sin \frac{\pi}{ag}.
\]  

(34)

We would like to find a solution that is analytic while the external force is so. One of the issues is the analytic continuation from \(t < 0\) to \(t > 0\) (or vice versa). This can be done by writing \(\ddot{v}_R(t)\) in terms of a hypergeometric function:

\[
\ddot{v}_R(t) = \begin{cases} \\
\frac{f_0}{m} \frac{F\left(1,1-1/ag,2-1/ag; 1-ag ; -e^{at}\right)}{(1-ag)e^{-at}} & \text{(for } t < 0), \\
\frac{f_0}{m} F\left(1, 1/ag, 1 + 1/ag; -e^{-at}\right) & \text{(for } t > 0). \end{cases}
\]  

(35)

We obtain two solutions from the above. One is

\[
\ddot{v}_{R-}(t) = \frac{f_0}{m} \frac{F\left(1, 1 - 1/ag, 2 - 1/ag; -e^{at}\right)}{(1 - ag)e^{-at}},
\]  

(36)

and the other is

\[
\ddot{v}_{R+}(t) = \frac{f_0}{m} F\left(1, 1/ag, 1 + 1/ag; -e^{-at}\right).
\]  

(37)

The behavior of \(\ddot{v}_{R-}(t)\) and \(\ddot{v}_{R+}(t)\) at the limit \(a \to \infty\) is obtained by using the connection formula

\[
F\left(1, 1/ag, 1 + 1/ag; -x\right) = \frac{1}{1 - ag} x^{-1} F\left(1, 1 - 1/ag, 2 - 1/ag; -x^{-1}\right) + \frac{\pi}{ag \sin \frac{\pi}{ag}} x^{-1}. \]

(38)
We see that $\dot{v}_{R-}(t)$ is the runaway solution
\begin{equation}
\lim_{a \to \infty} \dot{v}_{R-}(t) = \frac{f_0}{m} \theta(t)(1 - e^{t/g}), \tag{39}
\end{equation}
which diverges at the future infinity, $t \to \infty$. On the other hand, $\dot{v}_{R+}(t)$ is the pre-accelerating solution
\begin{equation}
\lim_{a \to \infty} \dot{v}_{R+}(t) = \frac{f_0}{m} \theta(t) + \theta(-t)e^{t/g}, \tag{40}
\end{equation}
which starts accelerating before the external force is applied.\(^1\)

Here we obtain the different solutions from the different methods of resummation. As we showed, it is possible to perform the resummation in a way that maintains both the regularity at the future infinity and causality. But then the solution contains a discontinuity even when the external force is very smooth. If one requires the solution to be analytic for a smooth external force, then the situation becomes quite similar to the ALD equation: keeping the causality, the solution contains the runaway at the future infinity; and keeping the regularity at future infinity, then the solution contains the pre-acceleration.

### 3.3. Fourier modes

Consider the external force $F(t)$ taking the following form:
\begin{equation}
F(t) = f_0 \text{Re}[e^{-i\omega t}]. \tag{41}
\end{equation}

The Landau–Lifshitz series is given by
\begin{equation}
\dot{v}_{LL}(t) = \frac{f_0}{m} \sum_{n=0}^{\infty} \text{Re}[(-i\omega)^n e^{-i\omega t}]. \tag{42}
\end{equation}

Unlike the previous two examples, $\dot{v}_{LL}(t)$ converges in the region $g\omega < 1$, and is same for the pre-accelerating solution:
\begin{equation}
\dot{v}_{LL}(t) = \frac{f_0}{m} \int_0^\infty \text{Re}[e^{-i\omega t - i\omega gs - s}] ds = \frac{f_0}{m} \text{Re} \left[ \frac{e^{-i\omega t}}{1 + i\omega g} \right]. \tag{43}
\end{equation}

For $\omega < 1/g$ the LL series is just the Taylor expansion of the pre-accelerating solution. Nothing special happens.

However, for $\omega > 1/g$, the LL series doesn’t converge. For an electron, $g$ is given by $\frac{e^2}{6\pi mc^2}$ and $\frac{1}{g} \sim 100$ MeV. The modes with frequency $\omega > 1/g$ are obviously out of the region of classical electrodynamics. It is these high-energy modes that cause the singular behavior of the LL series. If we cut off these high-energy modes, the LL series converges. This can be shown by considering the external

\(^1\) One can also obtain the pre-accelerating solution directly from Eq. (6):
\begin{equation}
\dot{v}_{\text{pre}}(t) = \frac{f_0}{m} \int_0^\infty ds \frac{e^{-s}}{1 + e^{-s + gs}} = \frac{f_0}{m} \int_0^1 dy \frac{1}{1 + e^{-yag}}
= \frac{f_0}{m} F(1, 1/ag, 1 + 1/ag; -e^{-at}) = \dot{v}_{R+}(t),
\end{equation}
where we changed the variable of integration by $y = e^{-s}$. 

force as

\[
F_{\text{ext}}(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} f(\omega) e^{i\omega t}.
\]

(44)

Then define the regularized external force \( F_R(t, \Omega) \) by

\[
F_R(t, \Omega) = \int_{-\Omega}^{\Omega} \frac{d\omega}{2\pi} f(\omega) e^{i\omega t},
\]

(45)

with the cutoff \( \Omega < 1/g \). Then the \( n \)th derivative of \( F_R(t, \Omega) \) satisfies

\[
g^n \frac{d^n}{dx^n} F_R(t, \Omega) = \int_{-\Omega}^{\Omega} \frac{d\omega}{2\pi} (i\omega)^n f(\omega) e^{i\omega t} \leq 2 \int_{0}^{\Omega} \frac{d\omega}{2\pi} \omega^n g^n f_{\text{max}} = \frac{f_{\text{max}}}{\pi} g^n \frac{\Omega^{n+1}}{n+1},
\]

(46)

where \( f_{\text{max}} \) denotes the maximum value of \( |f(\omega)| \) for \( -\Omega \leq \omega \leq \Omega \). Then LL series converges,

\[
\dot{v}_{\text{LL}}(t, \Omega) \sim \frac{1}{m} \sum_n g^n \frac{\Omega^{n+1} f_{\text{max}}}{(n+1)\pi},
\]

(47)

for \( \Omega < 1/g \).

4. Conclusions and discussion

In this paper, we investigated the analytic properties of the Landau–Lifshitz series. We showed that the LL series is an asymptotic series and investigated the methods of resummation. The Borel resummation gives us the pre-accelerating solution from the LL series. But a different resummation can also be performed and gives us the runaway solution. We showed this in two explicit examples and found that the runaway solution and the pre-accelerating solution correspond to the different regions of the asymptotic expressions.

In the third example, we showed that the singular behavior of the LL series is caused by high-energy modes of the external force. Since these modes are out of the region of classical dynamics, one can avoid the divergence of the LL series by simply cutting off these modes. However, this issue violates the Lorentz invariance. It is important to find methods of cutoff that can also be applied to the relativistic LL series.

The analysis in this paper focused on the non-relativistic case. The results can be extended to one-dimensional relativistic motion. However, the relativistic ALD equation is generally nonlinear, which may cause nontrivial effects. We would like to investigate these issues in future work.

As we have shown, each order of the LL series does not have the problems of pre-acceleration and runaway, but the series itself diverges and the resummation causes the problems. It is interesting to note that even though the perturbation is regular and causal, the resummation can lead to non-perturbative effects. It would very interesting to investigate the correspondence with quantum field theory. One natural approach to the problems of the ALD equation is to start from quantum field theory and derive the equation of radiation reaction perturbatively (see [14–18], for example). However, the perturbation of the quantum field theory corresponds to the perturbation of the LL series at \( \hbar \to 0 \). So to approach the problems of pre-acceleration and runaway from quantum field theory, one may have to sum up the series of all-order perturbation. And the classical limit of a quantum theory is given by \( \hbar \to 0 \), while it is known that the expansion of \( \hbar \) is also an asymptotic expansion. The problems of the radiation reaction are not only for electromagnetic dynamics, so it may be possible to find some simple models which contain the same problems but can be solved exactly.
Acknowledgements
S.Z. would like to thank S. Iso for encouragement and discussions, and J. Koga and B. Bulanov for discussions.

Appendix A. Equation of motion for a relativistic particle in one dimension
The equation of motion for a relativistic particle is complicated. However, if one considers that the motion of the particle is constrained to one dimension, the equation of motion can be simplified a lot.

For one-dimensional motion, the space coordinate and the time coordinate of the particle can be written by

$$\frac{dz^\mu}{ds} = (\cosh (\zeta(s)), \sinh (\zeta(s)), 0, 0). \quad (A1)$$

Then the radiation reaction term turns out to be

$$\ddot{z}^\mu + \dot{z}^\mu \ddot{z}^\nu \ddot{z}_\nu = \ddot{\zeta} (\sinh \zeta, \cosh \zeta, 0, 0), \quad (A2)$$

and the equation of motion (1) becomes

$$\frac{d\zeta}{d\tau} - \frac{e^2}{6\pi mc^3} \frac{d^2\zeta}{d\tau^2} = \frac{F_{\text{ext}}(\tau)}{mc}, \quad (A3)$$

where $F_{\text{ext}}(\tau) = eF^{10}(\tau(\tau))$. By replacing $\zeta = \frac{V(\tau)}{c}$, one obtains the equation

$$\frac{dV}{d\tau} - \frac{e^2}{6\pi mc^3} \frac{d^2V}{d\tau^2} = \frac{F_{\text{ext}}}{m}, \quad (A4)$$

which takes the same form as the non-relativistic equation (5).

It is worth noting that although the above equation takes the same form as the non-relativistic case, the external force $F_{\text{ext}}(\tau)$ may have a dependence on the coordinates $z^\mu$ through $\tau$. This dependence may cause nontrivial effects.

Appendix B. Notes on asymptotic expansions
A sequence $\{\varphi_n(x)\}$ is called an asymptotic sequence at $x = a$ when it satisfies

$$\varphi_{n+1}(x) = o(\varphi_n(x)), \quad \text{at} \ x \to a. \quad (B1)$$

We say that a function $f(x)$ is expanded in an asymptotic series

$$f(x) \sim \sum_{n=0}^\infty a_n \varphi_n(x) \quad (B2)$$

at $x \to a$ when it is satisfied that

$$f(x) - \sum_{n=0}^N a_n \varphi_n(x) = o(\varphi_N(x)), \quad \text{at} \ x \to a. \quad (B3)$$

Generally, two different functions can have the same asymptotic expansion.

References