The meson as a bound system in 2D quantum chromodynamics

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We investigate the Bethe–Salpeter-like amplitude, which involves a system comprising a quark and an anti-quark combined by gauge fields, to satisfy gauge invariance in two dimensions. We derive the equation of motion for this system and construct a series of singular integral-differential equations. By solving these equations, we obtain an algebraic equation that determines the mass spectra for the \((n+1)\)th non-zero eigenvalue (where \(n = 0\) or a positive integer).

In this algebraic equation, the solution for large values of \(n\) is written in the same form as G. ’t Hooft’s W-K-B solution, and the zero mass solution arises under a boundary condition equivalent to the boundary condition that ’t Hooft derived. Thus, we show convincingly that zero mass is actually a physical mass.

1. Introduction

In 1974, Gerard ’t Hooft determined the 2D mass spectra of a meson [1]. He insisted that these spectra include the zero mass spectrum. Although his work produced spectra for only two dimensions, his solution demonstrated the possibility of solving the \(U(1)\) problem, at least in principle. Therefore, many papers [2–5] on this subject were published in the late 1970s. However, an explicit solution for this spectrum has not yet been found. The absence of a solution indicates that we must investigate ’t Hooft’s zero mass spectrum more deeply; even in 2D studies, Peccei [6] and Pak [7] arrived at different conclusions regarding the existence of the zero mass spectrum. In addition, we wish to emphasize that ’t Hooft obtained the zero mass spectrum as an eigenvalue under a different boundary condition than the boundary conditions applied to other mass spectra. In this respect, we must consider the work of Hornbostel et al., who used the quantization of the \(1+1\)-dimensional light-cone gauge theory [8]. They concluded that a meson has zero mass for quark mass \(m = 0\), even though the case involving the lightest state is ambiguous. Their reasoning was based on the fact that the continuum limit achieves the same form as ’t Hooft’s equation, which gives \(\mu^2 = 0\) for the \(\phi = 1\) solution when \(m = 0\), where \(\mu^2\) is the eigenvalue and \(\phi\) is the eigenfunction, which is the wave function in the \(P_0 - P_1\) momentum space. However, as mentioned earlier, this \(\phi = 1\) solution violates the boundary condition required for a physical state. In addition, the \(\phi(P) = 1\) solution in momentum space is a \(\delta\)-function in \(X\) space (specifically, in \(T - X\) space) and is written as \(\delta(X)\). Subsequently, the quark and anti-quark always occupy the same space-time position. This result is clearly unacceptable as a physical solution. Therefore, their assertion that zero mass can be treated as a physical mass was inconclusive. It remains crucial to consider whether ’t Hooft’s zero mass spectrum can be regarded as a physical mass spectrum.
In this paper, we consider the equation of motion based on the Bethe–Salpeter-like amplitude and obtain an algebraic equation that determines the meson mass spectra group consistent with 't Hooft's mass spectra group. In addition, we obtain the zero mass spectrum as an eigenvalue under the same boundary conditions applied to the other mass spectra. Because our boundary condition is equivalent to 't Hooft's boundary condition, we can argue convincingly that the zero mass spectrum is actually a physical mass.

2. Formulation

In order to derive equations within the framework of 't Hooft's model [1], we introduce the following hadronic operator, which was proposed by Suura [8]. For a confined system, Suura defined the Bethe–Salpeter-like amplitude as

$$X(1, 2) = \langle 0| q(1, 2) | P \rangle.$$  (1)

where the gauge-invariant bi-local operator $q(1, 2)$ is defined as

$$q_{\alpha \beta}(1, 2) = T^c q^+_{\beta}(2) P \exp \left( i g \int^2_1 d\vec{x} \vec{A}(\vec{x}) \frac{\lambda_a}{2} \right) q_{\alpha}(1).$$  (2)

Here, $\alpha$ and $\beta$ denote Dirac indices. The above definition is formulated in the time axial gauge ($A_0 = 0$) because we are interested in the Hamiltonian formalism. This description is extended to the non-Abelian gauge field by

$$q_{\alpha \beta}(1, 2) = T^c q^+_{\beta}(2) P \exp \left( i g \int^2_1 d\vec{x} \vec{A}_a(x) \frac{\lambda_a}{2} \right) q_{\alpha}(1).$$  (3)

$P$ denotes the path ordering, and the $\frac{\lambda_a}{2}$ components are the generators of the adjoint representation of the SU($N$) color gauge group. The trace is calculated for color spin $a$. Because the physics for the observable color singlet should be path-independent, Suura chose a straight line for zeroth order consideration. Justification of this choice is given in Ref. [9] based upon the Tamm–Dancoff method [10,11]. However, because we are presently interested in the 't Hooft model (which is 2D), choosing a straight line as the path produces an exact result.

From the definition in Eq. (3), the 2D version of the gauge-invariant bi-local operator is described by

$$q_{\alpha \beta}(1, 2) = T^c q^+_{\beta}(2) P \exp \left( i g \int^2_1 dx A^a(x) \frac{\lambda_a}{2} \right) q_{\alpha}(1).$$  (4)

In two dimensions, the initial Lagrangian densities are given as follows:

$$L = -\frac{1}{4} F^a_{\mu \nu} F^{a \mu \nu} + \bar{q}(x) \left\{ i \gamma^\mu \partial_\mu + g \gamma^\mu A^a_{\mu}(x) \left( \frac{\lambda_a}{2} \right) + m \right\} q(x),$$  (5)

$$F^a_{\mu \nu} = \partial_\mu A^a_{\nu}(x) - \partial_\nu A^a_{\mu}(x) + g f^{abc} A^b_{\mu}(x) A^c_{\nu}(x),$$

where $a$ represents color indices, $\gamma$ are the Dirac $\gamma$-matrices, and $\mu$ and $\gamma$ are two components of space-time.

Because we employ the $A_0 = 0$ gauge, the momentum canonically conjugate to $A_1(x)$ is $\dot{A}_1(x)$ which is $-E_1(x)$, and the momentum canonically conjugate to $\psi(x)$ is $i \dot{\psi}^+(x)$. Henceforth, we drop subscript 1.
Therefore, we obtain the following Hamiltonian:

\[
H = \int dx \left\{ \frac{1}{2} (E^a E^a) + \psi^\dagger(x) (-i \alpha^i D_{Ai} + m \beta) \psi(x) \right\},
\]

(6)

\[
D_{Ai} = \partial_i O^a - i g f^{abc} A^b_i O^c \quad O: \text{any operator},
\]

(7)

Here, \(D_{Ai}\) is the covariant derivative, and \(i = 1\) in this case.

We employ the metric system and \(\gamma\)-matrices in the following way, which was also employed by Casher et al. [12]:

\[
g^{00} = 1 \quad g^{11} = -1 \quad \gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \gamma^1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \alpha^1 = \alpha = \gamma^0 \gamma^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

The usual canonical quantization procedure leads to

\[
\begin{bmatrix} A^a(x, t), E^b(x', t) \end{bmatrix} = -i \delta^{ab} \delta(x - x'),
\]

(8)

\[
\{ \psi(x, t), \psi^\dagger(x', t) \} = \delta(x - x').
\]

(9)

Both the equal time commutations between the components of \(A\) and between the components of \(E\) are zero; the equal time anti-commutations between the two \(\psi\) and the two \(\psi^\dagger\) are also zero. These Hamiltonian density and commutation relations are invariant under a time-independent \(SU(N)\) transformation such that

\[
A^a(x) \rightarrow U A^a(x) U^+, \quad E^a(x) \rightarrow U E^a(x) U^+, \quad \psi(x) \rightarrow U \psi,
\]

where \(U = U(x)\) can be any \(N \times N\) unitary matrix function in which \(x\) satisfies \(\det U = 1\). Because the \(U(x)\) are time independent, the invariance group \(U(x)\) is generated by the \(x\)-dependent operators \(G^a\), which are conserved such that

\[
G^a(x) \equiv D_1 E^a(x) - g \rho^a(x),
\]

(10)

which subsequently produces the following:

\[
U \equiv \exp \left( i \int dx \lambda^a(x) \omega^a(x) \right).
\]

One easily obtains

\[
\begin{bmatrix} J^a(x, t), J^b(x', t) \end{bmatrix} = f^{abc} \delta(x - x') J^c(x, t),
\]

(11)

\[
\begin{bmatrix} G^a(x, t), G^b(x', t) \end{bmatrix} = -g f^{abc} \delta(x - x') G^c(x, t),
\]

(12)

\[
\begin{bmatrix} G^a(x, t), \psi(x', t) \end{bmatrix} = -g \delta(x - x') \frac{\lambda^a}{2} \psi(x, t),
\]

(13)

\[
\begin{bmatrix} G^a(x, t) E^b(x', t) \end{bmatrix} = -g f^{abc} \delta(x - x') E^c(x, t),
\]

(14)

\[
\begin{bmatrix} G^a(x, t) A^b(x', t) \end{bmatrix} = -g f^{abc} \delta(x - x') A^c(x, t) - i \delta^{ab} \partial_1 \delta(x - x').
\]

(15)

Consequently, \(G^a\) commutes with \(H\), and therefore \(G^a = 0\).
By commuting \( H \) with \( \psi \), we obtain the equation of motion for \( \psi \). Likewise, by commuting \( A^a \), we derive the equation of motion:

\[
D_{\alpha \mu} F^{\mu \nu}(x) = J^{\alpha \nu}(x) \quad (\mu = 0, 1; \nu \neq 0).
\]

However, one cannot obtain the equation corresponding to Gauss’s law \((D \cdot B^a(x) = 0\) may be obtained in the 4D case). Hence, Gauss’s law must be imposed as a subsidiary condition on a physical state (namely the gauge-invariant states):

\[
G^a|\text{Phy} > = 0. \tag{16}
\]

Henceforth, we denote the string part with

\[
U(2, 1) \equiv P \exp \left( ig \int_1^2 dx A^a(x) \frac{\lambda_a}{2} \right). \tag{17}
\]

In order to obtain the equation of motion for \( q(1, 2) \), we must consider the effect of applying the gradient to the string part. By applying to the path integral the same treatment applied to the time ordering, one can write

\[
U(2, 1) = P \sum_{N=0}^{\infty} \frac{1}{N!} \left[ \int_0^1 ds \frac{dx(s)}{ds} M(x(s)) \right]^N, \tag{18}
\]

where

\[
M(x(s)) = ig A^a(x(s)) \frac{\lambda_a}{2},
\]

\[
x(s) = x(1) + s(x(2) - x(1)).
\]

The gradient \( \partial(2) \) is generated by the infinitesimal displacement of \( x(2) \), while the point \( x(1) \) is fixed.

We then obtain

\[
\partial(2)U(2, 1) = \lim_{\Delta \to 0} \frac{U(2 + \Delta, 1) - U(2, 1)}{\Delta}
\]

\[
= \lim_{\Delta \to 0} \frac{1}{\Delta} \left( P \sum_{N=0}^{\infty} \frac{1}{N!} \left[ \int_0^{1+\Delta} ds \frac{dx(s)}{ds} M(x(s)) \right]^N \right)
\]

\[
- P \sum_{N=0}^{\infty} \frac{1}{N!} \left[ \int_0^1 ds \frac{dx(s)}{ds} M(x(s)) \right]^N \right)
\]

\[
= \lim_{\Delta \to 0} \frac{1}{\Delta} \left( P \sum_{N=0}^{\infty} \frac{1}{N!} \left[ \int_0^1 ds \frac{dx(s)}{ds} M(x(s)) + \int_0^{1+\Delta} ds \frac{dx(s)}{ds} M(x(s)) \right]^N \right)
\]

\[
- P \sum_{N=0}^{\infty} \frac{1}{N!} \left[ \int_0^1 ds \frac{dx(s)}{ds} M(x(s)) \right]^N \right)
\]

\[
= \lim_{\Delta \to 0} \frac{1}{\Delta} \left( P \sum_{N=0}^{\infty} \frac{1}{N!} \left[ \int_0^1 ds \frac{dx(s)}{ds} M(x(s)) \right]^N \right)
\]

\[
= \lim_{\Delta \to 0} \frac{1}{\Delta} \left( P \sum_{N=0}^{\infty} \frac{1}{N!} \left[ \int_0^1 ds \frac{dx(s)}{ds} M(x(s)) \right]^N \right)
\]

\[
= P ig A^a(2) \frac{\lambda_a}{2} \sum_{N=1}^{\infty} \frac{1}{(N - 1)!} \left[ \int_0^1 ds \frac{dx(s)}{ds} M(x(s)) \right]^{N-1}
\]

\[
= ig A^a(2) \frac{\lambda_a}{2} U(2, 1).
\]
To obtain the third line from the second line, we used a path order argument. Because the path order is defined as the time order, a gauge field for a higher position is always situated on the left-hand side of a gauge field for a lower position. For example, we may more precisely write

\[ PA^a(x_{n-1}) \frac{\lambda_a}{2} A^b(x_n) \frac{\lambda_b}{2} A^c(x_{n+1}) \frac{\lambda_c}{2} = PA^c(x_{n+1}) \frac{\lambda_c}{2} A^b(x_n) \frac{\lambda_b}{2} A^a(x_{n-1}) \frac{\lambda_a}{2}, \]

where \( x_{n-1} < x_n < x_{n+1} \).

Therefore, the integral from 1 to 1 + \( \Delta \) is always on the left-hand side of the integral from 0 to 1. Similarly, applying \( \partial(1) \) to \( U(2, 1) \) gives

\[ \partial(1)U(2, 1) = -U(2, 1)i \gamma A_a(1) \frac{\lambda_a}{2}. \]

The equation of motion for quark fields is simply the usual Dirac equation with the vector gauge fields satisfying the gauge principle, which means that

\[ i\dot{q}(x) = -i\alpha \partial q(x) + g\alpha A_a(x) \frac{\lambda_a}{2} q(x). \]

Here we consider the case in which the quark mass is zero.

From the definition of the gauge-invariant bi-local operator \( q(1, 2) \) in two dimensions (given by Eq. (4)), the time derivative of this operator is

\[ i\partial_t q(1, 2) = -i\alpha \partial q(1, 2) - q(1, 2)i\alpha \partial(1) + g \int_1^2 dx q E(1, 2 : x), \]

for which we introduce the notation

\[ q_o(1, 2 : x) = T_0^c q^+(2) U(2, x) O^a \frac{\lambda_a}{2} U(x, 1) q(1). \]

Note that the gauge fields arising from the quark fields are canceled out by the gradient of the string part.

From the definition of \( q E(1, 2 : x) \), we find \( q E(1, 2 : \mp \infty) = E(\mp \infty) q(1, 2) \).

The term sandwiched between the physical states in this equation will vanish according to the cluster expansion theorem [13]. We therefore find

\[
q E(1, 2 : x) = q E(1, 2 : x) - \frac{1}{2} q E(1, 2 : -\infty) - \frac{1}{2} q E(1, 2 : +\infty) = \frac{1}{2} \int_{-\infty}^{x} dz \partial_z q E(1, 2 : z) - \frac{1}{2} \int_{x}^{\infty} dz \partial_z q E(1, 2 : z).
\]

The integrand can be rewritten as

\[
\partial_z q E(1, 2 : z) = T_0^c q^+(2) \partial_z U(2, z) E^a(z) \left( \frac{\lambda_a}{2} \right) U(z, 1) q(1)
= T_0^c q^+(2) U(2, z) \left( D_A E^a(z) \left( \frac{\lambda_a}{2} \right) \right) U(z, 1) q(1)
= g q_{\text{Phy}}(1, 2 : z).
\]

The last line results from the fact (Appendix A) that

\[
\left< 0 | T_0^c q^+(2) U(2, z) G^a(z) \left( \frac{\lambda_a}{2} \right) U(z, 1) q(1) | \text{Phy} \right> = 0.
\]

Therefore, the last line is exact when physical states are considered.
Consequently, Eq. (21) becomes
\[ q_E(1, 2 : z) = -\frac{g}{2} \int_{-\infty}^{\infty} dx \varepsilon(x - z) q_{J0}(1, 2 : x). \] (23)

In order to regroup the charge vertex term, we use the Fierz identity for SU(N) [14] as follows:
\[ \sum_a \left( \frac{\lambda_a}{2} \right)_{ij} \left( \frac{\lambda_a}{2} \right)_{kl} = \frac{1}{2} \delta_{ij} \delta_{jk} - \frac{1}{2N} \delta_{ij} \delta_{kl}. \]

We thereby obtain
\[ q_{J0}(1, 2 : x) = \frac{1}{2} q(1, x) q(x, 2) - \frac{1}{2N} J^0(x) q(1, 2). \]

The final form of the equation of motion for \( q(1, 2) \) is thus
\[ i \partial_t q(1, 2) = -i \partial(2) q(1, 2) - q(1, 2) i \partial(1) + \frac{g^2}{4N} \int_1^2 dz \int_{-\infty}^{\infty} d\varepsilon(x - z) J^0(x) q(1, 2) \]
\[ - \frac{g^2}{4} \int_1^2 \int_{-\infty}^{z} d\varepsilon(x - z) q(1, x) q(x, 2) \] (24)

Placing Eq. (24) between the vacuum and a physical state and then taking the center-of-mass system to be
\[ X(1, 2) \equiv \langle 0 | q(1, 2) | \text{Phy} \rangle, \]
\[ G = \frac{1}{2} (x(1) + x(2)), \quad r = x(2) - x(1), \]
results in
\[ X(1, 2) = e^{-i P_0 t} e^{i P_1 G} X(r). \]

After factoring out the phase part (the exponential part), Eq. (24) becomes
\[ P_0 X(r) = \{ \alpha P_1, X(r) \} - \left[ i \alpha N - \frac{1}{2 \partial r}, X(r) \right] + \frac{g^2}{4N} \int_1^2 dz \int_{-\infty}^{\infty} d\varepsilon(x - z) \langle 0 | J^0(x) | P \rangle S(1, 2) \]
\[ + \frac{g^2}{8} \int_{-\infty}^{\infty} d\varepsilon e^{i P_1 \varepsilon(1)} \langle 0 | x(2) \rangle - | x - x(1) \rangle | S(1, x), X(x(2) - x) \rangle, \] (25)

where \( S(1, x) = \langle 0 | q(1, x) | 0 \rangle \).

The derivation of the last term is given in Appendix B.

Recalling the fact that \( \alpha = \sigma_3 \) and then taking the decomposition
\[ X(1, 2) = X_0(1, 2) + (i \sigma_3) X_1(1, 2) + (\sigma_2) X_2(1, 2) + (\sigma_1) X_3(1, 2), \] (26)
we obtain the following series of equations:
\[ P_0 X_0(r) = i P_1 X_1(r), \]
\[ P_0 X_1(r) = -i P_1 X_0(r) + \frac{g^2}{8 \pi N} \int_1^2 dx (| x - x(1) \rangle - | x(2) - x \rangle) \langle 0 | J^0(x) | P \rangle, \]
\[ W_0 X_2(r) = \frac{2}{8 \pi} \frac{\partial X_3(r)}{\partial r} - \frac{g^2}{8 \pi} \int_{-\infty}^{\infty} dx | r - x' \rangle - | x \rangle X_3(r - x'), \]
\[ W_0 X_3(r) = -2 \frac{\partial X_2(r)}{\partial r} + \frac{g^2}{8 \pi} \int_{-\infty}^{\infty} dx | r - x' \rangle - | x \rangle X_2(r - x'). \] (29) (30)

Here, we set \( S(r) = (i \sigma_3) P \int_0^1 \frac{dr}{r^2} \) because Schwinger suggested [15] that the propagator function behaves like a free propagator with small values of \( r \) (see Appendix E). Additionally, we set \( P_1 = 0 \).
for the $X_2$ and $X_3$ cases in order to obtain a closed form for the equations; consequently, $P_0$ becomes $W_0$, which is the mass of this system.

3. **The 't Hooft model**

Because the Schwinger model and the 't Hooft model are $CP = -1$ and $CP = +1$ models, respectively, we expect to obtain the 't Hooft model’s mass spectra from Eqs. (29) and (30).

Changing the variables in the integral such that $r - x' = s$ and, henceforth, using the notation $r = x$, Eqs. (29) and (30) become, respectively,

$$W_0 X_2(x) = 2i \partial_x X_3(x) - \frac{g^2}{8\pi} \int_{-\infty}^{\infty} ds \frac{|s| - |x - s|}{x - s} X_3(s),$$

$$W_0 X_3(x) = -2i \partial_x X_2(x) + \frac{g^2}{8\pi} \int_{-\infty}^{\infty} ds \frac{|s| - |x - s|}{x - s} X_2(s).$$

In order to write the equations in closed form, we take $X_{\mp} = X_2 \mp i X_3$ and then Eqs. (31) and (32) become, respectively,

$$W_0 X_+(x) = -2i \partial_x X_+(x) + i \frac{g^2}{8\pi} \int_{-\infty}^{\infty} ds \frac{|s| - |x - s|}{x - s} X_+(s),$$

$$W_0 X_-(x) = 2i \partial_x X_-(x) - i \frac{g^2}{8\pi} \int_{-\infty}^{\infty} ds \frac{|s| - |x - s|}{x - s} X_-(s).$$

To address the integral term, we use the definition of absolute value, which results in

$$\int_{-\infty}^{\infty} ds \frac{|s| - |x - s|}{x - s} X_+(s)$$

$$= \int_{-\infty}^{0} ds \frac{-s}{x - s} X_+(s) + \int_{0}^{\infty} ds \frac{s}{x - s} X_+(s) - \int_{-\infty}^{x} ds \frac{x - s}{x - s} X_+(s) + \int_{x}^{\infty} ds \frac{x - s}{x - s} X_+(s)$$

$$= - \int_{-\infty}^{0} ds \varepsilon(s) X_+(s) - x \int_{-\infty}^{\infty} ds \varepsilon(s) X_+(s) + \int_{-\infty}^{\infty} ds \varepsilon(x - s) X_+(s),$$

where $\varepsilon$ is the step function defined as 1 for $x \geq 0$, and as $-1$ for $x < 0$.

Multiplying $\varepsilon(x)$ on both sides of the equation and considering the new function to be $F_+(x) = X_+(x)\varepsilon(x)$, Eq. (33) becomes

$$W_0 F_+(x) = -2i \partial_x F_+(x) + 4i \delta(x) F_+(x) \varepsilon(x) - i \frac{g^2}{8\pi} \varepsilon(x) x \int_{-\infty}^{\infty} ds \frac{F_+(s)}{s - x}$$

$$+ i \frac{g^2}{8\pi} \varepsilon(x) \int_{-\infty}^{\infty} ds F_+(s) \varepsilon(s) \varepsilon(x - s) - i \frac{g^2}{8\pi} \varepsilon(x) \int_{-\infty}^{\infty} ds F_+(s).$$

Here, for the second and fourth terms, we have used the fact that $(\varepsilon(x))^2 = 1$.

The $\delta$-function appeared because of the fact $\partial_x \varepsilon(x) = 2\delta(x)$.

Considering the $x > 0$ region (for which $\varepsilon(x) = 1$) and taking the derivative with respect to $x$ on both sides, Eq. (35) becomes

$$W_0 \partial_x F_+(x) = -2i \partial_x^2 F_+(x) + 4i \partial_x (\delta(x) F_+(x)) - i \frac{g^2}{8\pi} \int_{-\infty}^{\infty} ds \frac{F_+(s)}{s - x}$$

$$- i \frac{g^2}{8\pi} x \int_{-\infty}^{\infty} ds \frac{dF_+(s)}{s - x} + i \frac{g^2}{4\pi} F_+(x).$$

Here, to obtain the last term, we have used $\partial_x \varepsilon(x - s) = 2\delta(x - s)$. 

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For the singular integral term, we use the slightly modified Sokhotsky formula [16]:

\[
\frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{F_+(s)}{s-x} \, ds = a_1 \Phi_+^{(+)}(x) + a_2 \Phi_+^{(-)}(x),
\]

\[
F_+(x) = a_1 \Phi_+^{(+)}(x) - a_2 \Phi_+^{(-)}(x).
\]

where \( \Phi_+^{(+)}(x) \) is the value resulting from unlimitedly approaching \( x \) from the upper hemisphere, \( \Phi_+^{(-)}(x) \) is the value resulting from unlimitedly approaching \( x \) from the lower hemisphere, and \( a_1 \) and \( a_2 \) are non-zero arbitrary constants.

Here, we have modified the original form of the Sokhotsky formula by changing \( \Phi_+^{(+)} \) to \( a_1 \Phi_+^{(+)} \) and \( \Phi_+^{(-)} \) to \( a_2 \Phi_+^{(-)} \), which will be used in a later argument.

Note that \( s \) and \( x \) are real values.

Subsequently, Eq. (36) is written as

\[
a_1 \left[ \frac{\partial^2 \Phi_+^{(+)}(x)}{\partial x^2} + \left[ \left( \frac{W_0}{2i} \right) - \frac{g^2}{8} \frac{1}{2i} x \right] \frac{\partial \Phi_+^{(+)}(x)}{\partial x} + \left[ - \left( \frac{g^2}{8\pi} \right) - \left( \frac{g^2}{8} \right) \frac{1}{2i} \right] \Phi_+^{(+)} - 2\partial_x \left( \delta(x) \Phi_+^{(+)} \right) \right]
\]

\[
= a_2 \left[ \frac{\partial^2 \Phi_+^{(-)}(x)}{\partial x^2} + \left[ \left( \frac{W_0}{2i} \right) + \frac{g^2}{8} \frac{1}{2i} x \right] \frac{\partial \Phi_+^{(-)}(x)}{\partial x} + \left[ - \left( \frac{g^2}{8\pi} \right) + \left( \frac{g^2}{8} \right) \frac{1}{2i} \right] \Phi_+^{(-)} - 2\partial_x \left( \delta(x) \Phi_+^{(-)} \right) \right].
\]

In order to find the solutions for both \( \Phi_+^{(+)} \) and \( \Phi_+^{(-)} \), we set both sides of Eq. (39) to be equal to zero. Because \( a_1 \) and \( a_2 \) are non-zero constants, we can independently obtain the differential equations for \( \Phi_+^{(+)} \) and \( \Phi_+^{(-)} \) and then solve these differential equations.

To address the derivative of the \( \delta \)-function term, we integrate twice successively from \( \infty \) to \( x \). Note here that we consider the range \( x > 0 \) while \( \Phi_+^{(+)}(\infty) = 0 \) and \( \partial \Phi_+^{(+)} / \partial x |_{x=\infty} = 0 \); specifically, we are seeking solutions that satisfy the previous condition.

Subsequently, we can see that

\[
\int_{-\infty}^{\infty} dx' \delta(x') \Phi_+^{(+)}(x') = 0.
\]

Taking the derivative twice successively, we obtain the following differential equations:

\[
\frac{\partial^2 \Phi_+^{(+)}(x)}{\partial x^2} + \left[ \left( \frac{W_0}{2i} \right) - \frac{g^2}{8} \frac{1}{2i} x \right] \frac{\partial \Phi_+^{(+)}(x)}{\partial x} + \left[ - \left( \frac{g^2}{8\pi} \right) - \left( \frac{g^2}{8} \right) \frac{1}{2i} \right] \Phi_+^{(+)} = 0,
\]

\[
\frac{\partial^2 \Phi_+^{(-)}(x)}{\partial x^2} + \left[ \left( \frac{W_0}{2i} \right) + \frac{g^2}{8} \frac{1}{2i} x \right] \frac{\partial \Phi_+^{(-)}(x)}{\partial x} + \left[ - \left( \frac{g^2}{8\pi} \right) + \left( \frac{g^2}{8} \right) \frac{1}{2i} \right] \Phi_+^{(-)} = 0.
\]

\[\text{\textsuperscript{1}}\text{ We use the definition of the } \delta \text{-function as}
\]

\[
\delta(z) = \begin{cases} \frac{1}{2\pi} & (-\varepsilon \leq z \leq \varepsilon) \\ 0 & \text{otherwise} \end{cases}\quad \text{when } \varepsilon \text{ approaches 0.}
\]

We may then conclude that

\[
\int_{-\varepsilon}^{\varepsilon} dz \delta(z) f(z) = 0 \quad \text{if } x \text{ is always larger than } \varepsilon.
\]

This conclusion is always true (even though \( x \) approaches zero) as long as \( x \) is larger than \( \varepsilon \) when \( \varepsilon \) approaches zero.
The solutions satisfying the condition that both the function and its first derivative at infinity must be zero are as follows (the derivation is given in Appendix C):

\[
\Phi_+^{(+)}(x) = e^{-\frac{1}{4}x^2} e^{-\frac{1}{2} \beta x} 2^{-\frac{1}{2} - \frac{x}{2}} \left( \sqrt{\alpha x} - \frac{\beta}{\sqrt{\alpha}} \right)^{-\frac{1}{2}} W_{\frac{1}{4} - \frac{x}{2}, -\frac{1}{4}} \left( \frac{1}{2} \left( \sqrt{\alpha x} - \frac{\beta}{\sqrt{\alpha}} \right)^2 \right),
\]

\[
\Phi_+^{(-)}(x) = e^{-\frac{1}{4}x^2} e^{-\frac{1}{2} \beta x} 2^{-\frac{1}{2} + \frac{x}{2}} \left( i \left( \sqrt{\alpha x} + \frac{\beta}{\sqrt{\alpha}} \right) \right)^{-\frac{1}{2}} W_{\frac{1}{4} + \frac{x}{2}, -\frac{1}{4}} \left( -\frac{1}{2} \left( \sqrt{\alpha x} + \frac{\beta}{\sqrt{\alpha}} \right)^2 \right),
\]

where

\[\alpha = \frac{g^2}{8 \sqrt{2}i}, \quad \beta = \frac{W_0}{2i},\]

\(W_{\kappa, \mu}(x)\) is the Whittaker function.

In order to determine the eigenvalue of \(W_0\), we must consider the boundary condition.

The 't Hooft model's boundary condition was established in momentum space (specifically, in \(P_0 - P_1\) space). Because 't Hooft divided this quantity by the maximum momentum, his boundary condition was essentially identical to the condition that, at \(P_1 = 0\), the wave function is zero and that, at \(P_1 = \infty\), the wave function is zero.

Because this derivation is conducted in the normal space-time, this boundary condition is equivalent to the condition that, at \(x = 0\), the wave function is zero and that, at \(x = \infty\), the wave function is zero (\(x = 0\) corresponds to \(P_1 = \infty\), and \(x = \infty\) corresponds to \(P_1 = 0\)).

It is important to note that \(F_+(x)\) in Eq. (38) is essentially \(\chi_+(x) = \chi_2(x) + i \chi_3(x)\). This component is one of the off-diagonal elements in our Bethe–Salpeter-like amplitude (the other component is \(\chi_-\)). Consequently, the wave function automatically approaches zero at \(x = \infty\) because the Bethe–Salpeter-like amplitude is represented by a linear combination of (For \(\chi_-\), it is given by \(\Phi_+^{(+)}(x)\) and \(\Phi_+^{(-)}(x)\), \(\Phi_-^{(+)}(x)\) and \(\Phi_-^{(-)}(x)\).)

In order to satisfy the other boundary condition, we invoke the following wave function:

\[
F_+(x) = 2^{\frac{i}{\pi}} \Gamma \left( 1 + \frac{i}{\pi} \right) \Phi_+^{(+)}(x) - 2^{-\frac{i}{\pi}} \Gamma \left( 1 - \frac{i}{\pi} \right) \Phi_+^{(-)}(x).
\]

Here, we have taken

\[a_1 = 2^{\frac{i}{\pi}} \Gamma \left( 1 + \frac{i}{\pi} \right) \quad \text{and} \quad a_2 = 2^{-\frac{i}{\pi}} \Gamma \left( 1 - \frac{i}{\pi} \right)\]

in Eq. (38). These values were determined by imposing the condition that the constant term in \(F_+(0)\) (when \(\frac{(2W_0)^2}{g^2}\) is variable) must be canceled out in order to satisfy the boundary condition (see Appendix D). The expansion of \(F_+(x)\) is then expressed by Eq. (45) when which is based on \(\left( \sqrt{\alpha x} - \frac{\beta}{\sqrt{\alpha}} \right)^2 \rightarrow 0\), \(\left( i \left( \sqrt{\alpha x} + \frac{\beta}{\sqrt{\alpha}} \right) \right)^2 \rightarrow 0\), and \(\frac{\beta^2}{\alpha} \rightarrow 0\), the fact that \(W_{\kappa, \mu}\) is defined by a linear combination of the Whittaker functions for \(M_{\kappa, \mu}\) and \(M_{\kappa, -\mu}\) (see Appendix D):

\[
F_+(x) \rightarrow \frac{\Gamma \left( 1 + \frac{i}{\pi} \right) \Gamma \left( -\frac{1}{2} \right)}{2^{\frac{i}{\pi}} \Gamma \left( \frac{1}{2} + \frac{i}{\pi} \right)} \left( \left( \sqrt{\alpha x} - \frac{\beta}{\sqrt{\alpha}} \right)^2 \right)^{\frac{1}{2}}
\]

\[\quad - \frac{\Gamma \left( 1 - \frac{i}{\pi} \right) \Gamma \left( -\frac{1}{2} \right)}{2^{\frac{i}{\pi}} \Gamma \left( \frac{1}{2} - \frac{i}{\pi} \right)} \left( i \left( \sqrt{\alpha x} + \frac{\beta}{\sqrt{\alpha}} \right) \right)^{\frac{1}{2}}.
\]
Thus, using the fact that $F_+(x) = \varepsilon(x) X_+(x)$, we find that

$$X_+(x) = \varepsilon(x) F_+(x) = \varepsilon(x) \left[ 2^{\frac{3}{2}} \Gamma \left(1 + \frac{i}{\pi}\right) \Phi^{(+)}_+(x) - 2^{-\frac{i}{2}} \Gamma \left(1 - \frac{i}{\pi}\right) \Phi^{(-)}_+(x) \right], \quad (46)$$

because (as we defined earlier) $\varepsilon(x) = -1$ for $x < 0$, and $+1$ for $x \geq 0$; thus, $(\varepsilon(x))^2 = 1$.

Therefore, our boundary condition is represented such that at $x = 0$, $\chi_+(0) = 0$ and, at $x = \infty$, $\chi_+(\infty) = 0$.

As previously mentioned, $\chi_+$ automatically satisfies the $x = \infty$ boundary condition and, therefore, we need only to consider the $x = 0$ boundary condition.

The condition that $\chi_+(0) = 0$ gives the following equation:

$$\left( -i \left( \frac{2 W_0}{g^2} \right)^2 \right)^{-\frac{1}{4}} \left[ i^{\frac{3}{2}} \Gamma \left(1 + \frac{i}{\pi}\right) W^{-\frac{1}{2} + \frac{i}{\pi}} \left( -i \Gamma \left(1 - \frac{i}{\pi}\right) \Phi^{(-)}_+(x) \right) \right] = 0. \quad (47)$$

We must consider two additional cases: where $W_0^2/g_0^2$ approaches zero, and where this value is non-zero.

For the first case, we immediately obtain a zero mass solution from Eq. (45) (see Appendix D).

Thus, $W_0^2/g_0^2 = 0$. For the second case, we use the integral representation of $W_{+\mu}$ in the following manner [17,18]:

$$W^{-\frac{1}{2} + \frac{i}{\pi}, -\frac{1}{4}} \left( -i \left( \frac{2 W_0}{g^2} \right)^2 \right) = \frac{e^{\frac{W_0^2}{g^2}} \left( -i \frac{2 W_0}{g^2} \right)^{-\frac{1}{4} - \frac{i}{4}}}{\Gamma \left( \frac{1}{2} - \frac{i}{\pi} \right)} \int_0^\infty dt e^{-t} t^{-\frac{1}{2} - \frac{i}{\pi}} \left( 1 + i \frac{t}{\left( \frac{2 W_0}{g^2} \right)} \right)^{-1 - \frac{i}{\pi}},$$

$$W^{\frac{1}{2} + \frac{i}{\pi}, -\frac{1}{4}} \left( -i \left( \frac{2 W_0}{g^2} \right)^2 \right) = \frac{e^{\frac{W_0^2}{g^2}} \left( i \frac{2 W_0}{g^2} \right)^{-\frac{1}{4} + \frac{i}{4}}}{\Gamma \left( \frac{1}{2} + \frac{i}{\pi} \right)} \int_0^\infty dt e^{-t} t^{\frac{1}{2} + \frac{i}{\pi}} \left( 1 - i \frac{t}{\left( \frac{2 W_0}{g^2} \right)} \right)^{-1 + \frac{i}{\pi}}.$$

Therefore, after adjusting the coefficients, Eq. (47) becomes

$$\left( \frac{2 W_0^2}{g^2} \right)^{-\frac{1}{2} + \frac{i}{\pi}} e^{\frac{W_0^2}{g^2}} \left[ i^{\frac{3}{2}} \Gamma \left(1 + \frac{1}{\pi}\right) \left( \frac{2 W_0^2}{g^2} \right)^{\frac{2}{\pi}} \int_0^\infty dt e^{-t} t^{\frac{1}{2} - \frac{i}{\pi}} \left( 1 + i \frac{t}{\left( \frac{2 W_0}{g^2} \right)} \right)^{-1 - \frac{i}{\pi}} \right]$$

$$- i^{\frac{1}{2}} \Gamma \left(1 - \frac{i}{\pi}\right) e^{-\frac{2 W_0^2}{g^2}} \int_0^\infty dt e^{-t} t^{-\frac{1}{2} + \frac{i}{\pi}} \left( 1 - i \frac{t}{\left( \frac{2 W_0}{g^2} \right)} \right)^{-1 + \frac{i}{\pi}} = 0. \quad (48)$$

\[2\] In Ref. [18], the parabolic cylinder function is referred to as the Weber function.
From Eq. (48), we obtain the following equation:

$$\exp \left( i \left( 2x - \frac{2}{\pi} \log 2x + \frac{\pi}{2} \right) + \log \left( \frac{\Gamma \left( 1 + \frac{i}{\pi} \right)}{\Gamma \left( 1 - \frac{i}{\pi} \right)} \right) \right) \times \int_0^\infty dt \ e^{-t t^{-\frac{1}{2} - \frac{i}{\pi}}} \left( 1 + i \frac{t}{2\pi} \right)^{-1 - \frac{i}{\pi}} = 1 = \exp(i2N\pi), \quad (49)$$

where $x = \frac{w_i^2}{g^2}$.

We use the following representation of the $\Gamma$-function [17,18]:

$$\log \Gamma(z) = \frac{1}{2} \log \left( \frac{\pi}{z \sin(\pi z)} \right) + (1 - \gamma) z - \sum_{n=1}^\infty \frac{\zeta(2n+1) - 1}{2n+1} z^{2n+1},$$

where $\gamma$ is the Euler–Mascheroni constant, and $\zeta$ is the Riemann $\zeta$-function.

We can then express the $\Gamma$-function as follows:

$$\log \left( \frac{\Gamma \left( 1 + \frac{i}{\pi} \right)}{\Gamma \left( 1 - \frac{i}{\pi} \right)} \right) = i \left[ -\arctan \left( \frac{3\pi}{2\pi^2 - 1} \right) + \arctan \left( \frac{3}{\pi^2} \right) + \arctan \left( \frac{8\pi}{3\pi^2 - 4} \right) - \sum_{n=0}^\infty \frac{\zeta(2n+1) - 1}{2n+1} \right]$$

$$\times 2 \left( \left( 1 + \left( \frac{1}{\pi} \right)^2 \right)^{\frac{n+1}{2}} \sin \left( (2n+1)\arctan \left( \frac{1}{\pi} \right) \right) - \left( \left( \frac{1}{\pi} \right)^2 + \left( \frac{1}{\pi} \right) \right)^{\frac{n+1}{2}} \right)$$

$$\times \sin \left( (2n+1)\arctan \left( \frac{2}{\pi} \right) \right) \right]. \quad (50)$$

In order to address the integral, we use the following expression:

$$(a + ib)^{c+id} = e^{z \log (a^2 + b^2) - i\arctan \left( \frac{b}{a} \right)} e^{i \left( \frac{t}{2} \log (a^2 + b^2) + \arctan \left( \frac{b}{a} \right) \right)}.$$ 

Therefore, the integral becomes

$$\int_0^\infty dt e^{-t t^{-\frac{1}{2} - \frac{i}{\pi}}} \left( 1 + i \frac{t}{2W_0^2} \right)^{-1 - \frac{i}{\pi}}$$

$$= \int_0^\infty dt \exp \left( -t - \frac{1}{2} \log t - \frac{1}{2} \log \left( 1 + \frac{t^2}{2W_0^2} \right) + \frac{1}{\pi} \arctan \left( \frac{1}{2W_0^2} \right) \right)$$

3 In Ref. [17], the formula is given by $\ln \Gamma(z + 1)$ instead of $\ln \Gamma(z)$; however, because $\Gamma(z + 1) = z\Gamma(z)$, the expressions are identical.
\[
\times \exp \left(-i \left(\frac{1}{\pi} \log t + \frac{1}{2\pi} \log \left(1 + \frac{t^2}{(2w_0^2/g^2)^2}\right) + \arctan\left(\frac{t}{(2w_0^2/g^2)\right)}\right)\right) \right), \tag{51}
\]

\[
\int_0^\infty dt e^{-t - \frac{1}{2} + \frac{1}{\pi} \left(1 - i \frac{t}{(2w_0^2/g^2)}\right)^{-1 + \frac{i}{\pi}}} = \int_0^\infty dt \exp \left(-t - \frac{1}{2} \log t - \frac{1}{2} \log \left(1 + \frac{t^2}{(2w_0^2/g^2)^2}\right) + \frac{1}{\pi} \arctan\left(\frac{t}{(2w_0^2/g^2)\right)}\right) \times \exp \left(i \left(\frac{1}{\pi} \log t + \frac{1}{2\pi} \log \left(1 + \frac{t^2}{(2w_0^2/g^2)^2}\right) + \arctan\left(\frac{t}{(2w_0^2/g^2)\right)}\right)\right). \tag{52}
\]

Because the real part in the exponent is unchanged (while the imaginary part in the exponent differs only by a sign), Eqs. (51) and (52) can be written in the following manner:

right-hand side of Eq. (51) = \(A(x) - iB(x)\),

right-hand side of Eq. (52) = \(A(x) + iB(x)\).

where

\[
A(x) = \int_0^\infty dt \exp \left(-t - \frac{1}{2} \log t - \frac{1}{2} \log \left(1 + \frac{t^2}{4x^2}\right) + \frac{1}{\pi} \arctan\left(\frac{t}{2x}\right)\right) \times \cos \left(\frac{1}{\pi} \log t + \frac{1}{2\pi} \log \left(1 + \frac{t^2}{4x^2}\right) + \arctan\left(\frac{t}{2x}\right)\right), \tag{53}
\]

\[
B(x) = \int_0^\infty dt \exp \left(-t - \frac{1}{2} \log t - \frac{1}{2} \log \left(1 + \frac{t^2}{4x^2}\right) + \frac{1}{\pi} \arctan\left(\frac{t}{2x}\right)\right) \times \sin \left(\frac{1}{\pi} \log t + \frac{1}{2\pi} \log \left(1 + \frac{t^2}{4x^2}\right) + \arctan\left(\frac{t}{2x}\right)\right), \tag{54}
\]

and \(x = \frac{w_0^2}{g^2}\).

The ratio in the integral from Eq. (49) then becomes

\[
\frac{A(x) - iB(x)}{A(x) + iB(x)} = \exp(-2i \arctan\left(\frac{B(x)}{A(x)}\right)). \tag{55}
\]
Using the description of Eq. (55) for the ratio in the integral from Eq. (49), we obtain an algebraic equation that determines the non-zero mass spectra of the bound states to be the following:

\[
x - \frac{1}{\pi} \log x - \arctan \left( \frac{B(x)}{A(x)} \right) - \frac{1}{\pi} \log 2 + \frac{\pi}{4} - \frac{1}{2} \arctan \left( \frac{3\pi}{2\pi^2 - 1} \right) + \frac{1}{2} \arctan \left( \frac{2}{\pi} \right) \\
+ \frac{1}{2} \arctan \left( \frac{8\pi}{3\pi^2 - 4} \right) - \sum_{n=0}^{\infty} \frac{\xi(2n + 1) - 1}{2n + 1} \left( 1 + \left( \frac{1}{\pi} \right)^2 \right)^{n+\frac{1}{2}} \sin \left( (2n + 1) \arctan \left( \frac{1}{\pi} \right) \right) \\
- \left( \left( \frac{1}{2} \right)^2 + \left( \frac{1}{\pi} \right)^2 \right)^{n+\frac{1}{2}} \sin \left( (2n + 1) \arctan \left( \frac{2}{\pi} \right) \right) = N\pi,
\]

(56)

Here \( N = 0 \) or a positive integer, \( x = \frac{W^2_0}{g^2} \) and \( A(x) \) and \( B(x) \) are given in Eqs. (53) and (54), respectively.

To compare the eigenvalue of our solution with 't Hooft’s eigenvalue, we consider the case in which \( N \) is large. Because the value of \( \arctan \left( \frac{B(x)}{A(x)} \right) \) is between \(-\frac{\pi}{2}\) and \(\frac{\pi}{2}\), we can ignore this term and, instead, express the constant terms on the left-hand side of Eq. (56) by \( \rho \).

Setting \( x/\pi = N + \alpha \log N + \eta \), we can then show that

\[
N + \alpha \log N + \eta - \frac{1}{\pi^2} \log (N + \alpha \log N + \eta) - \frac{\rho}{\pi} - \frac{1}{\pi^2} \log \pi = N + \left( \alpha - \frac{1}{\pi^2} \right) \log N + \frac{1}{\pi^2} \log \left( 1 + \frac{1}{N} \log (N + \eta) \right) + \eta - \frac{\rho}{\pi} - \frac{1}{\pi^2} \log \pi \\
= N - \frac{1}{\pi} \sum_{n=1}^{\infty} \left( \frac{1}{N} \log (N + \eta) \right)^n = \text{Right-hand side} = N.
\]

To obtain the second to last line we used \( \alpha = 1/\pi^2 \) and \( \eta = \frac{\rho}{\pi} + \frac{1}{\pi^2} \log \pi \).

Thus, \( W^2_0 = g^2 \pi \left( N + \frac{1}{\pi^2} \log N + \frac{\rho}{\pi} + \frac{1}{\pi^2} \log \pi \right) \) is the first approximate solution of Eq. (56). This solution is in exactly the same form as the W-K-B solution in the 't Hooft model.

For the equation representing \( X_- \), when we replace \( W_0 \) with \(-W_0\), this equation is then identical to the equation representing \( X_+ \), which becomes apparent when we compare Eq. (33) with Eq. (34). Consequently, the formula obtained for \( X_- \) (which determines the mass spectra) is identical to Eqs. (47) and (48). Therefore, the equations obtained to determine the mass spectra of Eq. (56) and the zero mass spectrum are consistent in this argument. Here, we again insist that our zero mass spectrum is obtained from the same boundary condition used for the mass spectra group in Eq. (56); therefore, we conclude that the zero mass spectrum truly represents a physical mass.

Finally, we discuss whether our approach is equivalent to 't Hooft’s method. Our approach is quite similar to 't Hooft’s methodology but is not exactly identical. The reason for this difference is that the system we consider maintains gauge invariance, whereas 't Hooft’s system is not gauge invariant when the quark and anti-quark are at different space-time positions. Even in momentum space, this difference results in a discrepancy; in particular, the contour of the singular integral is not closed in 't Hooft’s case but is closed in our case. For this reason, 't Hooft’s integral equation cannot be solved analytically, whereas our case can be solved analytically (we solved this case in \( x \)-space). For comparison, we show the equation of motion in momentum space in Appendix F.
4. Conclusion

Using the bound system proposed by Suura, we have investigated ’t Hooft’s mass spectra. We applied the equation of motion and thereby obtained an algebraic equation to determine a mass spectra group that is consistent with ’t Hooft’s model. In addition, we obtained the zero mass spectrum from the same boundary condition employed for the previous mass spectra group. This application of boundary condition is crucial because ’t Hooft’s zero mass spectrum was obtained using a different boundary condition than the condition applied to the numerically calculated mass spectra group. Our boundary condition is equivalent to ’t Hooft’s boundary condition. Therefore, we can conclude that the zero mass spectrum is indeed a physical mass.

Appendix A.

We provide the proof of the equation

$$\left\langle 0 | T^a c q^+ (2) U(2, z) G^a(z) \left( \frac{\lambda_a}{2} \right) U(z, 1) q(1) | \text{Phy} \right\rangle = 0$$

Proof.

$$T^a c q^+ (2) U(2, z) G^a(z) U(z, 1) q(1)$$

$$= T^a c q^+ (2) \lim_{\Delta \to 0} P \sum_{n=0}^{\infty} \frac{1}{n!} \left[ \int_{z+\Delta}^{2} dx \text{ig} A^b(x) \left( \frac{\lambda_b}{2} \right) \right]^n G^a(z) \left( \frac{\lambda_a}{2} \right) U(z, 1) q(1)$$

$$= G^a(z) T^a c q^+ (2) \lim_{\Delta \to 0} P \sum_{n=0}^{\infty} \frac{1}{n!} \left[ \int_{z+\Delta}^{2} dx \text{ig} A^b(x) \left( \frac{\lambda_b}{2} \right) \right]^n \left( \frac{\lambda_a}{2} \right) U(z, 1) q(1)$$

$$= G^a(z) T^a c q^+ (2) U(2, z) \left( \frac{\lambda_a}{2} \right) U(z, 1) q(1).$$

By situating the last term between physical states or vacuum, this term becomes zero according to the definition of $G^a(z)$.

Appendix B.

$$- \frac{g^2}{4} \int_1^2 dz \int_{-\infty}^{\infty} dx \, \varepsilon(x - z) q(1, x) q(x, 2)$$

$$= - \frac{1}{8} g^2 \int_1^2 dz \int_{-\infty}^{\infty} dx \, \varepsilon(x - z) q(1, x) q(x, 2)$$

$$- \frac{1}{8} g^2 \int_1^2 dz \int_{-\infty}^{\infty} dx \, \varepsilon(x - z) q(2, x) q(x, 1).$$

For the second term, we exchange 1 and 2 after we divide the initial equation into two equal parts. The second term then becomes

$$- \frac{1}{8} g^2 \int_2^1 dz \int_{-\infty}^{\infty} dx \, \varepsilon(x - z) q(1, x) q(x, 1) = \frac{1}{8} g^2 \int_1^2 dz \int_{-\infty}^{\infty} dx \, \varepsilon(x - z) q(2, x) q(x, 1).$$

To calculate these integrals, we exchange the order of integration and then integrate with respect to $z$ first. Note that in this case $x$ can occupy three regions: (1) below 1, (2) between 1 and 2, (3) above 2.
(1) Below 1:
As the $\varepsilon$-function is always $-1$, the integration becomes $x(1) - x(2)$:

$$x(1) - x(2) = x(1) - x - (x(2) - x) = |x - x(1)| - |x(2) - x|.$$  

(2) Between 1 and 2:

$$\int_1^2 dz \varepsilon(x - z) = \int_1^x dz (+1) + \int_x^2 dz (-1) = x - x(1) - (x(2) - x)$$

$$= |x - x(1)| - |x(2) - x|.$$  

(3) Above 2:
As the $\varepsilon$-function is always $+1$, the integral becomes $x(2) - x(1)$

$$x(2) - x(1) = x - x(1) - (x - x(2)) = |x - x(1)| - |x(2) - x|.$$  

Therefore, after situating the equation between vacuum and a physical state, we obtain the following:

right-hand side $= -\frac{1}{8} g^2 \int_{-\infty}^\infty dx (|x - x(1)| - |x(2) - x|)[S(1, x), X(x, 2)]$

After taking the center-of-mass system for $X(x, 2)$, we obtain

right-hand side $= \frac{1}{8} g^2 \int_{-\infty}^\infty dx (|x(2) - x| - |x - (x(1))|) e^{-iP_0^t e^{iP_1 \frac{x(2) + x}{2}} [S(1, x), X(x(2) - x)]}$

$$= e^{-iP_0^t e^{iP_1 \frac{x(2) + x(1)}{2}}} \frac{1}{8} g^2 \int_{-\infty}^\infty dx e^{iP_1 \frac{x - x(1)}{2}}$$

$$\times (|x(2) - x| - |x - x(1)|)[S(1, x), X(x(2) - x)]$$

**Appendix C.**

Equation (40) is expressed as follows:

$$\frac{d^2 \Phi_+^{(+)}}{dx^2} + (\beta - \alpha x) \frac{d \Phi_+^{(+)}}{dx} + \left( - \left( \frac{g^2}{8\pi} \right) - \alpha \right) \Phi_+^{(+)} = 0, \quad (A1)$$

where $\alpha = \frac{g^2}{8\pi}$, $\beta = \frac{w_0}{2\pi}$.

After first taking

$$\Phi_+^{(+)}(x) = e^{\frac{i}{2} \alpha x^2} u(x),$$

Eq. (A1) then becomes

$$\frac{d^2 u}{dx^2} + \beta \frac{du}{dx} + \left[ -\frac{1}{2} \alpha - \left( \frac{g^2}{8\pi} \right) - \frac{1}{4} (\alpha x - \beta)^2 + \frac{1}{4} \beta^2 \right] u = 0. \quad (A2)$$

When we next take $u(x) = e^{-\beta x} v(x)$,

Eq. (A2) then becomes

$$\frac{d^2 v}{dx^2} + \left[ -\frac{1}{2} \alpha - \left( \frac{g^2}{8\pi} \right) - \frac{1}{4} (\alpha x - \beta)^2 \right] v(x) = 0. \quad (A3)$$

Using the change of variable $\frac{1}{2}(\alpha x - \beta) = z$, 

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Eq. (A3) then becomes
\[ \frac{d^2v}{dz^2} + \left[ -\frac{1}{2} \alpha - \left( \frac{g^2}{8\pi} \right) \frac{z^2}{4\alpha^2} \right] v(z) = 0. \]  
(A4)

Here we take \( \gamma = -\frac{1}{2} \left( \frac{g^2}{8\pi} \right) \) and apply a change of variable such that \( t = \frac{2}{\sqrt{\alpha}} z \).

Eq. (A4) then becomes
\[ \frac{d^2v}{dt^2} + \left[ \alpha \gamma - \frac{t^2}{4} \right] v(t) = 0. \]  
(A5)

By comparing Eq. (A5) with the standard form of the parabolic cylinder function [16], we find \( \frac{\alpha \gamma}{4} = \lambda + \frac{1}{2} \).

Using the definition of \( \alpha \), we obtain \( \lambda = -1 - \frac{2i}{\pi} \).

The solution of Eq. (A5) is then
\[ v(t) = D_{\lambda}(t) = 2^{\frac{1}{4} + \frac{1}{4}} t^{-\frac{1}{2}} W_{\frac{1}{2} + \frac{1}{4}, -\frac{1}{4}} \left( \frac{t^2}{2} \right). \]  
(A6)

By substituting the value obtained for \( \lambda \) into Eq. (A6), we find
\[ v(t) = 2^{-\frac{1}{4} - \frac{1}{4}} t^{-\frac{1}{2}} W_{-\frac{1}{4} - \frac{1}{4}, -\frac{1}{4}} \left( \frac{t^2}{2} \right). \]  
(A7)

Here, \( D_{\lambda}(t) \) and \( W_{k,\mu}(t) \) are the parabolic cylinder function and the Whittaker function, respectively, and these functions are defined in Ref. [14].

For \( \Phi_+^{(-)} \), Eq. (40) is represented by \( \alpha \) and \( \beta \) such that
\[ \frac{d^2\Phi_+^{(-)}}{dx^2} + (\beta + \alpha x)) \frac{d\Phi_+^{(-)}}{dx} + \left( -\frac{g^2}{8\pi} + \alpha \right) \Phi_+^{(-)} = 0. \]

By using the same argument applied previously, we obtain
\[ \frac{d^2v}{dt^2} + \left[ \alpha \gamma' - \frac{t^2}{4} \right] v(t) = 0, \]

where
\[ \gamma' = \frac{-\frac{1}{2} - \left( \frac{g^2}{8\pi} \right)}{1 - \frac{1}{4} \alpha^2}. \]

In this case, \( \lambda = -\frac{2i}{\pi} \).

Because the solutions to the parabolic cylinder function equation are \( D_{\lambda}(t) \) and \( D_{-\lambda-1}(it) \) [14], we select \( D_{-\lambda-1}(it) \) in this case. The solution is then
\[ v(t) = D_{-\lambda-1}(it) = 2^{-\frac{1}{4} + \frac{1}{4}} (it)^{-\frac{1}{2}} W_{\frac{1}{2} + \frac{1}{4}, -\frac{1}{4}} \left( -\frac{t^2}{2} \right). \]  
(A8)

From Eqs. (A7) and (A8), we can obtain the solutions to Eqs. (42) and (43).
Appendix D.

In order to investigate the zero mass solution, we return to the definition of the Whittaker function $W_{\kappa, \mu}(z)$:

$$W_{\kappa, \mu}(z) = \frac{\Gamma(-2\mu)}{\Gamma(-\mu - \kappa + \frac{1}{2})} M_{\kappa, \mu}(z) + \frac{\Gamma(2\mu)}{\Gamma(\mu - \kappa + \frac{1}{2})} M_{\kappa, -\mu}(z). \quad (A9)$$

The definition of the Whittaker function $M_{\kappa, \mu}(z)$ is as follows:

$$M_{\kappa, \mu}(z) = z^{\mu + \frac{1}{2}} e^{-\frac{z}{2}} \sum_{n=0}^{\infty} \frac{\Gamma(2\mu + 1) \Gamma(\mu - \kappa + n + \frac{1}{2})}{\Gamma(2\mu + n + 1) \Gamma(\mu - \kappa + \frac{1}{2})} z^n n!. \quad (A10)$$

Thus, when $z$ approaches 0, $M_{\kappa, \mu}(z)$ and $M_{\kappa, -\mu}(z)$ behave as follows:

$$M_{\kappa, \mu}(z) \to z^{\mu + \frac{1}{2}}, \quad M_{\kappa, -\mu}(z) \to z^{-\mu + \frac{1}{2}}. \quad (A11)$$

As our case is

$$\mu = -\frac{1}{4}, \quad \kappa = -\frac{1}{4} - \frac{i}{\pi} \quad \text{and} \quad \mu = -\frac{1}{4}, \quad \kappa = -\frac{1}{4} + \frac{i}{\pi},$$

when $z$ goes to zero, the Whittaker function $W_{\kappa, \mu}(z)$ behaves as follows:

$$W_{-\frac{1}{4} - \frac{i}{\pi}, -\frac{1}{4} + \frac{i}{\pi}}(z) \to \frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(1 + \frac{1}{2}\right)} z^{\frac{1}{2}} + \frac{\Gamma\left(-\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2} + \frac{1}{2}\right)} z^{\frac{3}{2}}, \quad (A12)$$

$$W_{-\frac{1}{4} + \frac{i}{\pi}, -\frac{1}{4} - \frac{i}{\pi}}(z) \to \frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(1 - \frac{1}{2}\right)} z^{\frac{1}{2}} + \frac{\Gamma\left(-\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2} - \frac{1}{2}\right)} z^{\frac{3}{2}}, \quad (A13)$$

where $z = \frac{1}{2} \left(\sqrt{\alpha x} - \frac{\beta}{\sqrt{\alpha}}\right)^2$ for the first Whittaker function $z = \frac{1}{2} \left(i \left(\sqrt{\alpha x} + \frac{\beta}{\sqrt{\alpha}}\right)\right)^2$ for the second Whittaker function.

By applying the description of Eqs. (A12) and (A13) to the Whittaker function, we can expand the form of $F_+(x)$ around $x = 0$ when $x$ is very small, and we thereby obtain

$$F_+(x) = a_1 \Phi_+^{(+)}(x) - a_2 \Phi_+^{(-)}(x)$$

$$= \frac{2^{-\frac{1}{2}} \Gamma\left(\frac{1}{2}\right)}{\Gamma\left(1 + \frac{1}{2}\right)} \left[ a_1 \frac{2^{-\frac{1}{2}}}{\Gamma\left(\frac{1}{2} + \frac{1}{2}\right)} \left(\left(\sqrt{\alpha x} - \frac{\beta}{\sqrt{\alpha}}\right)^2\right)^{\frac{1}{2}} + 2^{-1} \Gamma\left(-\frac{1}{2}\right) \left[ a_1 \frac{2^{-\frac{1}{2}}}{\Gamma\left(\frac{1}{2} + \frac{1}{2}\right)} \left(\left(\sqrt{\alpha x} + \frac{\beta}{\sqrt{\alpha}}\right)^2\right)^{\frac{1}{2}} - a_2 \frac{2^{\frac{3}{2}}}{\Gamma\left(\frac{1}{2} - \frac{1}{2}\right)} \left(\left(i \left(\sqrt{\alpha x} + \frac{\beta}{\sqrt{\alpha}}\right)\right)^2\right)^{\frac{1}{2}} \right] \right]. \quad (A14)$$

In order to satisfy the boundary condition that $X_+(0) = 0$ at $x = 0$, the first term must be zero because $X_+(x) = \varepsilon(x) F_+(x)$. 

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We then obtain
\[ a_1 = 2 \pi \Gamma \left(1 + \frac{i}{\pi} \right), \quad a_2 = 2 \pi \Gamma \left(1 - \frac{i}{\pi} \right). \tag{A15} \]
By using these values for \(a_1\) and \(a_2\), when \(x = 0\) in Eq. (A14), the boundary condition results in the following equation:
\[ \left( \frac{\Gamma \left(1 + \frac{i}{\pi} \right)}{\Gamma \left(\frac{1}{2} + \frac{i}{\pi} \right)} \right)^{1/2} - \left( \frac{\Gamma \left(1 - \frac{i}{\pi} \right)}{\Gamma \left(\frac{1}{2} - \frac{i}{\pi} \right)} \right)^{1/2} \left( \frac{2W_0^2}{\alpha g^2} \right)^{1/2} = 0, \tag{A16} \]

because \(\frac{\alpha^2}{\alpha} = (2W_0)^2\).

We then obtain \(W_0^2 = 0\).

**Appendix E.**

We first consider the behavior of the Green’s function that is defined as
\[ G(x, x') = \langle 0 | \Psi(x) \Psi^+(x') | 0 \rangle, \]
where \(\Psi\) is a fermion field (quark field), and \(x\) and \(x'\) have finite separation.

This Green’s function satisfies the following equation:
\[ (i \alpha^\mu \partial_\mu + \alpha^\mu gA_\mu) G(x, x') = \delta(x - x'). \tag{A15} \]
The solution can be written in the form
\[ G(x, x') = G_0(x, x') \exp(i\varphi(x) - i\varphi(x')), \tag{A16} \]
where
\[ i \alpha^\mu \partial_\mu G_0(x, x') = \delta(x - x'), \tag{A17} \]
\[ i \alpha^\mu \partial_\mu \phi(x) = \alpha^\mu gA_\mu(x). \tag{A18} \]
The solution to Eq. (A14) is
\[ G_0(x, x') = \frac{1}{2\pi} \int_0^\infty dp \exp\left(i \alpha^\mu (x_\mu - x'_\mu)\right) \quad (x_0 > x'_0) \]
\[ - \frac{1}{2\pi} \int_{-\infty}^0 dp \exp\left(i \alpha^\mu (x_\mu - x'_\mu)\right) \quad (x_0 < x'_0), \tag{A19} \]
and for equal time,
\[ G_0(x, x') = \frac{1}{2\pi i} \frac{\alpha^1}{x_1 - x'_1}. \tag{A20} \]

To find the solution to Eq. (A18), we rewrite Eq. (A18) as follows:
\[ -i \alpha^0 \partial_0 \phi(x) + \alpha^0 gA_0(x) = i \alpha^1 \partial_1 \phi(x) - \alpha^1 gA_1(x) = 0. \tag{A21} \]

Because we apply in the equal-time case and the \(A_0 = 0\) gauge, \(\phi(x)\) is included only as a spatial coordinate function in Eq. (A21).

Thus \(\phi(x)\) is expressed as
\[ \varphi(x) = \int_{-\infty}^{x_1} dz gA_1(z), \tag{A22} \]
where \(x_1\) is a spatial coordinate.
We can then describe the Green’s function $G(x, x')$ as follows:

$$G(x, x') = \frac{1}{2\pi i} \alpha^1 \exp \left( \frac{ig}{4\pi} \int_{x_1}^{x_1'} dz A_1(z) \right).$$  \hspace{1cm} (A23)

The vacuum expectation value of our amplitude is given by the following:

$$S(x, x') = \left\langle 0 \left| q(x)q^+ (x') \exp \left( \int_{x_1}^{x_1'} dz igA_1(z) \right) \right| 0 \right\rangle.$$  \hspace{1cm} (A24)

Inserting $|0><0|$ between $q^+ q$ and the exponent in Eq. (A21), we obtain

$$S(x, x') = G(x, x') \left\langle 0 \left| \exp \left( \int_{x_1}^{x_1'} dz igA_1(z) \right) \right| 0 \right\rangle.$$  \hspace{1cm} (A25)

$G(x, x')$ is given in Eq. (A23) and we notice that the phase factor cancels. We then obtain

$$S(x, x') = \frac{1}{2\pi i} \alpha^1 \frac{x_1 - x_1'}{x_1 - x_1'}. $$ \hspace{1cm} (A26)

For our case ($x_1' > x_1$ and $\alpha^1 = \sigma_3$), use of the following expression is justified:

$$S(r) = (i\sigma_3)Pr \frac{1}{2\pi r}.$$ \hspace{1cm} (A27)

Appendix F.

Here, we show the equation of motion in momentum space.

The initial equation is Eq. (33):

$$W_0X_+(x) = -2i\partial_x X_+(x) + i \frac{g^2}{8\pi} \int_{-\infty}^{\infty} ds \frac{|s| - |x - s|}{x - s} X_+(s).$$  \hspace{1cm} (A28)

Here, we consider the following two Fourier transforms:

$$T_+(p) \overset{\text{def}}{=} \frac{1}{2\pi i} \int_{-\infty}^{\infty} dx \exp(ipx)X_+(x),$$ \hspace{1cm} (A29)

$$\tilde{T}_+(p) \overset{\text{def}}{=} \frac{1}{2\pi i} \int_{-\infty}^{\infty} dx \exp(ipx)\varepsilon(x)X_+(x).$$ \hspace{1cm} (A30)

Equation (A28) can be rewritten as

$$W_0X_+(x) = -2i\partial_x X_+(x) - i \frac{g^2}{8\pi} \int_{-\infty}^{\infty} ds \frac{\varepsilon(s)s}{s - x} X_+(s)$$

$$- i \frac{g^2}{8\pi} \left[ \int_{-\infty}^{x} dsX_+(s) - \int_{x}^{\infty} dsX_+(s) \right].$$ \hspace{1cm} (A31)

Multiplying by $\exp(ipx)$ and taking a Fourier transform of Eq. (A31), we obtain the following equation:

$$W_0T_+(p) = -2pT_+(p) - \frac{g^2}{8\pi} 2\pi i \frac{\partial}{\partial p} \tilde{T}_+(p) + \frac{g^2}{8\pi} 2\pi T_+(p).$$ \hspace{1cm} (A32)

Note that the Fourier transform of Eq. (A31) becomes a mixture of an Eq. (A29) type Fourier transform and an Eq. (A30) type Fourier transform.
In order to obtain the second term on the right-hand side of Eq. (32), we can use Cauchy’s theorem for a singular integral because the integration range is \(-\infty \) to \(\infty \); we can thus take a closed contour by adding the arc of a semi-circle with an infinitely large radius. Note that the contribution of the arc of the semi-circle’s path to the integral is zero because \(X_+(x)\) at \(\infty\) is zero:

\[
\int_{-\infty}^{\infty} ds \frac{s(\epsilon(s)X_+(s))}{s-x} = \text{closed contour} L. \int_{-\infty}^{\infty} ds \frac{s(\epsilon(s)X_+(s))}{s-x} = 2\pi i \epsilon(x)X_+(x).
\]

For the last line, we used Cauchy’s theorem to obtain

\[
\frac{1}{2\pi i} \int_{-\infty}^{\infty} dx \exp(ipx) x \epsilon(x)X_+(x) = \frac{1}{2\pi} \frac{\partial}{\partial p} \left[ \int_{-\infty}^{\infty} dx \exp(ipx) \epsilon(x)X_+(x) \right] = \frac{1}{i} \frac{\partial}{\partial p} \bar{T}_+(p).
\]

Another expression is obtained by multiplying \(\epsilon(x)\) and \(\exp(ipx)\) and then taking a Fourier transform:

\[
W_0 \bar{T}_+(p) = -2p \bar{T}_+(p) - \frac{g^2}{8\pi} \frac{\partial}{\partial p} \bar{T}_+(p) + \frac{g^2}{8\pi} \bar{\Phi}_1(p) - \frac{g^2}{8\pi} \text{const.} \quad \text{(A33)}
\]

where

\[
\text{const.} = \int_{-\infty}^{\infty} dx \epsilon(x)X_+(x) = \int_{-\infty}^{\infty} dx F_+(x).
\]

Here we invoke the condition that the constant is zero. However, this condition is not peculiar because our solution satisfies this condition, as shown in the following discussion.

\(F_+(x)\) is given by Eq. (44) and is expressed as

\[
F_+(x) = a_1 \Phi_+^{(+)}(x) - a_2 \Phi_+^{(-)}(x),
\]

where

\[
\Phi_+^{(+)}(x) = C_1 e^{\frac{i}{\alpha}(x - \frac{\beta}{\alpha})^2} e^{-\frac{p^2}{4\pi \alpha}} x^{-\frac{1}{4}} \left( \frac{x - \beta}{ \alpha } \right)^2 \frac{1}{\alpha} W_{\frac{1}{2} - \frac{1}{4}, \frac{1}{2}} \left(\frac{\alpha}{2} \left( \frac{x - \beta}{ \alpha } \right)^2 \right), \quad \text{(A34)}
\]

\[
\Phi_+^{(-)}(x) = C_2 e^{\frac{-i}{\alpha}(x + \frac{\beta}{\alpha})^2} e^{-\frac{p^2}{4\pi \alpha}} x^{-\frac{1}{4}} \left( \frac{x + \beta}{ \alpha } \right)^2 \frac{1}{\alpha} W_{\frac{1}{2} + \frac{1}{4}, \frac{1}{2}} \left(\frac{-\alpha}{2} \left( \frac{x + \beta}{ \alpha } \right)^2 \right), \quad \text{(A35)}
\]

\(C_1, C_2\), are constants.

The above functions are given in Eqs. (42) and (43) in a slightly different form.

First, we consider the case of the integral for \(\Phi_+^{(+)}(x)\):

\[
\int_{-\infty}^{\infty} dx \Phi_+^{(+)}(x) = \int_{-\infty}^{0} dx \Phi_+^{(+)}(x) + \int_{0}^{\infty} dx \Phi_+^{(+)}(x).
\]

For the first term, when we change the variable \(x\) to \(-x\), the first integral then becomes

\[
I_1 = \int_{\infty}^{0} dx \Phi_+^{(+)}(-x) = \int_{0}^{\infty} dx \Phi_+^{(+)}(-x).
\]

Note that we change the variable integration range from the range of \(-\infty\) to \(0\) to the range of \(\infty\) to \(0\).
For Eq. (A34), \( x \) now becomes \(-x\), and by changing the variable such that \((-x - \frac{\beta}{\alpha})^2 = t\), we find
\[
-x - \frac{\beta}{\alpha} = \sqrt{t} \quad \text{and} \quad dx = \frac{dt}{2\sqrt{t}}.
\]
The integration range changes from the range of 0 to \( \infty \) to the range of \((-\frac{\beta}{\alpha})^2\) to \( \infty \).

Thus, the first term becomes
\[
I_1 = \int_{(-\frac{\beta}{\alpha})^2}^{\infty} \frac{dt}{2\sqrt{t}} e^{-\frac{2}{\alpha^2} t} \alpha^{-\frac{1}{2}} \frac{1}{\alpha^2} \bar{W}_{\frac{1}{2} + \frac{1}{\alpha^2} - \frac{1}{2}} \left( \frac{\alpha}{2} t \right).
\]
For the second term, taking \((-x - \frac{\beta}{\alpha})^2 = t\) and using the same argument that we applied to the first term, we obtain
\[
I_2 = \int_{(-\frac{\beta}{\alpha})^2}^{\infty} \frac{dt}{2\sqrt{t}} e^{-\frac{2}{\alpha^2} t} \alpha^{-\frac{1}{2}} \frac{1}{\alpha^2} \bar{W}_{\frac{1}{2} + \frac{1}{\alpha^2} - \frac{1}{2}} \left( \frac{\alpha}{2} t \right).
\]
Therefore \(I_1 + I_2 = 0\).

For the case, \(\Phi^{(+)}(x)\) we can use the same argument applied earlier to conclude that this part also results in zero. Thus, the constant is 0.

Summarizing Eqs. (A32) and (A33) for the condition that the constant is zero, we notice that the function \(\bar{T}_+(p) = T_+(p) + \bar{T}_+(p)\) satisfies the closed differential equation:
\[
W_0 \bar{T}_+(p) = -2p \bar{T}_+(p) - \frac{g^2}{8\pi} 2\pi i \frac{\partial}{\partial p} \bar{T}_+(p) + \frac{g^2}{8\pi} 2 \bar{T}_+(p). \tag{A36}
\]
For the second term on the right-hand side, we can apply Cauchy’s theorem, and then the second term becomes
\[
\frac{g^2}{8\pi} 2\pi i \frac{\partial}{\partial p} \bar{T}_+(p) = \frac{g^2}{8\pi} \text{C.C.L.} \int ds \frac{\partial \bar{T}_+(s)}{\partial s},
\]
where C.C.L. indicates the closed integral contour including the 0 to \(\infty\) path.

Because
\[
\text{C.C.L.} \int ds \frac{\partial \bar{T}_+(s)}{\partial s} = \text{C.C.L.} \int ds \frac{\bar{T}_+(s)}{(s - p)^2},
\]
we obtain the following equation when we change the variables such that \(8p = p'\) and \(8s = s'\):
\[
\frac{\pi}{g^2} W_0 \bar{T}_+(p') = -\frac{1}{4} \frac{\pi}{g^2} p' \bar{T}_+(p') + \frac{2}{p'} \bar{T}_+(p') - \text{C.C.L.} \int ds' \frac{\bar{T}_+(s')}{(s' - p')^2}. \tag{A37}
\]
In order to compare our result with the ’t Hooft case, we must remember that ’t Hooft’s equation was derived using the cone gauge. Therefore, ’t Hooft’s variable \(x\) is as follows:
\[
x = \frac{P_0 - P_1}{\text{max. momentum}} = \frac{\sqrt{P_1^2 + M^2} - P_1}{M}.
\]
Here, \(x\) approaches \(\frac{M}{P_1}\) as \(P_1 \to \infty\), and \(x\) approaches \(1 - \frac{P_1}{M}\) as \(P_1 \to 0\).

Thus, the first two terms on the right-hand side of ’t Hooft’s equation are proportional to \(P_1\) and \(\frac{1}{P_1}\) as \(P_1 \to \infty\) and \(P_1 \to 0\), respectively. This result implies that \(x\) approaches 0 as \(P_1 \to \infty\), and \(x\) approaches 1 as \(P_1 \to 0\).
Because this \( P_1 \) is identical to momentum \( p' \) in Eq. (A37), Eq. (A37) is essentially the same as \'t Hooft’s equation, except that the contour of the singular integral is closed.

Although we consider the comparison to be sufficient up to this point, we discuss the comparison further because the range of \( p' \) in Eq. (A37) is \( 0 \) to \( \infty \), but the range of \( x \) in \'t Hooft’s equation is \( 0 \) to \( 1 \).

Taking the change of variables to be \( p' = \frac{x}{1-x} \) and \( s' = \frac{x}{1-y} \) for \( x = 0 \) to \( 1 \), the range of \( p' \) changes to \( 0 \) to \( \infty \) (the same change occurs for \( s' \) and \( y \)).

We obtain the following form:

\[
\frac{\pi}{g^2} W_0 \tilde{T}_+(x) = -\frac{1}{4} \frac{\pi}{g^2} x \tilde{T}_+(x) + \frac{2(1-x)}{x} \tilde{T}_+(x) - \text{C.C.L.'} \int dy \frac{(1-x)^2 \tilde{T}_+(s)}{(y-x)^2}, \tag{A38}
\]

where C.C.L.’ indicates a closed integral contour including the 0 to 1 path:

\[
\tilde{T}_+ \left( \frac{x}{1-x} \right) = \tilde{T}_+(x).
\]

In order to compare our equation with \'t Hooft’s integral equation, we consider the case in which \( x \) is small (the \( x \to 0 \) case). This condition is actually the \( p' \to 0 \) case.

Then Eq. (A38) becomes

\[
\frac{\pi}{g^2} W_0 \tilde{T}_+(x) = 2 \frac{x}{\tilde{T}_+(x)} - \text{C.C.L.'} \int dy \frac{\tilde{T}_+(y)}{(y-x)^2}. \tag{A39}
\]

As mentioned previously, \'t Hooft’s \( x \) is

\[
x = \frac{P_0 - P_1}{\text{max. momentum}} = \frac{\sqrt{P_1^2 + M^2 - P_1}}{M} \to 1.
\]

Remembering that this \( P_1 \) is identical to our \( p' \), we must change the variable \( x \) to \( x' = 1 - x \) (and \( y \) to \( y' = 1 - y \) in the singular integral) in \'t Hooft’s equation because \( x \) goes to zero as \( p' \) approaches zero in our case. Then, \'t Hooft’s equation becomes

\[
\mu^2 \tilde{\varphi}(x') = \frac{\alpha_1}{1} \varphi(x') + \frac{\alpha_2}{x'} \varphi(x') - P \int_0^1 dy' \frac{\varphi(y')}{(y' - x')^2}. \tag{A40}
\]

Because this resemblance occurs only in the case where \( x \) and \( x' \) are small, we can conclude that our case is quite similar to \'t Hooft’s case but is not exactly identical. Furthermore, we must insist that the contour of the singular integral is closed for our case but not for \'t Hooft’s case.

**Appendix G.**

In order to obtain the Schwinger mass spectra, we use cluster decomposition instead of the Fierz identity because the Schwinger model is applied to the region where distance \( r \) is negligible.

Subsequently, we may conclude that

\[
q_{f0}(1, 2 : x) = q(1, x)q(x, 2) - J^0 q(1, 2).
\]

Thus we must rewrite the integral’s coefficient in Eq. (28) from \( \frac{g^2}{8\pi N} \) to \( \frac{g^2}{4\pi} \).
Because \( r \) is negligible, if we use the fact that
\[
\lim_{r \to 0} \frac{1}{r} [x(1) - x] - |x(2) - x| = 2\varepsilon(x - x(1)),
\]
write that
\[
< |J^0(x)| P > = e^{iP_1 x} X_0(0),
\]
and recall that the definition of the center-of-mass system is represented as
\[
X_i(1, 2) = -i P_0' e^{iP_1 x} X_i(r),
\]
the integral of Eq. (28) then becomes
\[
\lim_{\varepsilon \to 0} 2 \left( \int_{x(1)-\varepsilon}^{x(1)} dx (-e^{iP_1 x}) + \int_{x(1)}^{x(1)+\varepsilon} dx e^{iP_1 x} \right) = \frac{4}{i P_1} e^{iP_1 x(1)}.
\]
When \( r \) is negligible, i.e., \( x(2) \to x(1) \), we can factor out the center-of-mass term, and then Eqs. (27) and (28) become, respectively,
\[
P_0 X_0(0) = i P_1 X_1(0),
\]
\[
P_0 X_1(0) = -i P_1 X_0(0) + \frac{g^2}{\pi} \frac{1}{i P_1} X_0(0).
\]
By writing the above equation for \( X_0(0) \) only, we obtain
\[
\left( P_0^2 - P_1^2 - \frac{g^2}{\pi} \right) X_0(0) = 0.
\]
Hence, the eigenvalue is \( P_0^2 - P_1^2 = \frac{g^2}{\pi} \), which corresponds to the Schwinger mass spectra.

References