It has been around fifty years since R. K. Sachs and A. M. Wolfe predicted the existence of anisotropy in the cosmic microwave background (CMB) and ten years since the integrated Sachs–Wolfe effect (ISW) was first detected observationally. The ISW effect provides us with a unique probe of the accelerating expansion of the Universe. The cross-correlation between the large-scale structure and CMB is the most promising way to extract the ISW effect from the data. In this article, we review the physics of the ISW effect and summarize recent observational results and interpretations.

Subject Index  E51, E56, E63, E64, E65

1. Overview

After the discovery of the isotropic radiation of the cosmic microwave background (CMB) of the Universe by A. A. Penzias and R. W. Wilson in 1965 [101], R. K. Sachs and A. M. Wolfe predicted the existence of anisotropy in the CMB associated with a gravitational redshift in 1967 [117]. They fully integrated the geodesic equation in a perturbed Friedmann–Robertson–Walker (FRW) metric in the fully general relativistic framework. The Sachs–Wolfe (SW) article was the first paper to predict the presence of anisotropy in the CMB, which now plays an important role in constraining cosmological models, the nature of dark energy, modified gravity, and the non-Gaussianity of the primordial fluctuation.

Let us begin by reviewing the history of the topic. In the first twenty years after the Sachs–Wolfe paper, most of the subsequent work focused on the extension of the Sachs–Wolfe calculation to non-linear collapsed objects [97,114], or non-standard cosmological models such as topological defects [68]. Partridge and Wilkinson (1967) first gave a glimpse of the existence of inhomogeneity in the CMB temperature by using the Dicke radiometer [144]. They found a temperature excess in the direction of the known quasar cluster position and considered it to be the Rees–Sciama (RS) effect [114]. In the age of the COBE satellite, the Sachs–Wolfe paper attracted a huge amount of attention. Most of the subsequent papers focused on the theoretical prediction that was related to the observation: prediction of the amplitude of the quadrupole power for the SW effect [19,48–50]. Crittenden and Turok (1996) pointed out that the gravitational potential may decay in the $\Lambda$-dominated Universe at $z < 1$ to produce the ISW signal [26]. They also proposed a novel method to detect the ISW effect...
by cross-correlating the large-scale structure (LSS) with the CMB. Kneissl et al. (1997) made an attempt to extract the ISW effect by cross-correlating the CMB observed by COBE with the ROSAT X-ray background [70], Boughn and Crittenden (2002) used the NVSS radio galaxies for the cross-correlation [13], and Boughn et al. (1998) used the HEAO1 A2 X-ray background [16], but none of them made a significant detection.

The Sachs–Wolfe paper attracted renewed attention in the WMAP era. The first detection of the ISW was finally achieved by cross-correlating the WMAP first-year data with the number count of radio galaxies from the NVSS data, as well as with the HEAO1 A1 X-ray data [14]. Subsequently, many detections with various mass tracers have been reported. In the early 2000s, much work was focused on obtaining the cosmological constraint on dark energy models from the ISW effect, while, in the late 2000s to the present, more and more works have studied the various systematic effects that may enter in different ways for different measurement methods.

In this paper, we review the ISW effect from theoretical derivation of the basic equations to the present cosmological interpretations. The paper is organized as follows. In Sect. 2, we revisit the derivation of the CMB anisotropy induced by the perturbation of the background geometry decomposed into scalar, vector, and tensor modes. In Sect. 3, we discuss the statistical properties of the ISW effect and the method to measure it in the cross-correlation with the large-scale structure. We also discuss the possible systematic effects that affect our interpretations. In Sect. 4, we provide cosmological applications of the ISW effect, including constraints on dark energy and primordial non-Gaussianity. In Sect. 5, we give a summary.

2. Theory of the ISW effect in the standard cosmology

In this section, we derive the basic equations of the ISW effect based on the original paper [117]. We first write the line element in a spatially flat FRW metric:

\[ ds^2 = g_{\mu\nu} dx^\mu dx^\nu = a^2(\tau) \tilde{g}_{\mu\nu} dx^\mu dx^\nu, \]

where \( a \) is the scale factor that depends solely on the conformal time \( \tau \) and \( g_{\mu\nu} \) and \( \tilde{g}_{\mu\nu} \) are the metric and a conformally transformed metric, respectively. The metric consists of perturbed and unperturbed parts, i.e. \( \tilde{g}_{\mu\nu} = \eta_{\mu\nu} + \tilde{h}_{\mu\nu} \), with the unperturbed metric \( \eta_{\mu\nu} = \text{diag}(-, +, +, +) \) and \( \tilde{h}_{\mu\nu} \ll 1 \).

Here we ignore all quantities of order \( \mathcal{O}(h^2) \) and higher. Introducing two affine parameters \( \tau \) and \( \lambda \) to characterize the photon geodesic in the \( g_{\mu\nu} \) and \( \tilde{g}_{\mu\nu} \) metric, respectively, we have \( d\tau = a^2 d\lambda \), since the action of the geodesic should be invariant under the rescaling of \( g_{\mu\nu} \rightarrow a^2 g_{\mu\nu} \) and \( d\tau \rightarrow a^2 d\lambda \).

To simplify the calculation, we first work on the \( \tilde{g}_{\mu\nu} \) system and then translate it to the \( g_{\mu\nu} \) system.

The metric perturbation can be decomposed into scalar, vector, and tensor modes. \( h_{0i} \) can be divided into contributions from the scalar and vector modes while \( h_{ij} \) can be divided into contributions from the scalar, vector, and tensor modes as follows:

\[ \tilde{h}_{00} = -2A^{(s)} \]
\[ \tilde{h}_{0i} = -\partial_i B^{(s)} - B^{(v)}_i \]
\[ \tilde{h}_{ij} = -2\left[ D^{(s)} \delta_{ij} - \left( \partial_i \partial_j - \frac{1}{3} \nabla^2 \right) C^{(s)} \right] + (\partial_j C^{(v)}_i + \partial_i C^{(v)}_j) + C^{(t)}_{ij}, \]

where \( A, B, C, \) and \( D \) are arbitrary functions and the superscripts in parentheses, \( (s), (v), \) and \( (t) \), stand for the scalar, vector, and tensor quantities, respectively. The derivative \( \partial_i \) denotes the 3D covariant derivative. The scalar perturbation is not generated from the vector or tensor mode and the vector...
perturbation is not generated from the tensor mode; thus, we have constraints as
\[ \partial^i B_i^{(v)} = 0, \]
\[ \partial^i C_i^{(v)} = 0, \quad C_i^{(0)i} = 0, \quad \partial^i C_i^{(0)} = 0. \]

2.1. **Scalar mode linear perturbation**

We now need to fix the gauge degree of freedom. For the scalar perturbation, the conformal Newtonian gauge (longitudinal gauge) is useful [e.g. 80,89]. In the Newtonian gauge, all the gauge degrees of freedom are used to eliminate the off-diagonal components of the perturbed metric. Then the variables are fixed as \( B^{(s)} = C^{(s)} = 0 \), and the metric perturbation can be fully described by the two scalar quantities of \( A = \Phi, D = \Psi \), which are already gauge invariant. The metric turns out to be
\[ ds^2 = a^2[−(1 + 2\Phi)d\tau + (1 − 2\Psi)\delta_{ij}dx^i dx^j], \]
where \( \Phi \) and \( \Psi \) are the Newtonian potential and curvature perturbation respectively. Then we naturally obtain the non-vanishing connections,
\[ \tilde{\Gamma}_0^{0i} = \Phi', \quad \tilde{\Gamma}_0^{ij} = \partial_i\Phi, \quad \tilde{\Gamma}_ij = −\Psi\delta_{ij}, \]
where \( ' \equiv \partial/\partial \tau \). The photon geodesic \( x(\lambda) \) can be obtained by solving the geodesic equation:
\[ \frac{d\tilde{k}^\mu}{d\lambda} + \tilde{\Gamma}_{\mu\nu\lambda} \tilde{k}^\nu \tilde{k}^\lambda = 0, \]
where we introduce the 4-momentum \( \tilde{k}^\mu = \frac{dx^\mu}{d\lambda} \). This can be decomposed into unperturbed and perturbed geodesics as \( \tilde{k}^\mu = \tilde{k}^\mu + \delta \tilde{k}^\mu \). The photon energy is measured by an observer moving with the fluid, thus the observed energy needs to be projected with the 4-velocity in the un-tilde \( g_{\mu\nu} \) system:
\[ E = g_{\mu\nu}u^\mu u^\nu. \]
By definition, \( g_{\mu\nu}u^\mu u^\nu = 1 \), and the unperturbed component is \( u^\mu = a^{-1}(1, 0, 0, 0) \). The 4-velocity is written as \( u^\mu = a^{-1}(1 − \Phi, \nu^i) \). The spatial 3D velocity is already a first-order quantity, hence we do not need to explicitly solve the spatial part of the geodesic equation (9). The solution for the unperturbed background is trivial, i.e. \( \tilde{k}^0 = 1, \tilde{k}^i = \nu^i \), where \( \nu = (1, \nu^i) \) is the 4-tangent vector of the geodesic. The time part of the first-order solution of Eq. (9) is integrated as
\[ \frac{\delta \tilde{k}^0}{a^2} = \frac{\delta \tilde{k}^0}{a^2} \bigg|_{\tau_0}^{\tau_s} + 2[\Phi(\tau_s) − \Phi(\tau_0)] + \int_{\tau_0}^{\tau_s} (\Phi' + \Psi') d\tau, \]
where \( \tau_s \) denotes the conformal time of the decoupling time, and \( \tau_0 \) today. Now the redshift is defined by the photon energy ratio between emitter and receiver, \( 1 + z \equiv E(\tau_s)/E(\tau_0) \), and, using the fact that, under the rescaling of \( \tilde{g}_{\mu\nu} \rightarrow g_{\mu\nu} \), the 4-momentum scales as \( \tilde{k}^\mu \rightarrow a^2 k^\mu \),
\[ 1 + z = \frac{\tilde{k}^\mu u_\mu|_{\tau_s}}{k^\nu u_\nu|_{\tau_0}}. \]
Since the temperature drops with redshift as \( T = T_s/(1 + z) \), using Eqs. (10), (11), and (12), the observed temperature fluctuation over the sky is
\[ \frac{\delta T^{(s)}}{T} = \frac{\delta T^{(s)}}{T} \bigg|_{\tau_s}^{\tau_0} - \Phi(\tau_0) + \Phi(\tau_s) + [\nu \cdot e]_{\tau_0}^{\tau_s} + \int_{\tau_0}^{\tau_s} (\Phi' + \Psi') d\tau, \]
apart from the isotropic temperature. The first term is the intrinsic photon fluctuation at the last scattering surface other than those induced by the metric perturbation; \( \delta T / T \big|_{\tau_s} = \delta \gamma(\tau_s)/4 \). The second
The term is the gravitational redshift due to our gravitational potential; this is the monopole contribution and cannot be observed. The third term represents the temperature anisotropy caused by the gravitational redshift due to the potential fluctuations at the decoupling epoch, which is called the naive or ordinary Sachs–Wolfe effect in the literature. Since it also has spatial dependence, we shall write \( \Phi(\tau_s) = \Phi(\tau_s, \hat{n}) \), where \( \hat{n} \) denotes the angular position on the sky. The fourth term is the Doppler effect, which is induced by the relative motion between the observer and the CMB last scattering surface. The final integral term represents the temperature anisotropy caused by the time variation of the gravitational potential integrated along the line of sight, and this is the ISW effect. Since the gravitational potential is static in the matter-dominated Universe, i.e. the Einstein–de-Sitter (EdS) Universe, the ISW effect vanishes in the linear perturbation limit. Thus the ISW effect induces temperature fluctuation in the radiation-dominated era or a dark-energy or curvature-dominated Universe. The former is called the early ISW and the latter the late ISW effect. We note that, from the current observations, the matter radiation equality time, \( z_{\text{eq}} \approx 3300 \), is well before the decoupling, \( z_{\text{dec}} = 1090 \), and thus the temperature fluctuation of the early ISW is regarded as part of the primary anisotropy. We also note that careful authors include the visibility function to the last scattering surface in the integrand, \( e^{-\tau} \), where \( \tau \) is the optical depth; however, in the flat \( \Lambda \) CDM Universe, the redshift where the ISW effect becomes important is at \( z < 1 \), and thus \( e^{-\tau} = 1 \) is a good approximation.

### 2.2. Vector and tensor mode perturbations

The photon geodesic is also perturbed by the vector and tensor modes. For the vector mode, the geodesic perturbation can be characterized by \( B_i \) and \( C_i \). The temperature anisotropy induced by the vector mode is given by

\[
\frac{\delta T^{(v)}}{T} = \left. \frac{\delta T^{(v)}}{T} \right|_{\tau_s} + \left[ V_i e_i \right]_{\tau_0}^{\tau_s} + \frac{1}{2} \int_{\tau_0}^{\tau_s} d\tau \left( \partial_i V_j + \partial_j V_i \right) e^i e^j,
\]

where the first term is intrinsic vector-type temperature fluctuation, \( V_i \) is the rotational component of the velocity, and \( V_i = C_i^\prime + B_i \). In the standard inflation scenario, the vacuum fluctuation generates no super-horizon vector mode perturbation [76]. Even if it is generated with some exotic mechanisms, the vector mode has a only decaying mode solution, which can be negligible at later times; thus we shall assume that the vector mode does not exist [75]. In practice, it makes a negligible contribution to the CMB temperature observation, so we assume that it is absent.

The tensor mode, in other words the gravitational wave, is given by the traceless transverse rank two tensor, \( C_{ij} \) in Eq. (4). The temperature fluctuation induced by the tensor mode metric perturbation is

\[
\frac{\delta T^{(t)}}{T} = \left. \frac{\delta T^{(t)}}{T} \right|_{\tau_s} - \frac{1}{2} \int_{\tau_0}^{\tau_s} d\tau C^{\alpha (0)}_{ij} e^i e^j.
\]

The tensor mode decays on scales smaller than the horizon at the decoupling, say \( \sim 1 \) degree, and the significant contribution comes from the largest scales. Recently, the BICEP2 Collaboration reported that they had detected a signature of large amplitude of the gravitational wave, which was observed by the B-mode polarization power spectrum of the CMB [8]. This is indeed the tensor ISW effect! The best-fit value of the tensor-to-scalar ratio is \( r = 0.16 \) after removing the foreground components.

### 2.3. Spectrum of the ISW effect

In this section, we consider the power spectrum for the scalar mode perturbation. Here we assume that the energy contents of the Universe have no anisotropic stress, which relates \( \Phi \) and \( \Psi \) as \( \Phi = \Psi \).
Then the integrated Sachs–Wolfe effect is written as

$$\Theta_{\text{ISW}}(\hat{n}) = \frac{\delta T_{\text{ISW}}(\hat{n})}{T} = 2\int \frac{\partial \Phi(\hat{n}, \tau)}{\partial \tau} d\tau. \quad (16)$$

Now the potential $\Phi$ can be related to the density fluctuation through the Poisson equation. Working in the Fourier space makes things simple. In the Fourier space, the Poisson equation in the flat FRW universe is

$$\Phi(\mathbf{k}, \tau) = \frac{3}{2} \frac{\Omega_m}{a} \left( \frac{H_0}{k} \right)^2 \left[ \frac{3aH}{k^2} \theta(\mathbf{k}, \tau) + \delta(\mathbf{k}, \tau) \right] \quad (17)$$

where $\delta$ and $\theta$ are the density contrast and divergence of velocity of the matter, respectively, and $\Omega_m$ and $H_0$ are the matter density and Hubble parameter today, respectively. In the Newtonian limit where the scale of interest is much smaller than the horizon, i.e. $k \gg aH$, Eq. (17) is reduced to the well known form, $\Phi = 3\Omega_m/2a(H_0/k)^2\delta$. We usually expand the Fourier basis into the spherical harmonics and spherical Bessel function to obtain the full-sky expression of the fluctuation in the direction $\hat{n}$:

$$\Theta_{\text{ISW}}(\hat{n}) = 12\pi \Omega_m H_0^2 \sum_{lm} (-i)^l \int d\tau \int \frac{dkd\Omega_k}{(2\pi)^3} \frac{\partial}{\partial \tau} \left( \frac{\delta(\mathbf{k}, \tau)}{a} \right) j_l(kr) Y_{lm}^*(\hat{n}) Y_{lm}(\hat{k}). \quad (18)$$

The spherical harmonic counterpart is obtained using the orthogonality of the spherical harmonics:

$$\Theta_{lm} = \int d\Omega_k \Theta(\hat{n}) Y_{lm}^*(\hat{n}) = 12\pi \Omega_m H_0^2 (-i)^l \int d\tau \int \frac{dkd\Omega_k}{(2\pi)^3} \frac{\partial}{\partial \tau} \left( \frac{\delta(\mathbf{k}, \tau)}{a} \right) j_l(kr) Y_{lm}^*(\hat{k}). \quad (19)$$

The angular power spectrum is then calculated as

$$C_l^{\text{ISW}} = \langle \Theta_{lm} \Theta_{lm}^* \rangle = \frac{18}{\pi} \Omega_m^2 H_0^4 \int dk P(k) \left[ \int dr D(r - 1) \mathcal{H} j_l(kr) \right]^2, \quad (20)$$

where $f$ is the velocity factor, $f \equiv d \ln D/d \ln a$, and $\mathcal{H} = a'/a$ is the conformal Hubble parameter. The numerical calculation of Eq. (20) would be formidable because of the oscillatory behavior of the Bessel function. The Hankel transform, also known as the Fourier–Bessel transform, can be helpful instead of direct integration [53, 98]. Limber’s approximation is used in the literature but it is accurate only up to $\sim 10\%$ on scales larger than $l < 10$ [e.g. 37].

Figure 1 shows the power spectra of the primary CMB temperature fluctuation and the ISW effect for the best-fit model to the Planck first-year data combined with the WMAP polarization and high-$\ell$ CMB experiments [108]. We also show the anisotropy generated from the tensor mode with the tensor-to-scalar ratio $r = 0.16$, which was recently suggested from the discoveries made by the BICEP2 experiment [8]. We use the publicly available CAMB code [9] to compute the scalar and tensor spectra. The ISW component can be easily calculated by a slight modification to the “equations.f90” in the code. Most of the signal at $l > 20$ comes from the primary anisotropy while, on large scales, a significant fraction of anisotropy is generated at low redshifts, $z < 0.5$. We also show the first-year low-$\ell$ Planck data [10].

Because the large $r$ value constrained by the B-mode power spectrum, $C_l^{BB}$, of the BICEP2 enhances the amplitude of the temperature fluctuation, $C_l^{TT}$, on large scales, it is necessary to make some modifications to the model to keep the current observational constraints from the WMAP or the Planck unchanged: either a smaller (i.e. negatively larger) value of running of the scalar spectral index $\alpha_s$ [8] or isocurvature component of the initial fluctuation is required [69] to suppress the large-scale...
Fig. 1. Temperature anisotropy at low multipoles. The thick solid line shows the total anisotropy integrated from today to the last scattering surface including scalar and tensor contributions. The thin dashed line shows the scalar contribution of the primary anisotropy at the last scattering surface, the thick dashed and dot-dashed lines show contributions from the ISW effect generated at $z < 0.5$ and $z < 100$, respectively, and the horizontal thick purple line below $100 [\mu K^2]$ shows the tensor contribution with $r = 0.16$. Also shown by points with error bars is the CMB power spectrum obtained by the first-year low-$\ell$ Planck data [107].

power of the CMB temperature power spectrum. However, the amplitude of the tensor mode with the $r = 0.16$ model is subdominant at $\ell < 10$ compared to the ISW component. Thus, there still remains some possibility that non-standard gravity models or nonlinearity of the local large-scale structure can alter the amplitude of the ISW effect, and thus the constraint on the negatively large value of the running or the fraction of the isocurvature perturbation component might be reduced.

2.4. Nonlinear ISW effect

M. J. Rees and D. W. Sciama extended the Sachs–Wolfe calculation to nonlinear collapsed objects in 1968 [114]. In 1967, they were investigating an apparent clustering of quasars on large scales [113] reported by Strittmatter and Faulkner [130]. They considered the possibility that inhomogeneity in the matter distribution inferred from the large-scale clustering of quasars creates anisotropy in the CMB, as predicted by Sachs and Wolfe, and estimated the amplitude of the temperature fluctuation induced by a spherically symmetric collapse of the objects in an expanding Universe [114]. Note that here we use the word *nonlinear* for nonlinear density fluctuations but the metric perturbations (and hence the geodesic perturbations) are kept at first order. The second-order cosmological perturbation is treated in a consistent manner in Refs. [95,135–137] but here we limit our discussion to the first-order geodesic equation and nonlinearity is only included in density perturbations. Using the Poisson equation, we see that there are two contributions to the time evolution of a gravitational potential:

$$\Phi' = \frac{3}{2} \Omega_m \left( \frac{H_0}{k} \right)^2 \frac{1}{a} (i k \cdot p - \mathcal{H} \delta).$$  \hspace{1cm} (21)

Here we have used the continuity equation $\delta' = i k \cdot p$, where $p$ is the momentum of the density field $p = v(1 + \delta)$. From Eq. (21), we see that the ISW and RS effects consist of two components: one
proportional to the Hubble flow and the other the momentum of the object. In the linear perturbation limit, the momentum is equal to the velocity field so the linear ISW effect in principle traces the statistical property of the large-scale velocity field [24]. In a weakly nonlinear regime or fully nonlinear regime, a halo model approach is used to describe the nonlinear time evolution of the gravitational potential, which was originally developed for describing the nonlinear clustering of the dark matter halo [24,81,121]. The dark matter power spectrum is described by the sum of two contributions: a two-halo term where the pair is in different halos, and a one-halo term where the pair is in the same halo [123]. Once we provide the mass function [e.g. 66,126,134,143] and profile of the dark matter halo [91] then we can immediately calculate the nonlinear clustering of the dark matter. Similarly, the velocity field can be decomposed into two components: the velocity due to virial motion about the center of mass of its parent halo, and that due to the motion of the parent halo itself [125], which provides the nonlinear momentum power spectrum [81].

Another approach is the higher-order perturbation theory. As we will see in Sect. 3, the cross-correlation between the ISW and density tracer is useful to isolate the ISW effect from the CMB [26]. Then the angular cross-correlation power spectrum between the ISW and any tracers of the density field can be some function of the cross-power spectrum of $\Phi'$ and $\Phi$:

$$P_{\Phi'\Phi}(k) = \frac{9}{4} \Omega_m^2 \left( \frac{H_0}{k} \right)^4 \frac{1}{a^2} (P_{\delta\delta} - \mathcal{H}P_{\delta\delta}),$$  \hspace{1cm} (22)

where we employ the notation $(X(k)Y^*(k')) \equiv (2\pi)^3 \delta^D(k-k') P_{XY}(k)$.

The continuity equation is written as $\delta' = -\theta (1 + \delta)$, where $\theta$ is the divergence of the velocity. Then the perturbed variables $\delta$ and $\theta$ can be expanded in a series:

$$\delta(k, \tau) = \sum_n D^n(\tau) \delta_n(k, \tau),$$  \hspace{1cm} (23)

$$\theta(k, \tau) = \mathcal{H} f \sum_n D^n(\tau) \theta_n(k, \tau),$$  \hspace{1cm} (24)

where the $n$th variable is written in terms of the product of linear fluctuations as

$$\delta_n(k, \tau) = \int \frac{d^3 q_1}{(2\pi)^3} \cdots \frac{d^3 q_n}{(2\pi)^3} \delta_1(q_1) \cdots \delta_1(q_n) F_n(q_1, \ldots, q_n) \delta^D \left( \sum_i q_j - k \right),$$  \hspace{1cm} (25)

$$\theta_n(k, \tau) = - \int \frac{d^3 q_1}{(2\pi)^3} \cdots \frac{d^3 q_n}{(2\pi)^3} \delta_1(q_1) \cdots \delta_1(q_n) G_n(q_1, \ldots, q_n) \delta^D \left( \sum_i q_j - k \right),$$  \hspace{1cm} (26)

where $\delta^D$ is the Dirac delta function. The functions $F_n$ and $G_n$ describe the mode coupling between different wavevectors, and are explicitly given by Refs. [65,82]. Keeping all the terms that are less than the fourth order of either $\delta_1$ or $\theta_1$, we have [93]

$$P_{\delta\delta}(k) = D^2 P(k) + 2D^4 \int \frac{d^3 q}{(2\pi)^3} P(q) \left[ P((k-q)) F_2^2(q, k-q) + 3P(k) F_3(q, -q, k) \right],$$  \hspace{1cm} (27)

$$P_{\delta\theta}(k) = \mathcal{H} f D^2 P(k) - \mathcal{H} f D^4 \left[ P(k) \int \frac{d^3 q}{(2\pi)^3} P(q)(F_3(q, -q, k) + 3G_3(q, -q, k)) + 2 \int \frac{d^3 q}{(2\pi)^3} P(q) P((k-q)) F_2^2(q, k-q) G_2^2(q, k-q) \right]$$

$$+ 2 \int \frac{d^3 q}{(2\pi)^3} P(q) P((k-q)) F_2^2(q, k-q) G_2^2(q, k-q)$$
Fig. 2. The dimensionless cross-power spectrum between $\Phi$ and $\Phi'$, $\Delta^2_{\Phi\Phi}(k)$ at different redshifts computed using Eqs. (22), (27), and (28). Dashed lines show a negative cross-power, while the solid lines are positive. The figure is adapted from Ref. [93].

\begin{equation}
+ 2 \int \frac{d^3q}{(2\pi)^3} \left( F_2(q, k - q)P(q)P(|k - q|) + G_2(q, q - k)P(k)P(|k - q|) \right) \\
+ F_2(k, -q)P(k)P(q) \left[ \frac{k \cdot q}{q^2} \right],
\end{equation}

where $P(k)$ is the linear power spectrum of $\delta$. More conveniently, $P_{\delta\delta'}$ can be approximated by $P_{\delta\delta'}(k, \tau) = \frac{1}{2} \frac{\partial}{\partial \tau} P_\delta(k, \tau)$ [94,140].

Figure 2 shows the 3D cross-correlation power spectrum of $\Phi$ and $\Phi'$ at different redshifts in the $\Lambda$CDM Universe. In the linear regime, the gravitational potential is negative, $\Phi < 0$, in the overdense region, and decays with time due to the accelerating expansion, so that $\Phi' > 0$. Thus $\Phi$ and $\Phi'$ show an anti-correlation. In the nonlinear regime, the gravitational potential grows, i.e. the potential well gets deeper, and thus $\Phi' < 0$, which gives $\Phi$ and $\Phi'$ a positive correlation. At redshift $z = 10$, the Universe is close to being matter dominated so that the linear ISW effect is small and the nonlinear RS effect appears relatively prominently. At lower redshifts, the linear ISW effect is significantly enhanced since the fraction of dark energy becomes more substantial, but we can still see the transition scale as the power turns from an anti-correlation to a correlation at $k \sim \mathcal{O}(0.1)$ Mpc/$h$. At $z = 0$, the linear ISW effect dominates at all scales.

The RS effect generates temperature anisotropy not only through the contraction and expansion of the structure but also the bulk motion of the structure perpendicular to the line of sight [3,4,88,116]. Birkinshaw and Gull [11,12] point out that the moving cluster, which has transverse velocity to the line of sight, may produce the local dipole structure on the CMB temperature. In order to explore the complete dynamics, N-body simulation with the ray-tracing method is quite useful. Tuluie and Laguna [138,139] carry out a ray-tracing simulation for a maximum 360 Mpc/$h$ box N-body simulation to see the cumulative temperature fluctuation from $z = 100$ to today. They separate the sources of anisotropy into the intrinsic change in gravitational potential of structure and the transverse bulk velocity of structure; however, the box size is not large enough to see the dynamics of clusters or voids whose sizes are larger than 100 Mpc/$h$. Cai et al. [22] use a Gigaparsec-size simulation to construct a full-sky map of the ISW. They argue that the nonlinear RS and moving halo contributions to the total are not significant, <10%, but the relative importance (relative to the linear ISW) is much higher if we go to a higher redshift that is consistent with Refs. [21,129] and with Fig. 2. This is simply
because the linear ISW is negligible at high redshifts where the Universe is still matter dominated, while the nonlinear RS effect exists regardless of dark energy.

### 3. Observing the ISW and RS effects

The ISW effect was first detected in the CMB measured by the WMAP cross-correlated with the large-scale structure data traced by the X-ray background radiation and radio galaxies [14]. Several attempts have been made to detect the ISW effect by cross-correlating the CMB map measured by COBE with the ROSAT X-ray background [70], NVSS radio galaxies [13], or HEAO1 X-ray background [16], but they could not detect the signal. However, the non-detection of the ISW effect can put an upper limit on the amount of dark energy (DE) and, surprisingly, Boughn and Crittenden [13] present the limit \( \Omega_{\Lambda} < 0.74 \), which is fairly close to the current limit of the dark energy parameter [109]. Subsequently, a number of papers appeared. Figure 3 shows the first detection of the ISW effect in 2004, and the latest detection in 2013. Table 1 presents a summary of the detection of the ISW effect today. Some papers reach consistent conclusions while others show significant inconsistencies. This is mainly due to either the wrong statistics being used or contamination of the sample due to an incomplete subtraction of systematics.

In the following subsections, we review the observations of the ISW and RS effects. In Sect. 3.1, we define the cross-correlation methods. In Sect. 3.2, we discuss the possible systematic effects that affect the significance of the detection of the ISW effect.

#### 3.1. Cross-correlation with the LSS

Since the ISW effect is generated when photons pass through a time-varying gravitational potential of the large-scale structure, it can be detected by cross-correlating with some tracers of the large-scale structure. We can write this idea as follows:

\[
\langle \Theta \delta_{\text{LSS}} \rangle = \langle (\Theta_{\text{dec}} + \Theta_{\text{fg}} + \Theta_{\text{ISW}} + \Theta_{\text{SZ}} + \Theta_{\text{lens}} + \cdots) \delta_{\text{LSS}} \rangle, \tag{29}
\]

where \( \Theta \) is the temperature fluctuation and \( \delta_{\text{LSS}} \) is the density fluctuation of some tracers of the large-scale structure, e.g. galaxy number count. Sources of CMB temperature fluctuation include \( \Theta_{\text{dec}} \), the primary anisotropy at or before the decoupling epoch, and \( \Theta_{\text{fg}} \), any astrophysical foreground contamination from the solar system or Galactic plane. The rest of the terms in Eq. (29) are attributed to the cosmological origin that generates secondary CMB anisotropy. \( \Theta_{\text{ISW}} \) is the ISW effect, \( \Theta_{\text{SZ}} \) is the Sunyaev–Zel’ dovich (SZ) effect [131] and \( \Theta_{\text{lens}} \) is the gravitational lens effect [46,61,74]. Here we assume that the large-scale structure is not correlated with the primary CMB at the decoupling epoch and we also assume that the ISW effect and the other secondary CMB sources can be distinguished, since the ISW effect is only important at the largest scales, while the other effects are dominant at much smaller scales.

#### 3.1.1. Angular power spectrum.

The observed quantities used in the literature can be classified into four types: angular power spectrum, angular correlation function, wavelet, and stacking. As we observe the CMB temperature fluctuation projected on the sky, it can be expanded into the spherical harmonic series

\[
a_{lm}^{\text{ISW}} = \int d\Omega_\hat{n} \Theta_{\text{ISW}}(\hat{n}) Y_{lm}^*(\hat{n}), \tag{30}
\]

where \( \hat{n} \) is a unit vector pointing toward a 2D position on the sky, \( Y_{lm} \) is the spherical harmonic function, \( d\Omega_\hat{n} \) is the volume element of the unit sphere and the integral is over the whole sky.
Table 1. A comparison of ISW detection. If the redshift of the large-scale structure tracer is measured spectroscopically, we denote the range by $z_s$. We use $z_p$ for the photometric redshift range, whereas we use $z$ when the redshift is inferred by other methods, such as integrating the luminosity function, fitting the amplitude of cross-correlation, or partial cross-matching with known redshift sources. The numbers after the survey name stand for the data release, e.g. WMAP1 is the WMAP first-year data release, and SDSS4 is the fourth SDSS data release and so on. a) Joint constraints with the Type Ia supernovae (SNIa) data. b) Includes a posteriori selection. c) WMAP5 + BAO + SN1a + ISW. d) CMASS + LOWZ sample.

<table>
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<th>Author</th>
<th>CMB</th>
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<th>Method</th>
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<th>DE constraints</th>
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<td>Kneissl et al. '97 [70]</td>
<td>COBE4</td>
<td>ROSAT XRB</td>
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<td>HEAO1 XRB</td>
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<td>CCF</td>
<td>No detection</td>
<td>$\Omega_\Lambda &lt; 0.74$</td>
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<td>WMAP1</td>
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<td>$z \simeq 0.3, 0.5$</td>
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<td>Cabre et al. '06 [20]</td>
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<td>$4.4\sigma$</td>
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<td>$0.075 &lt; \Omega_m &lt; 0.475(1\sigma)$</td>
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<td>Pietrobon et al. '06 [102]</td>
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<td>$0.41 &lt; \Omega_\Lambda &lt; 0.79(2\sigma)$</td>
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<td>2MASS XSC</td>
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<td>$w = -0.70_{-0.50}^{+0.05}(1\sigma)$</td>
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<td>SDSS6,8(main)</td>
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<td>CCF</td>
<td>$2.2\sigma, 2.2\sigma$</td>
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<td>Redshift Range</td>
<td>Reference</td>
<td>Source</td>
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<td>2.2( \sigma ), 2.5( \sigma )</td>
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<td>SDSS6(QSO)</td>
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<td>NVSS</td>
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<td>SDSS6(LRG)</td>
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<td>SDSS6(QSO)</td>
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<td>CCF</td>
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<td>Savangwit et al. '10 [118]</td>
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<td>NVSS</td>
<td>( 0 &lt; z &lt; 3 )</td>
<td>CCF</td>
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<td>( 0.2 &lt; z_s &lt; 0.5 )</td>
<td>CCF</td>
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<td>Goto et al. '12 [47]</td>
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<td>WISE pre.</td>
<td>( 0 &lt; z &lt; 0.3 )</td>
<td>APS</td>
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<td>SDSS6 (LRG)</td>
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<td>Planck et al. XIX '13 [106]</td>
<td>Planck1</td>
<td>BOSS8(^d)</td>
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<td>SDSS8(main)</td>
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<tr>
<td></td>
<td></td>
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<td>Ilic et al. '13 [64]</td>
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<td>SDSS6,7(Void)</td>
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<td>WISE full</td>
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<td>APS</td>
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<tr>
<td>Ferraro et al. '14 [33]</td>
<td>WMAP9</td>
<td>WISE full (gal)</td>
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<td>APS</td>
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<td></td>
<td></td>
<td>WISE full (AGN)</td>
<td>( z \sim 1.1 )</td>
<td>APS</td>
<td></td>
<td>combined</td>
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</table>
Fig. 3. Left panel shows the first detection of the ISW in 2004 by Boughn and Crittenden [14]. They use the WMAP first-year data cross-correlated with the NVSS radio galaxies and X-ray data and find the ISW effect with 3.0 \( \sigma \) detection. Right panel shows the latest detection in 2013 by Hernandez-Monteagudo et al. [but see also Ref. [33] 56]. They use the WMAP ninth-year data cross-correlated with the LRG and CMASS galaxies observed by the BOSS and find the ISW effect with 1.6 \( \sigma \) detection.

The expanded temperature fluctuation can be cross-correlated with the density tracer \( X \),

\[
a^X_{lm} = \int d\Omega \delta_X(\hat{n}) Y^*_{lm}(\hat{n}),
\]

(31)

where \( X \) is the matter tracer field projected on to the sky by

\[
\delta_X(\hat{n}) = \int_0^{r_s} dr \delta_m(\hat{n}r, \tau) W_X(r),
\]

(32)

with the projection kernel \( W_X \). The upper bound of the integral is defined by the maximum distance of the source that has a non-zero contribution to the projection. For the galaxy distribution, the projection kernel is

\[
W_{gal}(r) = b(r)r^2\phi(r) \left[ \int dr' r'\phi(r') \right]^{-1},
\]

(33)

where \( \phi \) is the radial selection function and \( b(r) \) is the galaxy bias that may depend on redshift but not on the scale (by assumption). We have assumed that, in the scale where the ISW effect is important, the galaxy number counts are linearly related to the underlying dark matter density but, strictly speaking, the linear relation is valid only on very large scales. It breaks down on smaller scales, introducing a possible source of systematics. However, in most of the ISW studies, it is typically enough to take into account the redshift evolution of the galaxy bias only and ignore the possible scale dependence of the bias. Weak lensing convergence is another powerful tool to trace the distribution of the dark matter, which has the kernel [see e.g. Ref. 6]

\[
W_\kappa(r) = \frac{3}{2} \Omega_m H_0^2 r \int d \tau' \frac{r'}{r-r'},
\]

(34)

where \( p(r) \) is the distribution of galaxies whose shapes are measured and it is normalized as \( \int dr p(r) = 1 \). From the statistical isotropy we can define the angular power spectrum (APS) as

\[
(a^X_{lm} a^{X*}_{l'm'}) = C^I_{ll'} \delta_{ll'} \delta_{mm'}. \]

(35)
Here the ensemble average is over possible realizations of the Universe; however, we have only one Universe in reality. Thus $C_l$ is estimated by averaging over $2l + 1$ modes for each multipole $l$-mode:

$$\hat{C}^{\text{ISW}-X}_l = \frac{1}{2l + 1} \sum_{m=-l}^{m=l} a_{lm}^{\text{CMB}} a_{lm}^{X,*},$$

(36)

In the practical observation, the whole sky is not observed. For example, we mask the region where the signal is not reliable or is significantly contaminated by the foreground such as dust and synchrotron emissions from the Galaxy, or zodiacal light of the solar system. The extra-galactic radio sources should also be masked. Then the cut sky harmonic coefficient, $\tilde{a}_{lm}$, is

$$\tilde{a}_{lm} = \int d\Omega_\mathbf{n} W(\mathbf{n}) \Theta_{\text{ISW}}(\mathbf{n}) Y_{lm}^*(\mathbf{n}),$$

(37)

where $W(\mathbf{n})$ is a window function (or a mask function) that takes 1 at the observed pixel and 0 at the masked pixel. The power spectrum convolved with the mask function $\tilde{C}_l \equiv \langle \tilde{a}_{lm} \tilde{a}_{lm}^* \rangle$ is called the pseudo power spectrum. It has a smaller amplitude, typically by $\tilde{C}_l \simeq f_{\text{sky}} C_l$, where $f_{\text{sky}}$ is the fraction of sky observed. The effect of the mask can be corrected in an unbiased manner with the pseudo $C_l$ estimator (PCL) [31,59] or quadratic maximum likelihood estimator (QML) [30,132].

### 3.1.2. Angular correlation function.

The cross-correlation function (CCF) is defined as the Legendre transform of the power spectrum

$$C^{\text{ISW}-X}(\theta) = \frac{1}{4\pi} \sum_l (2l + 1) C_l^{\text{ISW}-X} P_l(\cos \theta) b_l^{\text{CMB}} b_l^{X} p_l^2,$$

(38)

where $p_l$ and $b_l$ are the pixel and beam transfer functions, respectively. The inverse transform is

$$C_l^{\text{ISW}-X} = 2\pi \int d\cos \theta P(\cos \theta) C^{\text{ISW}-X}(\theta) \left( b_l^{\text{CMB}} b_l^{X} p_l^2 \right)^{-1}.$$

(39)

The estimator of the correlation function is nothing but

$$\hat{C}^{\text{ISW}-X}(\theta) \equiv \frac{\sum_{ij} \Theta_{\text{CMB}}(\hat{\mathbf{n}}_i) \delta_X(\hat{\mathbf{n}}_j) w_{\text{CMB}}(\hat{\mathbf{n}}_i) w_X(\hat{\mathbf{n}}_j)}{\sum_{ij} w_{\text{CMB}}(\hat{\mathbf{n}}_i) w_X(\hat{\mathbf{n}}_j)},$$

(40)

where $\cos \theta = \hat{\mathbf{n}}_i \cdot \hat{\mathbf{n}}_j$. The function $w$ is used to minimize the variance when the depth of the galaxy survey or effective sensitivity to the CMB at each pixel is not uniform, at the expense of increasing sample variance [35].

### 3.1.3. Wavelet.

Wavelet analysis is useful particularly when signals are localized in both configuration and frequency space. For a cross-correlation study, the covariance of the wavelet coefficients (WLT) can be used [85,102,141,142]. The WLT covariance at a given scale in configuration space $R$ is defined as

$$C^{\text{ISW}-X}_\psi(R) = \frac{1}{N_R} \sum_{\hat{\mathbf{n}}} \omega_{\text{CMB}}(R, \hat{\mathbf{n}}) \omega_X(R, \hat{\mathbf{n}}),$$

(41)

where $\omega(R, \hat{\mathbf{n}})$ is the wavelet coefficient at the position $\hat{\mathbf{n}}$. It can be obtained by convolving a map with a wavelet function. Here we assume that the wavelet function is given by a spherical Mexican
hat (SMH) wavelet given by

\[ \Psi(y, R) = \frac{1}{\sqrt{2\pi N(R)}} \left[ 1 + \left( \frac{y}{2} \right)^2 \right] \left[ 2 - \left( \frac{y}{R} \right)^2 \right] e^{-y^2/2R^2}, \tag{42} \]

where \( N(R) \) is a normalization constant, \( N(R) = R\sqrt{1 + R^2/2 + R^4/4} \). The distance on the tangent plane is given by \( y = 2 \tan(\theta/2) \). Then the WLT coefficient is

\[ \omega_X(R, \hat{n}) = \int d\Omega_\hat{n} X(\hat{n} - \hat{n}') \Psi(\theta', R), \tag{43} \]

where the volume element \( d\Omega_\hat{n} \) subsumes an infinitesimal solid angle pointing toward \( \hat{n} = (\theta, \phi) \).

The observed WLT covariance can be compared with the theoretical calculation with this formula:

\[ C_{\psi}^{\text{ISW} - X, \text{TH}}(R) = \sum_i \frac{2l + 1}{4\pi} \rho_i^2 \Psi_i^2(R) b_i^\text{CMB} b_i^X C_i^{\text{ISW} - X, \text{TH}}, \tag{44} \]

where \( \Psi_i \) is the spherical harmonic coefficient of the SMH and \( C_i^{\text{ISW} - X, \text{TH}} \) is the theoretical prediction of the cross-correlation APS between the ISW and mass tracer.

The choice of the wavelet function, Eq. (42), is not unique. Veilva et al. (2004) and Mukherjee and Wang (2004) [90,141] applied the SMH wavelet and Cruz et al. (2006) [27] used the elliptical Mexican hat wavelet to find a prominent cold spot in the CMB map with more than 2 \( \sigma \). However, Zhang and Huterer (2010) [148] claimed that the cold spot is found to be statistically significant only if the SMH wavelet is used, but not with the top-hat or Gaussian windows. As we have mentioned, the WLT is useful for finding localized features in the data, but we should be careful when interpreting the results, given the degree of freedom in choosing the wavelet functions.

3.1.4. Stacking. The stacking method is particularly useful for measuring the average temperature profile of the CMB around clusters and voids [23,51,52,64,106] and for finding the correspondence between the temperature fluctuation and the specific structure of the density field [51,57,99,100]. The stacked CMB temperature can be expressed as

\[ S(R) = \sum_i^n A_i^{-1} \int d\Omega_i \Delta T(\phi_i, \theta_i) M(\phi_i, \theta_i) \Xi(\theta_i, R), \tag{45} \]

where the normalization is \( A_i = \int d\Omega M(\phi_i, \theta_i) \), \( M \) is the composite mask of the CMB and large-scale structure, and \( \Xi \) is the filter function. For the compensated filter that eliminates the constant offset of the temperature, the filter function is given as (see e.g. Ref. [23] and A. J. Nishizawa and K. T. Inoue, manuscript in preparation)

\[ \Xi(\theta_i, R) = \begin{cases} 
1, & \theta_i < R \\
-1, & R \leq \theta_i < \cos^{-1}(2\cos R - 1) \\
0, & \text{otherwise}
\end{cases} \tag{46} \]

Here we take a local coordinate system in which the \( z \)-axis is pointing to the \( i \)th cluster or void center with \( \theta_i, \phi_i \), and \( d\Omega_i \) denoting the polar and azimuthal angles and the volume element in this local coordinate system, respectively. The radius \( R \) is the physical scale instead of the angle that satisfies the relation \( R = \phi_i \chi(z_i) \), where \( \phi_i \) is the angle that subtends the size of a cluster or a void lying at a comoving distance \( \chi(z_i) \). In other words, the CMB temperature profile is rescaled before stacking, which increases the statistical significance of the correlation signal [64] (see also A. J. Nishizawa...
and K. T. Inoue, manuscript in preparation). Stacking the CMB at the locations of clusters and voids gives a typical profile of the CMB temperature. Since the sign of the ISW temperature fluctuation is opposite at the voids and clusters, we take the difference between the stacked temperature at voids and clusters, i.e.

\[ S(R) = S_{\text{cluster}}(R) - S_{\text{void}}(R). \] (47)

The interpretation of the significance of the detection of the ISW with the stacking analysis needs careful treatment. As pointed out in Ref. [57], if we look at a particular scale, the significance is prominent and is inconsistent with the standard \( \Lambda \)CDM prediction, but the significance goes down if we combine all scales; this is the so-called a posteriori selection effect [7,57,64,106].

### 3.2. Systematic errors

As seen in Table 1, the reported detection significances vary among papers and are often not consistent with each other, even if they use the same CMB and large-scale structure data sets. This is partly because of the difference in the statistics they use and partly because of the imperfect control of the systematics. In this section, we discuss possible contamination that may affect the analysis of the ISW effect. The main contaminant of the CMB is foreground Galactic dust and synchrotron emissions. On the other hand, the galaxy distribution as a tracer of the large-scale structure also has uncertainty, including incomplete star–galaxy separation, magnification bias, and redshift distribution uncertainties. Furthermore, the SZ effect and point source contamination can be a source of systematics, which is discussed in a later section.

#### 3.2.1. Magnification bias.

Suppose that we correlate the CMB with the number count of galaxies. The observed number count of galaxies reflects the true, underlying clustering of galaxies, while it can also generate an apparent (artificial) clustering in the sky due to various effects. The gravitational lens effect alters the number count of galaxies through two effects. It locally changes the area of the sky observed and hence the number of galaxies observed. It also magnifies the light of distant faint galaxies, and thus enhances the number of galaxies observed in the vicinity of massive low-\( z \) galaxies [e.g. 17]. Given that we observe the galaxy number density at a certain redshift bin, \( z_i \), the observed number density can be written as a sum of two components [79]:

\[ \delta_{\text{gal}}^{\text{obs}}(\hat{n}, z_i) = \delta_{\text{gal}}^s(\hat{n}, z_i) + \delta_{\text{gal}}^\mu(\hat{n}, z_i), \] (48)

where \( \delta_{\text{gal}}^s \) is the intrinsic galaxy or, in other words, unlensed galaxy number density and \( \delta_{\text{gal}}^\mu \) is the magnification bias correction. Now we make a modification to Eq. (33). The underlying 3D dark matter fluctuation projected into a redshift bin \( z_i \) with the kernel is

\[ \delta_{\text{gal}}^s(\hat{n}, z_i) = \int dz b(z)W^s(z, z_i)\delta_m(\hat{n}\chi(z)), \] (49)

where \( b(z) \) is a galaxy bias that depends on redshift and \( W^s(z, z_i) \) is a kernel that projects galaxies at \( z \) onto a redshift bin of \( z_i \). The explicit form of the kernel depends on the model assumed. If we assume that each galaxy has a redshift measured with a Gaussian error of \( \sigma(z) \), the kernel can be written as

\[ W^s(z, z_i) = \frac{1}{2}N(z) \left[ \text{erfc} \left( \frac{(i - 1)\Delta_z - z}{\sqrt{2}\sigma(z)} \right) - \text{erfc} \left( \frac{i\Delta_z - z}{\sqrt{2}\sigma(z)} \right) \right], \] (50)

where \( \Delta_z \) is the width of the redshift bin, \( \text{erfc} \) is the complementary error function, and \( N(z) \) is a redshift distribution of galaxies that satisfies \( N(z)dz = p(r)dr = r^2\phi(r)dr \) normalized as \( \int dz N(z) = 1. \)
Now we derive the expression for $\delta^{\mu}_{\text{gal}}$. The gravitational lens deflects light and thus changes the position of galaxies as

$$\hat{n}^s = \hat{n} + \delta\hat{n},$$

where the superscript $s$ denotes the quantity for unlensed galaxies, and $\delta\hat{n}$ is the deflection angle.

The magnification can be given by the Jacobian of the mapping of $\hat{n}^s \rightarrow \hat{n}$, i.e.

$$A^{-1} \equiv \left| \frac{\partial \hat{n}^s}{\partial \hat{n}} \right|.$$  \hspace{1cm}  \text{(52)}$$

The galaxy flux is magnified by

$$f = Af^s,$$  \hspace{1cm}  \text{(53)}$$

where $f$ is the observed flux and $f^s$ is the intrinsic galaxy flux. In the weak lensing limit, the magnification can be approximated as $A \approx 1 + 2\kappa$, where $\kappa$ is the so-called “convergence field”, which satisfies $|\kappa| \ll 1$. In this limit, the observed number density of galaxies can be expressed in terms of the number density of unlensed galaxies as

$$\delta^{\text{obs}}_{\text{gal}}(\hat{n}, z) = \delta^{s}_{\text{gal}}(\hat{n} + \delta\hat{n}, z) + (5\alpha(z) - 2)\kappa [1 + \delta^{s}_{\text{gal}}(\hat{n} + \delta\hat{n}, z)],$$  \hspace{1cm}  \text{(54)}$$

where $\alpha$ is the logarithmic slope of the cumulative number counts of galaxies at the faint end:

$$\alpha = \frac{d \log N(>m)}{dm} \bigg|_{m_{\text{lim}}}.$$  \hspace{1cm}  \text{(55)}$$

This definition of $\alpha$ is only true for galaxy populations that have a linear slope in the relation between the logarithmic cumulative number and the magnitude. Though in most cases this is true, one can generalize the definition of $\alpha$ by introducing an efficiency function [63]. In the weak lensing limit, the deflection angle $\delta\hat{n}$ is also a first-order small quantity. Expanding $\delta^{s}_{\text{gal}}(\hat{n} + \delta\hat{n})$ and keeping the first-order term gives

$$\delta^{\text{obs}}_{\text{gal}}(z, \hat{n}) = \delta^{s}_{\text{gal}}(z, \hat{n}) + (5\alpha - 2)\kappa,$$  \hspace{1cm}  \text{(56)}$$

where the second term on the RHS denotes $\delta^{\mu}_{\text{gal}}(z, \hat{n})$. Using the expression of the convergence $\kappa$ for the sources distributed around the $z_i$ bin, we obtain

$$\delta^{\mu}_{\text{gal}}(\hat{n}, z_i) = \frac{3\Omega_m H_0^2}{2} (5\alpha(z_i) - 2) \int \frac{dz}{H(z)} g(z, z_i) (1 + z) \delta_m [\hat{n} \chi(z)],$$  \hspace{1cm}  \text{(57)}$$

where $\chi$ is the comoving distance and the lensing efficiency function $g$ is given by

$$g(z, z_i) = \chi(z) \int_z^{\infty} d\zeta' \frac{\chi(z') - \chi(z)}{\chi(z')} W(z', z_i).$$  \hspace{1cm}  \text{(58)}$$

It has been shown that the magnification bias in the ISW analysis does not affect the constraints on dark energy parameters inferred from low-$z$ samples; however, it can have a significant impact when we cross-correlate the CMB with galaxy samples at $z > 2$ [79]. Thus, we need to be careful when interpreting the null detection of the ISW at high redshift.

### 3.2.2. Extinctions

The Galactic dust extinction may be a major source of systematics in the measurement of the ISW effect [40,44]. In the region where the Galactic dust extinction is high, the galaxy colors tend to be red and alter the large-scale distribution of the galaxy sample, depending on the wavelength observed. In the high-extinction region, we also expect the observed temperature...
map to contain microwave emissions from the Galactic dust, which then produce a spurious correlation between the temperature map and the galaxy count data. The advantage of using the near- to mid-infrared range is minimization of the amount of dust extinction, which enables us to mitigate the systematics. Correcting the effect of dust is essential for the cross-correlation study because it allows us to use a greater sky area. As the ISW effect appears only on large angular scales where both the CMB and galaxy data are almost sample variance limited, increasing the usable sky area is the only way to increase the statistical significance of the cross-correlation signal \[45,103–105,119,147\]. However, the dust correction contains uncertainty, which may affect the precise measurement. A more conservative approach is to simply mask out the contaminated region of the sky. Giannantonio et al. \[43\] exclude high-extinction regions with \(A_r > 0.18\) and \(0.08\), leaving usable sky fractions of \(f_{\text{sky}} = 0.22\) and \(0.11\), respectively. The latter has less contamination but larger statistical error and is statistically consistent with the former to within \(1\) \(\sigma\). Therefore, the former cut appears to be sufficient for this data set.

3.2.3. **SZ and point sources.** A correlation between the CMB and the large-scale structure is also generated by the SZ effect and point sources. As a high-energy electron scatters a CMB photon through inverse Compton scattering in a high-temperature cluster of galaxies, the SZ effect distorts the shape of the blackbody CMB spectrum. For that reason, the temperature change induced by the SZ effect depends on the frequency that we observe; the temperature decreases at low frequency while it increases at high frequency, and no change occurs at 217 GHz. Thus, the SZ signal in the cross-correlation can in principle be isolated with multi-frequency observations. Furthermore, the characteristic angular scale that the SZ effect correlates with the large-scale structure is smaller than that of the ISW effect. In the harmonics space, the ISW peaks at \(l = 2\) and becomes negligible at \(l = 30\), while the SZ effect becomes important at \(l > 100\) \[1,58,120\]. The point sources found in the CMB map are extra-galactic sources such as AGNs and dusty galaxies. They are bright in low (AGNs) and high (dusty galaxies) frequency bands. They are also visible in optical and infrared bands, where many galaxy survey data are available. They also produce a correlation between the CMB and large-scale structure; however, it can be distinguished from the ISW signal because point sources produce a correlation at much smaller scales than that of the ISW effect \[1,32,62\]. This discrimination by scale is possible for the power spectrum in the harmonic space but impossible for the correlation function in the configuration space because, in the harmonic space, the multipole scales where ISW and point source signals become prominent are different, while, in the configuration space, different angular scales and different physical origins do not have one-to-one correspondence \[55,56\]. Although the point sources can be removed by masking if they are resolved by CMB experiments, unresolved galaxies may still give rise to an extra correlation signal.

4. **Application to cosmology**

4.1. **Constraints on dark energy in the \(\Lambda\)- and \(w\)-CDM models**

As first pointed out by Crittenden and Turok (1996) \[26\], the ISW effect can be used as a powerful probe of dark energy. In this section we describe the background Universe by the flat FRW metric with matter and generalized time-varying dark energy. Then we can write the Friedmann equation as

\[
H^2(z) = H_0^2 [\Omega_m(1 + z)^3 + \Omega_{DE}(z, w)],
\]

\[59\]
Fig. 4. Dependence of the ISW effect on the cosmological models, \((D/a)'\), as a function of redshift. A smaller amount of dark energy brings the Universe closer to the EdS model, and thus produces a weaker ISW effect (black solid and red dashed lines). A less negative equation of state, \(w\), (green dot-dashed line) causes dark energy to dominate at an earlier epoch and enhances the ISW effect, while more negative \(w\) (blue dotted line) causes dark energy to dominate at a later epoch and depress the ISW effect. This model can be tested by cross-correlation with high-z objects like quasi-stellar objects (QSOs) or radio galaxies.

where the function \(\zeta\) is given by

\[
\zeta(z, w) = \exp \left[ 3 \int_0^z dz' \frac{1 + w(z')}{1 + z'} \right],
\]

where \(w\) is in general a function of redshift but is parametrized as \(w = \text{const.} \) or \(w(z) = w_0 + w_a \frac{z}{1+z}\) where \(w_0\) and \(w_a\) are constants [e.g. 77]. The explicit cosmological dependence of the ISW effect originates from \(\partial_t (D/a) = \mathcal{H}D(f - 1)\) and the matter power spectrum. When we combine other cosmological data sets such as galaxy clustering or weak lensing, the ISW effect provides information on dark energy via the \(\mathcal{H}D(f - 1)\) term. This quantity is shown in Fig. 4 as a function of redshift. A smaller amount of dark energy brings the Universe closer to the EdS Universe, and thus produces a weaker ISW effect (the black solid and red dashed lines). A larger value of the equation of state, i.e. \(w > -1\) (the green dot-dashed line), causes the dark energy to be prominent at an earlier time, which enhances the ISW effect. Such an early dark energy models can be tested by cross-correlation with high-z density tracers like quasars, which lie at \(z < 3\) [115].

Table 1 shows a summary of the reported detections of the ISW effect and the constraints on dark energy models. Boughn and Crittenden (2002) [13] put the first constraint on the dark energy parameter by cross-correlating the NVSS galaxies number counts with the CMB temperature observed by COBE. Although they did not find a significant detection of the ISW effect, it gives an upper limit to the amount of dark energy, \(\Omega_\Lambda < 0.74\). The redshift distribution of the NVSS galaxies is inferred from an integral of a luminosity function of the radio galaxies [29]. With the inferred redshift distribution, the galaxy auto-correlation power spectrum can nicely account for the observed clustering of the radio galaxies but it is not sufficient to validate the true redshift distribution. Fosalba and Gaztanaga (2004) [35] present more stringent constraints on the \(\Lambda\)CDM model with the APM galaxy survey and the WMAP data. They fixed the Hubble parameter, \(\sigma_8\), and the constant bias parameter, assuming a flat geometry, and obtained \(0.53 < \Omega_\Lambda < 0.86\) (2\(\sigma\)). Fosalba et al. (2003) [36] found a similar result from a combined analysis of the APM galaxies, the SDSS main galaxies, and the LRG sample, \(0.69 < \Omega_\Lambda < 0.87\) (2\(\sigma\)). They utilized the cross-correlation function for the measurement of the ISW effect. The covariance matrix is estimated in two ways: jackknife resampling and a Monte Carlo simulation. The jackknife (JK) error is consistent with the Monte Carlo (MC) simulation on large scales, while it is significantly underestimated on scales smaller than \(\theta < 2\) degrees. Nolta et al.
Some works use the wavelet for constraining dark energy models. As we mentioned, a finite sound speed but do not find a meaningful constraint on the sound speed of dark energy. et al. (2005) extend the analysis to constrain the model where dark energy has clustering with a $\Delta l_n$ function, as well as the cross-correlation between them, all contribute to the covariance matrix as shown in Eq. (62).

For a joint analysis of the ISW and the galaxy-clustering APS, the logarithm of the likelihood function, $\chi^2 = -2 \ln L$, is given by

$$\chi^2 = (x_{\text{obs}} - \langle x\rangle) [C^{-1}]_{ij} (x_{\text{obs}} - \langle x\rangle),$$

where $x_{\text{obs}}$ is the observed APS, and $C_{ij}^{gT}$ and the index $i$ represent the angular $l$-bin and sample used, when we use multiple samples for the large-scale structure tracers. The covariance matrix $C$ can be estimated in different ways. The Fisher matrix formalism gives a theoretical estimate of the covariance matrix, while jackknife (JK) or bootstrap (BS) resampling give an estimate of the covariance based on the observational data themselves. The latter tends to underestimate the true error due to the lack of modes larger than the survey volume. The Monte Carlo simulation is particularly useful when the data are not Gaussian, or processes such as complicated radial or angular selection functions are taken into account. For the Gaussian case, the covariance matrix can be simply calculated by the Fisher formula:

$$C = \frac{1}{f_{\text{sky}} (2l + 1)} [\tilde{C}_{l}^{g} \tilde{C}_{l}^{gT} + (C_{l}^{gT})^2],$$

where $\tilde{C}_{l}^{g}$ is the power spectrum of galaxies with the shot noise, $\tilde{C}_{l}^{g} = C_{l}^{g} + \bar{n}^{-1}$, where $\bar{n}_g$ is the number density of galaxies, and $\tilde{C}_{l}^{T} = C_{l}^{T} + N_{l}^{T}$, where $N_{l}^{T}$ is the noise power spectrum of the CMB. Padmanabhan et al. (2005) use the fourth data release of the SDSS LRG photometric sample with accurate photometric redshifts $\Delta_{20}$ = 0.03. The cross-correlation APS is used as an estimator and they introduce a quadratic estimator, which allows a nearly maximum likelihood estimation. Gaztanaga et al. (2006) explore the wCDM model with multiple large-scale structure tracers: 2MASS, APM, SDSS, NVSS, and HEAO. They point out that the parameter degeneracy between $\Omega_{\Lambda}$ and $\Omega_{m}$ obtained from the ISW effect is perpendicular to those from SNIa observations and thus the combination of the ISW effect with other cosmological probes can be a powerful tool to constrain these parameters. Combining the ISW data with the SNIa data, they obtain the cosmological constraints of $\Omega_{\Lambda} = 0.7 \pm 0.05$ and $\nu = -1.02 \pm 0.17$. Using the same data set, Corasaniti et al. (2005) extend the analysis to constrain the model where dark energy has clustering with a finite sound speed but do not find a meaningful constraint on the sound speed of dark energy. Some works use the wavelet for constraining dark energy models. As we mentioned before, the wavelet analysis has the advantage of detecting localized signals. Therefore, it has higher significance than other methods in terms of the signal-to-noise ratio of finding a signal at a given scale. They find the signal of the ISW effect with a higher significance of $\sim 3 \sigma$ than the other methods. The error covariance for the wavelet is derived in a straightforward manner using Eqs. (44) and (62):

$$\Delta \left[ C_{l}^{\text{ISW} - \text{X.TH}} (R) \right] = \sum_{l} \frac{2l + 1}{16 \pi^2} p_l^4 \Psi_l^4 (R) \left( b_{l}^{\text{CMB}} b_{l}^{\text{X}} \right) \left[ C_{l}^{\text{CMB}} C_{l}^{\text{X}} + (C_{l}^{\text{ISW} - \text{X}})^2 \right].$$


Fig. 5. The cross-correlation power spectrum of the ISW and NVSS galaxy. The lines show the best-fit $f_{NL}$ model (black solid), $f_{NL} = 100$ (blue dotted), $f_{NL} = -100$ (red dashed), and $f_{NL} = 800$ (green dot-dashed) while keeping the other cosmological parameters unchanged. The figure is adapted from Slosar et al. (2008) [127].

They present consistent results with the current $\Omega_\Lambda$ constraint [e.g. 108] but slightly lower values are favored. The equation of state parameter favors larger (less negative) values but it is still consistent with the cosmological constant within the 1$\sigma$ level.

4.2. Non-Gaussianity

Although the recent observation of the CMB by the Planck Collaboration shows that the non-Gaussianity (NG) of the primordial fluctuation is consistent with zero, it is still worth discussing it for future observations, which should give much tighter constraints on (or potentially detect) the NG. In this section, we describe the possible impacts of the NG of the primordial fluctuation on the ISW effect. We assume that the NG has the form of [38,72]

$$\Phi = \phi + f_{NL}(\phi^2 - \langle \phi^2 \rangle)$$

with the parameter $f_{NL}$, which describes the amplitude of the NG of the primordial fluctuation, and $\phi$ is a random Gaussian variable.

The ISW effect basically plays two roles in the context of NG. First, the ISW effect can be a proxy of mass of the large-scale structure, especially on large scales. It is difficult to discriminate the primordial NG from those which originate from the nonlinear gravitational evolution of the structure. However, a non-zero $f_{NL}$ alters the number of rare objects formed responding to the initial gravitational potentials. The number of rare objects is enhanced if $f_{NL}$ is positive but depressed for negative $f_{NL}$. Thus the effect of a non-zero $f_{NL}$ on the halo or galaxy power spectrum becomes prominent on large scales [28]. As a result, the NG bias acquires a correction of $b \rightarrow b + \Delta b$, where

$$\Delta b(k) = 2(1 - b)f_{NL}\delta_c \frac{3\Omega_m H_0^2}{2D(a)k^2},$$

where $\delta_c$ is the critical density and $b$ is the usual Eulerian constant bias. Thus the NG can be constrained through the ISW–galaxy cross-correlation. The galaxy–galaxy auto-correlation is mainly used for constraining the NG and the ISW–galaxy cross-correlation has a relatively lower statistical power. However, the cross-correlation is less sensitive to systematic effects and thus gives a robust measurement; they are, therefore, complementary to each other. Figure 5 compares the cross-correlation power spectrum of the ISW and the NVSS galaxies with the standard $\Lambda$CDM prediction and NG predictions keeping all the parameters other than $f_{NL}$ unchanged [127]. Slosar et al.
(2008) [127] found no significant detection of the NG. They obtained a constraint of $f_{NL} = 105^{+647}_{-337}$ by using only the ISW–NVSS cross-correlation, while Afshordi and Tolley (2008) [2] found a 2 $\sigma$ hint of the NG. $f_{NL} = 236 \pm 127$. Xia et al. (2011) [145] also found weak evidence for a positive $f_{NL}$, $f_{NL} = 74 \pm 40(1\sigma)$ by combining the auto-correlation of the NVSS and the NVSS–ISW cross-correlation. They used the redshift distribution of the NVSS galaxies that was directly measured by the limited number of spectroscopic galaxies [18], whereas Slosar et al. (2008) [127] used one measured from the NVSS clustering itself, which made a significant difference to $f_{NL}$.

Recently, Giannantonio et al. (2014) [43] have pointed out that the NVSS catalog contains a serious systematic error: the number density of galaxies depends on declination and right ascension. This systematics largely affects auto-correlation of the NVSS galaxies, while the impact of the systematics on the cross-correlation is small. Giannantonio and Percival (2014) [42] revisit the NG constraints via cross-correlation of the ISW effect with a suite of galaxy samples and find no evidence of the NG, $f_{NL} = 46 \pm 68$, by using the ISW–galaxy cross-correlation only. They also take into account the cross-correlation of the CMB-lensing with galaxy samples and obtain a joint constraint of $f_{NL} = 12 \pm 21$ in the end. This is consistent with the CMB bispectrum analysis of the Planck Collaboration [110].

Second, the ISW effect may bring a systematic effect to the measurement of the CMB bispectrum through the ISW–lensing correlation [46,110]. The observed CMB temperature can be decomposed into three parts: primordial CMB at the last scattering surface, lensed-CMB, and the secondary anisotropy that is generated at low redshifts, such as the ISW or the SZ. For the Gaussian field, all the information is encoded into the power spectrum or two-point correlation function, so that the bispectrum or three-point function is the lowest statistics that describes the NG clustering. However, the ISW effect and the CMB-lensing are correlated with each other because the large-scale structure that makes CMB photon deflect via a gravitational lens also induces the ISW effect at low redshifts. Therefore, it mimics the primordial NG to produce a non-zero signal in the CMB bispectrum. The ISW effect is important on large scales and the CMB-lensing appears on much smaller scales, so the ISW–lensing bispectrum peaks at a squeezed configuration of the triangle. Thus the correlation between the CMB-lens and the ISW effect brings a bias to the measurement of the primordial NG, especially for the local type by $\Delta f_{NL} \simeq 10$ [22,71], which may seriously affect the current measurement, i.e. $\sigma_{f_{NL}} = 5.8$ [110]. On the other hand, the ISW–lensing bias to the other configurations of the bispectrum is negligible [54,67,83,84,124,128].

5. Summary

In this article we have reviewed the ISW effect induced by linear and nonlinear structures of the Universe. It is well known that the Sachs–Wolfe effect and the ISW effect are simultaneously derived from cosmological perturbation theory. The ISW effect can be generated by scalar, vector, and tensor mode fluctuations of the metric but the significant contribution comes from the scalar mode, which is related to the time variation of the matter-density fluctuation. The recent detection of tensor-mode CMB polarization claimed by the BICEP2 Collaboration [8] could indeed be the polarization generated by the tensor-mode ISW effect from primordial gravitational waves. The time variation of the gravitational potential in the standard $\Lambda$CDM Universe comes from: 1) the coherent decay of the gravitational potential due to the accelerating expansion of the Universe, 2) the isotropic clustering inflow of a mass lump toward the center-of-mass, and 3) the transverse motions of the clusters. The latter two effects are too tiny to detect with the current CMB experiments, but might be possible...
to detect in future experiments using finer angular resolutions such as ACTPol [92], SPTPol [5], or COrE [133].

We have also reviewed the observational studies of the ISW effect. The ISW effect was first detected by the cross-correlation of the CMB observed by the WMAP with the number counts of radio galaxies measured by the NVSS. The detection significance was 2–3σ [14]. Subsequently, a number of detections have been reported with various tracers of dark matter using a variety of statistical methods. As the ISW effect reflects the large-scale fluctuations in the Universe, it can be used to constrain the cosmological models, especially at low redshifts (z < 1). The statistical error of the ISW effect is mostly dominated by sample variance but the detection significances vary among papers and are often not consistent with each other. This is partly a consequence of the different statistics used, different treatments of the foreground, or different estimates or assumptions of the redshift distribution of galaxies. As we acquire more knowledge on these issues, we may be able to understand the inconsistencies seen in Table 1. All-sky polarization data from the Planck Collaboration will be useful for further study of the dust model of our Galaxy, as well as confirmation of the large amplitude of the tensor mode fluctuation discovered by BICEP2. In addition to this, complete BOSS spectroscopic galaxy and QSO samples can be used for calibrating the redshift distribution of galaxies and QSOs. Therefore, both the Planck and BOSS data that are going to be delivered soon will allow us to extensively study the ISW effect with better understandings of the systematics.

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