Casimir-like corrections to the classical tensions of strings and membranes

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We find the Casimir-like energies for strings and membranes. We show that the related Casimir forces can be interpreted as quantum corrections to the classical tensions of the strings and membranes. We see that these corrections always increase the tensions of the circular string as well as the spherical membrane, while, for the straight string, and rectangular and cylindrical membranes, these Casimir forces may increase or decrease the tensions. So we find that the quantum vacuum can break the (tensional) isotropy of the rectangular and cylindrical membranes. Also, obtaining the nonzero-temperature Casimir energy, we find relations for the tensions at nonzero temperature.

Subject Index A46, A47, A73, B30

1. Introduction

It is well known that the quantum zero-point oscillations of a field can result in some observable quantum effects such as the famous Casimir force. In fact, this force arises from the constraints on the normal modes of the vacuum state, due to some conditions imposed on the fields. So, generally, this effect depends highly, not on the microscopic, but on the geometrical properties of the field system.

In this paper we consider models of strings and membranes. In Refs. [1–6] (and also Ref. [7] as a review), the Casimir effect for uniform and nonuniform piecewise strings has been studied. Also, in Refs. [8,9], the authors have found relations for quantum vacuum-induced forces between some beads on strings and flat membranes. Here we find a direct physical interpretation of Casimir forces for strings and membranes at zero as well as nonzero temperature, which has not been given in other similar works. In Sect. 2, we study the Casimir effect for circular as well as straight strings, and in Sect. 3, for rectangular, cylindrical, and spherical membranes. Using the classical Lagrangian, we find the classical equation of motion for the displacement field. Hence one can find a harmonic oscillatory expansion (for each transverse direction) of the fields. Taking these oscillatory modes as the quantum oscillators, we find the vacuum state energy. Finally, in Sect. 4, we find the vacuum energy of these structures at nonzero temperature, by obtaining a mode expansion for the free energy.
2. The quantum-corrected tension of the strings

2.1. Straight strings

The classical solutions for small transverse vibrations of straight strings are well known (the longitu-
dinal vibrations can be neglected for very thin strings). These solutions, e.g., for the Dirichlet boundary condition at both ends (denoted as DD) are obtained as a superposition of independent classical harmonic oscillatory modes with frequencies $\nu_0 \frac{n\pi}{a}$ in which $a$ is the length of the string and $\nu_0$ is known as the velocity of the wave in the string, $\nu_0 = \sqrt{\frac{\mu_0}{\sigma_0}}$, in which the constants $\mu_0$ and $\sigma_0$ are the linear density (mass per unit length) and the classical tension of the string respectively. After quantization we can take these classical oscillation modes to be quantum oscillators. So the vacuum energy of the string is just the summation over the zero-point energies of the quantum harmonic oscillators of two transverse directions:

$$E_{\text{vac}} = 2 \sum_{n=1}^{\infty} \frac{1}{2} \frac{\hbar \nu_0}{a} \frac{n\pi}{a} = \frac{\pi \hbar \nu_0}{a} \sum_{n=1}^{\infty} n. \quad (1)$$

To obtain the physical finite part of the vacuum energy, i.e. the Casimir energy, we regularize the above series using the known Abel–Plana formula (see, e.g., Ref. [10]):

$$\sum_{n=1}^{\infty} n = \int_0^{\infty} u du - 2 \int_0^{\infty} \frac{u du}{\exp[2\pi u] - 1} = \int_0^{\infty} u du - \frac{1}{12}. \quad (2)$$

So, using Eq. (1), one obtains

$$E_{\text{vac}} = -\frac{\pi \hbar \nu_0}{12a} + \frac{\pi \hbar \nu_0}{a} \int_0^{\infty} u du. \quad (3)$$

Therefore the Casimir energy and Casimir force are found as

$$E_{\text{cas}}(a) = -\frac{\pi \hbar \nu_0}{12a},$$

$$F_{\text{cas}}(a) = -\frac{\partial E_{\text{cas}}}{\partial a} = -\frac{\pi \hbar \nu_0}{12a^2}. \quad (4)$$

Note that the series in Eq. (1) can be directly regularized by applying the known Riemann zeta function (see e.g. Ref. [11]). Now in order to find a physical interpretation of the Casimir force for a string of length $a$, suppose it is lengthened to $a + \delta a$. Then the change in the total ground state energy $\delta E_{\text{tot}}^g$, due to this lengthening, can be written as

$$\delta E_{\text{tot}}^g = \sigma_0 \delta a + \frac{\partial E_{\text{vac}}}{\partial a} \delta a = (\sigma_0 - F_{\text{cas}}) \delta a, \quad (5)$$

where in the second line we have absorbed the infinite part of the vacuum energy into the classical tension,

$$\left( \sigma_0 - \frac{\pi \hbar \nu_0}{a^2} \int_0^{\infty} u du \right) \to \sigma_0, \quad (6)$$

as a definition for the physical tension of the string. This is quite similar to the renormalization of classical bare parameters in the framework of the quantum field theory. Now logically we can think
of \(\sigma_0 - F_{\text{cas}}\) as the quantum-corrected tension of the string:

\[
\sigma_c(a) \equiv \sigma_0 - F_{\text{cas}}(a) = \sigma_0 + \frac{\pi \hbar}{12a^2\sqrt{\frac{\sigma_0}{\mu_0}}} \quad \text{for the DD condition.} \tag{7}
\]

So the infinite part of the vacuum energy can be absorbed as a renormalization term into the (bare) tension, while the finite part (i.e. the Casimir energy) gives a quantum correction to the tension of the string. Note that the vacuum energy can also contain constant terms (i.e. terms independent from the length parameter \(a\) of the string), but these terms would not be observable and do not affect the quantum correction term, because the correction is from the Casimir force (derivative with respect to \(a\)), not the Casimir energy. However, the uniqueness condition for the renormalized ground state energy \(E_{\text{g}}^{\text{ren}}\) can be considered as

\[
E_{\text{g}}^{\text{ren}} \rightarrow 0 \quad \text{if} \quad a\mu_0 \rightarrow \infty, \tag{8}
\]

because, for a very massive (\(\mu_0a \rightarrow \infty\)) string, the vacuum oscillations are expected to be negligible. As we see from Eq. (7), for the DD condition the string tension increases due to the quantum oscillations of the vacuum state. The same results could be found for Neumann boundary conditions at both ends (NN)\(^1\). For the DN boundary condition, the vacuum energy would simply be obtained as

\[
E_{\text{vac}} = \frac{\pi \hbar \nu_0}{a} \sum_{n=0}^{\infty} \left( n + \frac{1}{2} \right). \tag{9}
\]

The above series can be regularized directly using the Hurwitz zeta function \(\zeta_H\) (see, e.g., Ref. [11]) as

\[
\sum_{n=0}^{\infty} \left( n + \frac{1}{2} \right) = \zeta_H(-1, 1/2) = \frac{1}{24}. \tag{10}
\]

So we obtain

\[
E_{\text{cas}}(a) = \frac{\pi \hbar \nu_0}{24a} \\
F_{\text{cas}}(a) = \frac{\pi \hbar \nu_0}{24a^2}. \tag{11}
\]

Note that, for the DN condition, in contrast to the DD and NN conditions, the Casimir force has a positive value, thus the Casimir-like correction decreases the classical tension of the string:

\[
\sigma_c(a) = \sigma_0 - \frac{\pi \hbar}{24a^2\sqrt{\frac{\sigma_0}{\mu_0}}} \quad \text{for the DN condition.} \tag{12}
\]

2.2. Circular string

Consider a circular string with negligible thickness. For small circumference-vibrations of the string compared to the string radius \(r\), the radius can be assumed to be constant. Then the Lagrangian can

\(^1\)In the case of the NN condition or the periodical condition (e.g., for the circular string), the solution also has two non-oscillation terms that correspond to translational motion of the system. In the rest frame of the system, these translational terms can simply be neglected.
be written as
\[
L = \int_{0}^{2\pi} d\theta \left( \frac{H_0^2}{2} \left( (\partial_\theta R)^2 + (\partial_\theta Z)^2 \right) - \frac{\sigma_0}{2r} \left( (\partial_\theta R)^2 + (\partial_\theta Z)^2 \right) \right),
\]
(13)
in which \( \theta \) is polar coordinate, and also \( R(\theta, t) \) and \( Z(\theta, t) \) are configuration fields equivalent to radial and transverse vibratory displacements, respectively. The equation of motion would be
\[
\left[ \frac{\partial^2}{\partial t^2} - \frac{\nu_0^2}{r} \frac{\partial^2}{\partial \theta^2} \right] R(\theta, t) = 0,
\]
(14)
with \( \nu_0^2 \equiv \sqrt{\frac{\sigma_0}{\mu_0}} \), and the same equation for \( Z(\theta, t) \). The solutions for \( Z(\theta) \) and \( R(\theta) \) can be obtained (neglecting translational-motion terms) in terms of creation and annihilation operators of independent harmonic oscillatory modes. Then the vacuum energy of the string is just the expectation value of the corresponding Hamiltonian (obtained from the above Lagrangian) for the vacuum state:
\[
E_{\text{vac}} = \langle 0 | H | 0 \rangle = \hbar \nu_0 \sum_{n=1}^{\infty} \frac{n}{r},
\]
(14)
which is just the summation over the zero-point energies of the quantum harmonic oscillatory modes. Therefore \( E_{\text{cas}}(r) \) and \( F_{\text{cas}}(r) \) can be found as
\[
E_{\text{cas}}(r) = -\frac{\hbar \nu_0}{12r},
\]
\[
F_{\text{cas}}(r) = -\frac{\partial E_{\text{cas}}}{\partial r} = -\frac{\hbar \nu_0}{12r^2}.
\]
(15)
Note that, this time, the Casimir force is central. The variation \( \delta E_{\text{g}}^{\text{tot}} \), caused by the change \( 2\pi \delta r \) in the circumference of the string, can be written as
\[
\delta E_{\text{g}}^{\text{tot}} = 2\pi \sigma_0 \delta r + \frac{\partial E_{\text{cas}}}{\partial r} \delta r = 2\pi \left( \sigma_0 + \frac{\hbar \nu_0}{24\pi r^2} \right) \delta r.
\]
(16)
So
\[
\sigma_c(r) \equiv \sigma_0 + \frac{\hbar}{24\pi r^2} \sqrt{\frac{\sigma_0}{\mu_0}}
\]
(17)
can be interpreted as the quantum-corrected tension. As we see, quantum vacuum correction always increases the classical tension of the circular string.

3. **The quantum-corrected tension of the membranes**

3.1. **Rectangular flat membrane**

A rectangular flat membrane (Fig. 1) with negligible thickness and for small surface vibrations can be described by the known Lagrangian
\[
L = \frac{1}{2} \int_{a}^{b} dx \int_{0}^{d} dy \left[ \rho_0 (\partial_x Z)^2 - \tau_0 (\partial_x Z)^2 - \tau_0 (\partial_x Z)^2 \right],
\]
(18)
where the displacement field \( Z(x, y, t) \) corresponds to surface vibrations and \( \rho_0 \) and \( \tau_0 \) are surface-density (mass per unit area) and surface-tension (tension per unit length) of the membrane respectively. The equation of motion is obtained as the wave equation \( \left( \frac{\partial^2}{\partial t^2} - \nu_0^2 \frac{\partial^2}{\partial x^2} - \nu_0^2 \frac{\partial^2}{\partial y^2} \right) \)
Fig. 1. Rectangular flat membrane.

\[ Z(x, y, t) = 0, \text{ in which } \nu_0 = \sqrt{\frac{\kappa_0}{\rho_0}}. \] For the DD boundary condition, the solution can be written (up to coefficients) as

\[ Z_{n,m}(x, y, t) \sim \sin \left( \frac{n\pi}{b} x \right) \sin \left( \frac{m\pi}{a} y \right) \exp(-i\alpha_{n,m}t) + \text{c.c.}, \] (19)

in which \( \alpha_{n,m} = \nu_0 \sqrt{(\frac{n\pi}{b})^2 + (\frac{m\pi}{a})^2} \) and c.c. stands for the complex conjugate terms. So the vacuum energy would be obtained as

\[ E_{\text{vac}} = \frac{\hbar \nu_0}{2} \sum_{n,m=1}^{\infty} \sqrt{(\frac{n\pi}{b})^2 + (\frac{m\pi}{a})^2}. \] (20)

This series can be regularized by applying the Abel–Plana formula twice, and the infinite terms can be absorbed as some renormalization terms into the bare tension of the membrane. Then the Casimir energy and the Casimir forces of the \( x \)- and \( y \)-directions are found as \( b \geq a \)

\[ E_{\text{cas}}(a, b) = \hbar \nu_0 \left( \frac{\pi}{48a} - \frac{b}{16\pi a^2} \zeta_R(3) - \frac{1}{a} G \left( \frac{b}{a} \right) \right) \]

\[ F_{\text{cas}}^x(a, b) = -\frac{\partial E_{\text{cas}}}{\partial b} = \hbar \nu_0 \left( \frac{\zeta_R(3)}{16\pi} + G' \left( \frac{b}{a} \right) \right) \frac{1}{a^2} \]

\[ F_{\text{cas}}^y(a, b) = -\frac{\partial E_{\text{cas}}}{\partial a} = \hbar \nu_0 \left[ \frac{\pi}{48a^2} - \frac{b}{8\pi a^3} \zeta_R(3) - \frac{1}{a^2} G \left( \frac{b}{a} \right) - \frac{b}{a^3} G' \left( \frac{b}{a} \right) \right] \] (21)

where \( \zeta_R \) denotes the Riemann zeta function, \( G \left( \frac{b}{a} \right) \equiv \frac{1}{2} \sum_{n,m=1}^{\infty} \frac{\pi}{nm} K_1 \left( 2\pi nm \frac{b}{a} \right) \) in which \( K \) is a Bessel function of the second kind, and the prime denotes the derivative with respect to the argument; \( G' \left( \frac{b}{a} \right) \equiv \frac{\partial G \left( \frac{b}{a} \right)}{\partial \left( \frac{b}{a} \right)} \). With numerical computations, one can find 0 < \( G \left( \frac{b}{a} \right) \leq G(1) \approx 0.000495 \) and -0.00339 \( \approx G' \left( 1 \right) \leq G' \left( \frac{b}{a} \right) < 0 \), thus \( F_{\text{cas}}^x \) is always positive (i.e. repulsive), but \( F_{\text{cas}}^y \) may be negative (i.e. attractive). In practice, the terms \( G \left( \frac{b}{a} \right) \) and \( G' \left( \frac{b}{a} \right) \) can be ignored with a good degree of accuracy; therefore, we find that \( F_{\text{cas}}^y \) is repulsive if \( 1 \leq \frac{b}{a} \lesssim 1.37 \) and is attractive if \( \frac{b}{a} \gtrsim 1.37 \). Note that our relations in Eq. (21) are in agreement with comparable expressions of other
parallel works (see, e.g., Sect. 4 of Ref. [13]); however, our numerical results for the Casimir forces are not given in other works. Now, in order to interpret these results more physically, we write

\[ \delta E_g^{\text{tot}} = (b \tau_0) \delta a + \frac{\partial E_{\text{cas}}}{\partial a} (a \tau_0) \delta b + \frac{\partial E_{\text{cas}}}{\partial b} \delta b \]

\[ = b \left( \tau_0 - \frac{1}{b} F_y^{\text{cas}} \right) \delta a + a \left( \tau_0 - \frac{1}{a} F_x^{\text{cas}} \right) \delta b, \tag{22} \]

where \( a \tau_0 \) and \( b \tau_0 \) are total tensions of the \( x \)- and \( y \)-directions respectively. Then we can take the corrected tensions as

\[ \tau_{x}(a, b) \equiv \tau_0 - \frac{1}{a} F_x^{\text{cas}}(a, b) \]

\[ = \tau_0 - \frac{h}{\rho_0} \sqrt{\frac{\tau_0}{b}} \left[ \frac{\pi}{48ba^2} + \frac{\zeta_R(3)}{8\pi a^3} - \frac{1}{6} G \left( \frac{b}{a} \right) - \frac{1}{3} G' \left( \frac{b}{a} \right) \right]. \tag{23} \]

So \( \tau_x \) always decreases, i.e. the tensional stability of the membrane in the \( x \)-direction decreases, while \( \tau_y \) decreases if \( 1 \leq \frac{b}{a} \leq 1.37 \), and increases when \( \frac{b}{a} > 1.37 \). That is, as is seen \( \tau_x \neq \tau_y \), the vacuum oscillations generally break the (tensional) isotropy of a rectangular membrane. For a square membrane \((b = a)\), we find

\[ F_x^{\text{cas}}(a) = F_y^{\text{cas}}(a) \approx 0.02 \frac{h \nu_0}{a^2} \]

\[ \tau_x(a) = \tau_y(a) \approx \tau_0 - 0.02 \frac{h}{\rho_0} \frac{1}{a^3}, \tag{24} \]

so quantum corrections always decrease the surface tension of the square membrane. We see that the vacuum oscillations respect the isotropy of a square membrane. For the NN boundary condition, one can write

\[ Z_{n,m}(x, y, t) \sim \cos \left( \frac{n \pi x}{b} \right) \cos \left( \frac{m \pi y}{a} \right) \exp(-i \omega_{n,m} t) + \cos \left( \frac{n \pi x}{b} \right) \exp \left( -i \nu_0 \frac{n \pi}{b} t \right) \]

\[ + \cos \left( \frac{m \pi y}{a} \right) \exp \left( -i \nu_0 \frac{m \pi}{a} t \right) + \text{c.c.}, \tag{25} \]

in which \( \omega_{n,m} \) is defined as before. So the vacuum energy is obtained as

\[ E_{\text{vac}} = \frac{h \nu_0}{6} \left( \sum_{n,m=1}^{\infty} \sqrt{\left( \frac{n \pi}{b} \right)^2 + \left( \frac{m \pi}{a} \right)^2} + \sum_{n=1}^{\infty} \frac{n \pi}{b} + \sum_{m=1}^{\infty} \frac{m \pi}{a} \right), \tag{26} \]

and after regularization we find the Casimir energy:

\[ E_{\text{cas}}(a, b) = -\frac{h \nu_0}{3} \left[ \frac{\pi}{48a} + \frac{\pi}{24b} \frac{1}{2} G \left( \frac{b}{a} \right) \right]. \tag{27} \]

As is seen, the Casimir energy of the NN condition is different from that of the DD condition.
3.2. **Cylindrical membrane**

The Lagrangian for small surface vibrations of a cylindrical membrane (see Fig. 2) can be written as

\[ L = \frac{1}{2} \int_0^{2\pi} d\phi \int_0^l dz \left[ r \rho_0 (\partial_r R)^2 - r \tau_0 (\partial_z R)^2 - \frac{\tau_0}{r} (\partial_\phi R)^2 \right], \quad (28) \]

where the field \( R(z, \phi, t) \) corresponds to vibrational displacement (here, for small vibrations, \( r \) is taken to be constant). Now we have a boundary condition for the longitudinal variable \( z \), and a periodical condition for the azimuthal variable \( \phi \). For the DD boundary condition one obtains an oscillatory mode expansion as

\[ R_{n,m}(z, \phi, t) \sim \sin \left( \frac{n\pi}{l} z \right) \exp(+im\phi) \exp(-i\omega_{n,m} t) + \sin \left( \frac{n\pi}{l} z \right) \exp(-im\phi) \exp(-i\omega_{n,m} t) + \sin \left( \frac{n\pi}{l} z \right) \exp(-i\nu_0 \frac{n\pi}{l} t) + \text{c.c.}, \quad (29) \]

in which \( \omega_{n,m} \equiv \nu_0 \sqrt{(n\pi/l)^2 + (m/r)^2} \) with \( \nu_0 \equiv \sqrt{\tau_0/\rho_0} \). Therefore the vacuum energy of the membrane would be obtained as

\[ E_{\text{vac}} = \frac{\hbar \nu_0}{4} \left( \sum_{n,m=1}^{\infty} \sqrt{(n\pi/l)^2 + (m/r)^2} + \sum_{n=1}^{\infty} \frac{n\pi}{l} \right). \quad (30) \]

Similarly, as for the rectangular membrane, we find the Casimir energy, Casimir radial, and longitudinal forces as \( l \geq \pi r \)

\[ E_{\text{cas}}(r, l) = \frac{\hbar \nu_0}{2} \left( \frac{\pi}{24r} - \frac{\pi}{16\pi^3 r^2} \zeta_R(3) - \frac{1}{\pi r} G \left( \frac{l}{\pi r} \right) \right) \]

\[ F_{r,\text{cas}}(r, l) = \frac{\hbar \nu_0}{2} \left( \frac{\pi}{24l^2} + \frac{\zeta_R(3)}{16\pi^3 l^2} + \frac{1}{\pi^2 r^2} G' \left( \frac{l}{\pi r} \right) \right) \]

\[ F_{z,\text{cas}}(r, l) = \frac{\hbar \nu_0}{2} \left( \frac{1}{48r^2} - \frac{l}{8\pi^3 r^3} \zeta_R(3) - \frac{1}{\pi^2 r^2} G \left( \frac{l}{\pi r} \right) - \frac{l}{\pi^2 r^3} G' \left( \frac{l}{\pi r} \right) \right). \quad (31) \]

Then we write

\[ \delta E_{\text{tot}}^l = (l \tau_0)(2\pi \delta r) + \frac{\partial E_{\text{cas}}}{\partial r} \delta r + (2\pi r \tau_0) \delta l + \frac{\partial E_{\text{cas}}}{\partial l} \delta l \]

\[ = 2\pi l \left( \tau_0 - \frac{1}{2\pi l} F_{r,\text{cas}} \right) \delta r + 2\pi r \left( \tau_0 - \frac{1}{2\pi r} F_{z,\text{cas}} \right) \delta l, \quad (32) \]
with \(2\pi r \tau_0\) and \(l \tau_0\) as the total longitudinal and azimuthal tensions respectively. Hence the quantum-corrected tensions can be taken as

\[
\tau_{\text{azim}}(r, l) \equiv \tau_0 - \frac{1}{2\pi r} F^\text{cas}_r(r, l)
\]

\[
= \tau_0 - \hbar \sqrt{\frac{\tau_0}{\rho_0}} \left( \frac{1}{192\pi rl^2} - \frac{\zeta_R(3)}{32\pi^4 r^3} - \frac{1}{4\pi^2 r^2} G \left( \frac{l}{\pi r} \right) - \frac{1}{4\pi^3 r^3} G' \left( \frac{l}{\pi r} \right) \right)
\]

\[
\tau_{\text{long}}(r, l) \equiv \tau_0 - \frac{1}{2\pi r} F^\text{cas}_l(r, l)
\]

\[
= \tau_0 - \hbar \sqrt{\frac{\tau_0}{\rho_0}} \left( -\frac{1}{96rl} + \frac{\zeta_R(3)}{64\pi^4 r^3} + \frac{1}{4\pi^3 r^3} G' \left( \frac{l}{\pi r} \right) \right).
\]

(33)

So \(\tau_{\text{azim}}\) decreases if \(1 \leq \frac{l}{\pi r} \lesssim 1.37\), and increases if \(\frac{l}{\pi r} \gtrsim 1.37\), while \(\tau_{\text{long}}\) increases if \(1 \leq \frac{l}{\pi r} \lesssim 2.34\) and decreases if \(\frac{l}{\pi r} \gtrsim 2.34\). As seen, the quantum corrections can also break the isotropy of the cylindrical membrane. For the NN boundary condition, one can write

\[
R_{n,m}(z, \phi, t) \sim \exp(-im\phi) \exp \left( -i \nu_0 \frac{m}{r} t \right) + \exp(+im\phi) \exp \left( -i \nu_0 \frac{m}{r} t \right)
\]

\[
+ \cos \left( \frac{n\pi}{l} z \right) \exp(+im\phi) \exp \left( -i \omega_{n,m} t \right)
\]

\[
+ \cos \left( \frac{n\pi}{l} z \right) \exp(-im\phi) \exp \left( -i \omega_{n,m} t \right)
\]

\[
+ \cos \left( \frac{n\pi}{l} z \right) \exp \left( -i \nu_0 \frac{n\pi}{l} t \right) + \text{c.c.},
\]

(34)

so the vacuum energy can be obtained as

\[
E_{\text{vac}} = \frac{\hbar \nu_0}{6} \left( \sum_{n,m=1}^{\infty} \left( \frac{n\pi}{l} \right)^2 + \left( \frac{m}{r} \right)^2 + \sum_{n=0}^{\infty} \frac{n\pi}{l} + \sum_{m=1}^{\infty} \frac{m}{r} \right),
\]

(35)

and so the Casimir energy would be found as

\[
E_{\text{cas}}(r, l) = -\frac{\hbar \nu_0}{3} \left( \frac{1}{48r} + \frac{\pi}{24l} + \frac{l}{16\pi^3 r^2} \zeta_R(3) + \frac{1}{\pi} G \left( \frac{l}{\pi r} \right) \right).
\]

(36)

### 3.3. Spherical membrane

The Lagrangian of a spherical thin membrane of radius \(r\) with small surface vibrations can be written as

\[
L = \frac{1}{2} \int_2^\pi d\phi \int_0^\pi \sin \theta d\theta \left[ \rho_0 r^2 (\partial_\theta R)^2 - \tau_0 (\partial_\theta R)^2 - \frac{\tau_0}{\sin^2 \theta} (\partial_\phi R)^2 \right].
\]

(37)

The equation of motion is found as

\[
\left[ \left( \frac{r}{\nu_0} \right)^2 \frac{\partial^2}{\partial r^2} - \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) - \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right] R(\theta, \phi, t) = 0.
\]

(38)

The angular part of this equation just leads to the generalized Legendre equation, and therefore the solution is written in terms of spherical harmonics:

\[
R_{k,m}(\theta, \phi, t) \sim \exp \left( -i \frac{\nu_0}{r} t \sqrt{k(k+1)} \right) Y_{k,m}(\theta, \phi) + \text{c.c.},
\]

(39)
in which \( Y_{k,m} \) are the spherical harmonics. Then after some calculations one can find the vacuum energy as

\[
E_{\text{vac}} = \frac{\hbar \nu_0}{2r} \sum_{k=1}^{\infty} (2k + 1) \sqrt{k(k+1)}. \tag{40}
\]

The above series can be regularized using the known Euler–Maclaurin summation formula; see e.g. Ref. [12]. Here, ignoring irrelevant infinite terms, this formula can be approximated as

\[
\sum_{k=1}^{\infty} f(k) \simeq \int_{1}^{\infty} f(k) \, dk + \frac{1}{2} f(k = 1) - \frac{1}{12} \left( \frac{\partial f}{\partial k} \right)_{k=1} + \frac{1}{720} \left( \frac{\partial^3 f}{\partial k^3} \right)_{k=1}, \quad (41)
\]

where \( f \) is an analytic function in the integration region, and in the right-hand side \( k \) is taken as a continuous variable. So the series in the vacuum energy (40) can be regularized as

\[
\sum_{k=1}^{\infty} (2k + 1) \sqrt{k(k+1)} \simeq -\int_{0}^{1} dk \, (2k + 1) \sqrt{k(k+1)} + \frac{5867}{2560\sqrt{2}} \approx -0.265. \tag{42}
\]

The above result is in agreement with the corresponding result in Refs. [14,15] obtained from zeta function regularization. Then we find the Casimir energy and Casimir force:

\[
E_{\text{cas}}(r) \approx -0.133 \frac{\hbar \nu_0}{r},
\]

\[
F_{\text{cas}}(r) = -\frac{\partial E_{\text{cas}}}{\partial r} \approx -0.133 \frac{\hbar \nu_0}{r^2}. \tag{43}
\]

As we see, the Casimir force for a spherical membrane is always central. Again it is desirable to find an appropriate physical explanation for this force. We write

\[
\delta E_{\text{tot}}^\text{int} = \tau_0 \left( 2r \delta r \int_{0}^{2\pi} d\phi \int_{-1}^{1} d(cos \theta) \right) + \frac{\partial E_{\text{cas}}}{\partial r} \delta r = 8\pi r \left( \tau_0 - \frac{1}{8\pi r} F_{\text{cas}}(r) \right) \delta r, \tag{44}
\]

This relation can be partly understood by noting that the change in the spherical area of the membrane, due to the infinitesimal change in radius \( \delta r \), equals \( \delta (4\pi r^2) = 8\pi r \delta r \). Now the quantum-corrected (surface) tension can be taken as

\[
\tau(r) \equiv \tau_0 - \frac{1}{8\pi r} F_{\text{cas}}(r) \approx \tau_0 + 0.00527 \hbar \sqrt{\frac{\tau_0}{\rho_0}} \frac{1}{r^3}, \tag{45}
\]

so the Casimir-like correction always increases the surface tension of the spherical membrane. Also we see that the vacuum oscillations do not break the isotropy of the classical tension of the spherical membrane.

### 4. The tension at nonzero temperature

So far we have not considered the thermal effects in obtaining the Casimir forces, while a physical system actually involves thermal excitations. In the framework of the statistical mechanic, as we know, a thermal equilibrated system is considered as an ensemble of states having the same temperature, with a probability distribution. The regularized free energy of this ensemble then provides the Casimir energy at nonzero temperature [16]. Here we analyze the thermal equilibrated systems, applying the known Matsubara formalism. As we know, the thermodynamics of a system is described by a partition function having the functional integral representation \( Z \sim \int D\Psi \exp(-S_E[\Psi]) \) in which \( \Psi \) is the configuration space field and \( S_E[\Psi] \) is the Euclidean action of the system (we remind ourselves that \( S_E[\Psi] \) is obtained by Wick rotation of the time coordinate: \( t \to -it \)).
4.1. Strings at nonzero temperature

The Euclidean action of the straight string can be written as

$$S_E[Y] = i \int dt \int dx [Y(x, t) \Delta Y(x, t)]$$

with

$$\Delta = -\frac{1}{2} \mu_0 \partial_t^2 - \frac{1}{2} \sigma_0 \partial_x^2. \quad (46)$$

So the partition function will be

$$Z = \int DY \exp(iS_E) \sim \det(\Delta)^{-1}. \quad (47)$$

Then, ignoring a constant term, the free energy is obtained as

$$F \equiv -k_B T \ln Z = k_B T \text{Tr} \ln(\Delta), \quad (48)$$

in which $k_B$ is the Boltzmann constant and $T$ is the temperature of the thermal equilibrated system (here the string). Utilizing the Matsubara formalism, one can write the field modes (for the Dirichlet condition) as

$$Y_{nj}(x, t) = \frac{1}{\sqrt{2\pi}} \exp(-i \xi_j t) \sin \left(\frac{n\pi}{a} x\right), \quad (49)$$

which contains the Matsubara frequencies $\xi_j \equiv 2\pi j/k_B T/\hbar$. Since

$$\Delta Y_{nj} = \left[\frac{1}{2} \mu_0 \xi_j^2 + \frac{1}{2} \sigma_0 \left(\frac{n\pi}{a}\right)^2\right] Y_{nj}, \quad (50)$$

dropping an infinite constant term, the free energy of the string is found as

$$\mathcal{F}(a, T) = k_B T \sum_{j=-\infty}^{\infty} \sum_{n=1}^{\infty} \ln \left[\xi_j^2 + \frac{\sigma_0}{\mu_0} \left(\frac{n\pi}{a}\right)^2\right]. \quad (51)$$

Such an expression can be simplified as the sum of a zero-temperature part $\mathcal{F}_0$ and a thermal correction term $\mathcal{F}_T$ (see, e.g., Chap. 5 of Ref. [16]) as

$$\mathcal{F}(a, T) = \mathcal{F}_0 + \mathcal{F}_T$$

$$\mathcal{F}_0 = \frac{\hbar v_0}{a} \sum_{n=1}^{\infty} \frac{n\pi}{a} \quad (52)$$

$$\mathcal{F}_T = 2k_B T \sum_{n=1}^{\infty} \ln \left[1 - \exp \left(-\frac{\hbar v_0 n\pi}{k_B T a}\right)\right].$$

The above calculation may be explained in a simple physical way: As we realized before, the strings (as well as membranes) can be considered as a superposition of independent harmonic oscillatory modes. So a thermal equilibrated string of temperature $T$ can be regarded as a superposition of infinite number of independent quantum harmonic oscillators, each of which being equilibrated at the temperature $T$. For a harmonic oscillator of temperature $T$, the partition function is given as

$$\Lambda_n = \exp \left(-\frac{\hbar}{2k_B T \omega_n}\right) \left[1 - \exp \left(-\frac{\hbar}{k_B T \omega_n}\right)\right]^{-1}, \quad (53)$$

in which $\omega_n$ is the oscillation frequency. Taking $\omega_n = v_0 \frac{n\pi}{a}$, we can assign a free energy for each transverse direction of each mode of the straight string as

$$\mathcal{F}_n = -k_B T \ln \Lambda_n = \frac{\hbar v_0 n\pi}{2a} + k_B T \ln \left[1 - \exp \left(-\frac{\hbar v_0 n\pi}{k_B T a}\right)\right]. \quad (54)$$

The free energy of the string is then obtained just by summing over the free energies of the mode spectrum of all transverse directions $\mathcal{F} = 2 \sum_{n=1}^{\infty} \mathcal{F}_n$, and so Eqs. (52) will be directly obtained.
The zero-temperature part $F_0$ gives (after regularization) just the Casimir energy of the string:

$$F_0 = -\frac{\pi \hbar \nu_0}{12a} = E_{\text{cas}}(a). \quad (55)$$

As a result, in the framework of the Casimir effect, the free energy $F$ is taken as the nonzero-temperature Casimir energy [13,16]. So the nonzero-temperature Casimir energy of the string can be written as

$$F_{\text{cas}}(a, T) \equiv F(a, T) = E_{\text{cas}}(a) + 2k_B T \sum_{n=1}^{\infty} \ln \left[ 1 - \exp \left( -\frac{\hbar \nu_0 n\pi}{k_B T a} \right) \right], \quad (56)$$

and so the nonzero-temperature Casimir force can be given as

$$F_{\text{cas}}(a, T) \equiv -\frac{\partial F_{\text{cas}}}{\partial a}. \quad (57)$$

Physically we expect that for large $a$ (i.e. for a long string) the nonzero-temperature Casimir force tends to zero. But using the Abel–Plana formula one can find for large $a$

$$\sum_{n=1}^{\infty} \ln \left[ 1 - \exp \left( -\frac{\hbar \nu_0 n\pi}{k_B T a} \right) \right] = -\frac{\pi k_B T}{6\hbar \nu_0} a + O(\ln a). \quad (58)$$

From the first term in the right-hand side of the above relation, $F_{\text{cas}}(a, T)$ would contain a nonzero term at the limit $a \to \infty$. So we should subtract this term, to obtain the physical Casimir energy at nonzero temperature:

$$F_{\text{cas}}(a, T) = -\frac{\pi \hbar \nu_0}{12a} + 2k_B T \sum_{n=1}^{\infty} \ln \left[ 1 - \exp \left( -\frac{\hbar \nu_0 n\pi}{k_B T a} \right) \right] + \frac{\pi a}{3\hbar \nu_0} (k_B T)^2. \quad (59)$$

Now the nonzero-temperature Casimir force can be found as

$$F_{\text{cas}}(a, T) \equiv -\frac{\partial F_{\text{cas}}}{\partial a} = F_{\text{cas}}(a) \left( 1 - \sum_{n=1}^{\infty} \frac{24n \exp (-n\theta/T)}{1 - \exp (-n\theta/T)} + 4\pi^2 (\theta/T)^{-2} \right), \quad (60)$$

in which $\theta \equiv \pi \hbar \nu_0 / ak_B$, and we have used Eq. (4). Therefore, following Eq. (7), one can find the corrected tension of the string at nonzero temperature:

$$\sigma(a, T) \equiv \sigma_0 + \frac{\partial F_{\text{cas}}}{\partial a} = \sigma_0 + \frac{\pi \hbar \nu_0}{12a^2} \left( 1 - \sum_{n=1}^{\infty} \frac{24n \exp (-n\theta/T)}{1 - \exp (-n\theta/T)} + 4\pi^2 (\theta/T)^{-2} \right). \quad (61)$$

At low temperature we can approximate the series in Eq. (61) with its first term, so we can write

$$\sigma(a, T) \approx \sigma_0 + \frac{\pi \hbar \nu_0}{12a^2} \left( 1 - 24 \exp (-\theta/T) + 4\pi^2 (\theta/T)^{-2} \right); \quad T \ll \theta. \quad (62)$$

The corresponding results for the circular string can be found in a quite similar way.
4.2. Membranes at nonzero temperature

The temperature corrections for membranes are obtained similarly as we did for strings. For a rectangular flat membrane (with the Dirichlet boundary condition) we have

\[ \mathcal{F}_T = k_B T \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \ln \left( 1 - \exp \left[ - \frac{\hbar}{k_B T} \alpha_{n,m} \right] \right) \]

\[ \alpha_{n,m} \equiv \nu_0 \sqrt{ \left( \frac{n \pi}{a} \right)^2 + \left( \frac{m \pi}{b} \right)^2 } \]  \hspace{1cm} (63)

so

\[ \mathcal{F}_{\text{cas}}(a, b, T) = E_{\text{cas}}(a, b) + \mathcal{F}_T \]

\[ F^\text{cas}_x(a, b, T) = - \frac{\partial \mathcal{F}_{\text{cas}}}{\partial b} \quad \text{and} \quad F^\text{cas}_y(a, b, T) = - \frac{\partial \mathcal{F}_{\text{cas}}}{\partial a} \]  \hspace{1cm} (64)

with \( E_{\text{cas}}(a, b) \) given by Eq. (21). Then, applying the Abel–Plana formula twice, we find for large values of \( a \) and \( b \)

\[ \sum_{n,m=1}^{\infty} \ln \left( 1 - \exp \left[ - \frac{\hbar}{k_B T} \alpha_{n,m} \right] \right) \approx \frac{\pi}{12} \frac{k_B T}{\hbar \nu_0} (a + b) - \frac{\zeta_R(3)}{2\pi} \left( \frac{k_B T}{\hbar \nu_0} \right)^2 ab. \]  \hspace{1cm} (65)

Thus, requiring \( F^\text{cas}_x \) and \( F^\text{cas}_y \) to tend to zero at large \( b \) and \( a \) respectively, we will find the physical \( \mathcal{F}_{\text{cas}} \) as

\[ \mathcal{F}_{\text{cas}}(a, b, T) = E_{\text{cas}}(a, b) + k_B T \sum_{n,m=1}^{\infty} \ln \left( 1 - \exp \left[ - \frac{\hbar}{k_B T} \alpha_{n,m} \right] \right) \]

\[ - \frac{\pi}{12} \frac{(k_B T)^2}{\hbar \nu_0} (a + b) + \frac{\zeta_R(3) (k_B T)^3}{2\pi (\hbar \nu_0)^2} ab. \]  \hspace{1cm} (66)

Then the corrected tensions would be given as

\[ \tau_x(a, b, T) = \tau_0 - \frac{1}{a} F^\text{cas}_x(a, b, T) = \tau_0 + \frac{1}{a} \frac{\partial \mathcal{F}_{\text{cas}}}{\partial b} \]

\[ \tau_y(a, b, T) = \tau_0 - \frac{1}{b} F^\text{cas}_y(a, b, T) = \tau_0 + \frac{1}{b} \frac{\partial \mathcal{F}_{\text{cas}}}{\partial a}. \]  \hspace{1cm} (67)

Keeping the first term of the series in Eq. (66), we find a low-temperature correction to the tension of, e.g., the \( x \)-direction as

\[ \frac{\pi \hbar \nu_0}{ab^2} \exp \left[ \frac{-(\eta/T) \sqrt{1 + (b/a)^2}}{\sqrt{1 + (b/a)^2}} \right] - \frac{\pi}{12a} \frac{(k_B T)^2}{\hbar \nu_0} + \frac{\zeta_R(3) (k_B T)^3}{2\pi (\hbar \nu_0)^2}; \quad T \ll \eta. \]  \hspace{1cm} (68)

in which \( \eta \equiv \pi \hbar \nu_0 / bk_B \). The corresponding results are similar for the cylindrical membrane.

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