Hydrodynamics on non-commutative space: A step toward hydrodynamics of granular materials

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Hydrodynamics on non-commutative space is studied based on a formulation of hydrodynamics by Y. Nambu [1] in terms of Poisson and Nambu brackets. Replacing these brackets by Moyal brackets with a parameter $\theta$, a new hydrodynamics on non-commutative space is derived. It may be a step toward finding the hydrodynamics of granular materials whose minimum volume is given by $\theta$. To clarify this minimum volume, path integral quantization and the uncertainty relation of Nambu dynamics are examined.

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1. Introduction

In 1973, Y. Nambu proposed a generalized Hamiltonian dynamics, in which the usual phase space spanned by a canonical pair $(p, q)$ is generalized to that spanned by more than three canonical variables $(x_1, x_2, \ldots, x_n)$ [1]. The simplest generalization is a 3D phase space of $(x_1, x_2, x_3)$, where Hamilton’s equation of motion is written in terms of two Hamiltonians, $H_1$ and $H_2$, as follows:

$$\frac{dx_i}{dt} = \frac{\partial (x_i, H_1, H_2)}{\partial (x_1, x_2, x_3)} \quad (i = 1, \ldots, 3).$$

For the time development of an observable $O(x_1, x_2, x_3)$, we have

$$\frac{dO}{dt} = \frac{\partial (O, H_1, H_2)}{\partial (x_1, x_2, x_3)}.$$  \hspace{1cm} (2)

The right-hand sides are written in terms of Jacobians. In the usual Hamilton dynamics, the Liouville theorem states that the phase space volume $dp \wedge dq$ occupied by an ensemble of dynamical systems is preserved in time. The generalization of this to the $n$-dimensional phase space is easy. Therefore, in the generalized ($n$-dimensional) Hamiltonian dynamics, now called Nambu dynamics, the phase space volume $dx_1 \wedge dx_2 \ldots \wedge dx_n$ occupied by an ensemble of systems is temporarily preserved. The dynamics naturally incorporates the infinite-dimensional local symmetries of the volume-preserving diffeomorphisms whose transformations $(x_1, \ldots, x_n) \rightarrow (x'_1, \ldots, x'_n)$ preserve the Jacobian:

$$\frac{\partial (x'_1, x'_2, \ldots, x'_n)}{\partial (x_1, x_2, \ldots, x_n)} = 1.$$  \hspace{1cm} (3)
For the 2D phase space case, $\partial(A, B)/\partial(q, p)$ is the Poisson bracket, and for the case of the phase space having more than three canonical variables, we call $\partial(A_1, A_2, \ldots, A_n)/\partial(x_1, x_2, \ldots, x_n)$ the Nambu bracket.

The quantization of this generalized Hamiltonian dynamics, or the quantization of the Nambu bracket, was tried in the paper of 1973 [1]. Since then many people have tried to quantize the Nambu brackets by using various methods [2–37] (see also M. Flato and C. Fronsdal, unpublished data).

In the background of Nambu dynamics, there exists a volume-preserving diffeomorphism for an ensemble of dynamical systems, so that it naturally fits to the incompressible fluid dynamics, where an ensemble of fluid ingredients moves in time, keeping its occupying volume. Therefore, it is quite natural that Nambu recently reformulated the hydrodynamics in terms of Poisson brackets in two spacial dimensions and Nambu brackets in three spacial dimensions [38]. He considered, of course, an incompressible fluid.

In this paper, we investigate a hydrodynamics on non-commutative space based on the formulation of hydrodynamics by Nambu. We construct a new hydrodynamics on non-commutative space through the replacement of the Poisson and Nambu brackets by Moyal ones. This is a method invented by Moyal [39] regarding quantization, so that we use it to quantize the space or to find the quantum Nambu brackets. Since we have to clarify the meaning of the Moyal bracket, we discuss the relationship between the Moyal product and the path integral quantization of a toy model. In the toy model the Moyal product may reproduce the expectation value of the quantum theory.

Our final aim is to produce the hydrodynamics describing the motion of granular materials whose minimum volume is expressed by a model parameter $\theta$ in the Moyal bracket. The physics of granular materials is an interesting topic and is rapidly developing [40]. To clarify the minimum volume, we examine the quantization of the Nambu dynamics in the path integral formulation. In 3D phase space, the quantum Nambu dynamics is a closed string theory. In this way, the uncertainty relation, which gives the basis of the minimum volume, is clarified. We note that the extension of the Lagrangian formulation of non-commutative perfect fluids has been explored in Refs. [41–43], and diffusion in non-commutative geometries has been studied in Refs. [44,45]. In addition, uncertainty relations in non-commutative space-time [46] and an application of hydrodynamics like that by Nambu for D-branes [47] have been investigated.

It is true that different methods of quantization give different hydrodynamics. So, it is interesting to consider different hydrodynamics on different non-commutative spaces with different quantization methods, and compare the obtained results to the experimental data that seem to have been compiled so far for various granular materials. This is, however, beyond the scope of this paper.

The organization of this paper is as follows. In Sect. 2, we review the hydrodynamics by Nambu. In Sect. 3, we formulate a new hydrodynamics on non-commutative space, starting from the hydrodynamics by Nambu. In Sect. 4, we compare the Moyal product with the expectation value in the path integral quantization of a toy model. In Sect. 5, we examine the path integral quantization of Nambu dynamics in general and clarify its uncertainty relation. Our investigations are finally concluded in Sect. 6.

### 2. Nambu’s hydrodynamics

The continuity equation of a fluid is given in terms of the density $\rho(x; t)$ and velocity $v(x; t)$ of the fluid by

$$\dot{\rho}(x; t) + \nabla(\rho(x; t)v(x; t)) = 0,$$  \hspace{1cm} (4)
which becomes, in the incompressible case \((\rho = \text{const})\),
\[
\nabla \mathbf{v}(x; t) = 0. 
\]
(5)

Here, the dot denotes the time derivative of \(\partial/\partial t\), and \(\nabla\) is the differential operator as \(\nabla \equiv (\partial/\partial x_1, \partial/\partial x_2, \partial/\partial x_3)\). Then, we can introduce stream functions, one function \(\varphi(x_1, x_2; t)\) in two spatial dimensions and two functions \(\varphi_1(x_1, x_2, x_3; t)\) and \(\varphi_2(x_1, x_2, x_3; t)\) in three spatial dimensions, and express velocity fields so as to satisfy the continuity equation (5) as follows:
\[
v_i = \dot{x}_i = \{x_i, \varphi\}_p \quad (i = 1, 2 \text{ for } 2D),
\]
(6)
\[
v_i = \dot{x}_i = \{x_i, \varphi_1, \varphi_2\}_N \quad (i = 1, 2, 3 \text{ for } 3D),
\]
(7)

where the Poisson and Nambu brackets are defined by a Jacobian,
\[
\{A_1, A_2, \ldots, A_n\} = \frac{\partial(A_1, A_2, \ldots, A_n)}{\partial(x_1, x_2, \ldots, x_n)} \equiv \sum_{i_1, i_2, \ldots, i_n=1}^n \varepsilon^{i_1i_2\cdots i_n} \partial_{i_1} A_1(x; t) \partial_{i_2} A_2(x; t) \cdots \partial_{i_n} A_n(x; t),
\]
(8)

where \(\varepsilon^{i_1i_2\cdots i_n}\) is the Levi–Civita tensor or the totally anti-symmetric tensor. The case of \(n = 2\) is the Poisson bracket and that of \(n = 3\) is the Nambu bracket.

Nambu considered that the position of an element of a fluid \(x_i(t)\) \((i = 1, \ldots, n)\) at time \(t\) is parameterized by its initial (material) coordinates \((\sigma_1, \sigma_2, \ldots, \sigma_n)\) at \(t = 0\), i.e.,
\[
x_i(t) = x_i(\sigma_1, \sigma_2, \ldots, \sigma_n; t) \quad (i = 1, \ldots, n).
\]
(9)

Then, the incompressibility condition is given by
\[
\frac{\partial(x_1, x_2, \ldots, x_n)}{\partial(\sigma_1, \sigma_2, \ldots, \sigma_n)} = 1.
\]
(10)

In full usage of this condition he derived the Navier–Stokes equation, where the Jacobian in terms of \((\sigma_i; i = 1, \ldots, n)\) that appears at the beginning is replaced finally by the Jacobian in terms of \((x_i; i = 1, \ldots, n))\), Poisson and Nambu brackets, due to (10).

The equations of motion of a 2D incompressible fluid \((i = 1, 2)\) so derived by Nambu are
\[
\rho \left( \{x_i, \varphi\} + \{x_i, \varphi, \varphi\} \right) + \varepsilon^{ij} \{p, x_j\} - \eta \Delta \{x_i, \varphi\} = 0,
\]
(11)
while in a 3D fluid \((i = 1, 2, 3)\) they read
\[
\rho \left( \{x_i, \varphi_1, \varphi_2\} + \{x_i, \varphi_1, \varphi_2\} + \{x_i, \varphi_1, \varphi_2\} + \{x_i, \varphi_1, \varphi_2\} \right)
+ \frac{1}{2} \varepsilon^{ijk} \{p, x_j, x_k\} - \eta \Delta \{x_i, \varphi_1, \varphi_2\} = 0,
\]
(12)

where \(\rho\) is the pressure, but the external potential \(V\) may be included in \(\rho\) as \(\rho + V\), \(\Delta\) is the Laplacian, and the index of shear viscosity \(\eta\) is introduced. These equations are identical to the usual Navier–Stokes equations,
\[
\rho \frac{\partial \mathbf{v}}{\partial t} + \nabla p - \eta \Delta \mathbf{v} = 0.
\]
(13)
where the Lagrangian derivative is

\[
\frac{Dv}{Dt} = \frac{\partial v}{\partial t} + (v \cdot \nabla)v = \frac{\partial v}{\partial t} + \nabla \left( \frac{1}{2}v^2 \right) + \omega \times v, \tag{14}
\]

and \( \omega = \nabla \times v \) is the vorticity. In two dimensions we have to choose \( \omega = (0, 0, \omega) \) as usual.

It is instructive to derive the Nambu equations (12) explicitly, starting from the Navier–Stokes equations (13).

3. Hydrodynamics on non-commutative space

Now, we introduce the Moyal product and the Moyal bracket and are going to replace the Poisson and Nambu brackets by the Moyal brackets.

The Moyal product or *-product is defined with a constant parameter \( \theta_{ab} \) by

\[
A(x) * B(x) = \exp \left( \frac{i}{2!} \theta_{ab} \frac{\partial^2}{\partial y^a \partial z^b} \right) A(y) B(z) \bigg|_{y,z \to x}, \tag{15}
\]

and its natural generalization to the three *-product with a parameter \( \theta_{abc} \) is

\[
A(x) * B(x) * C(x) = \exp \left( \frac{i}{3!} \theta_{abc} \frac{\partial^3}{\partial y^a \partial z^b \partial u^c} \right) A(y) B(z) C(u) \bigg|_{y,z,u \to x}. \tag{16}
\]

By taking simply \( \theta_{ab} = \epsilon_{ab} \theta_2 \), and \( \theta_{abc} = \epsilon_{abc} \theta_3 \), then what we have introduced is a parameter with the dimension of area for \( \theta_2 \), or volume for \( \theta_3 \).

The Moyal bracket is defined as follows:

\[
[A(x), B(x)]_M = \sum_{A,B} \epsilon_{AB} A(x) * B(x), \tag{17}
\]

and

\[
[A(x), B(x), C(x)]_M = \sum_{A,B,C} \epsilon_{ABC} A(x) * B(x) * C(x). \tag{18}
\]

Now we are going to replace the Poisson bracket in the 2D hydrodynamics and the Nambu bracket in the 3D hydrodynamics by the corresponding Moyal brackets as follows:

\[
\{A, B\}_P \to \frac{1}{i \theta_2} [A, B]_M. \tag{19}
\]

\[
\{A, B, C\}_N \to \frac{1}{i \theta_3} [A, B, C]_M. \tag{20}
\]

Then, we will arrive at a new hydrodynamics having a parameter \( \theta_2 \) or \( \theta_3 \), which may be related to the size of the granular materials comprising the fluid.
The result of the replacement is that all the single Moyal brackets are identical to the Poisson bracket or the Nambu bracket, and the difference arises only in the double Moyal brackets, i.e., for the 2D hydrodynamics,

$$
[[x_i, \varphi(x)]_M, \varphi(x)]_M = \{[x_i, \varphi(x)], \varphi(x)\} - \frac{(\theta_2)^2}{24} \left( \frac{\partial y_1 \partial z_2 - \partial y_2 \partial z_1}{y_i(y) \varphi(z)} \right)_{y,z \to x} + O \left( \theta_2^4 \right),
$$

and, in the 3D hydrodynamics, the difference appears in

$$
[[x_i, \varphi_1(x), \varphi_2(x)]_M, \varphi_1(x), \varphi_2(x)]_M
= \{[x_i, \varphi_1(x), \varphi_2(x)], \varphi_1(x), \varphi_2(x)\}
- \frac{(\theta_3)^2}{3!} \epsilon_{v, \varphi_1, \varphi_2} \left( \left( \frac{\partial^3}{\partial y_a \partial z_b \partial u_c} \right)^3 v_i(y) \varphi_1(z) \varphi_2(u) \right)_{y,z,u \to x} + O \left( \theta_3^4 \right).
$$

Now the Navier–Stokes equations of motion in the non-commutative space with $O(\theta^2)$ corrections are given by

$$
\frac{\rho}{D} \frac{Dv}{Dt} + \nabla p - \eta \Delta v = K,
$$

where the $O(\theta^2)$ correction $K$ reads

$$
K = \frac{(\theta_2)^2}{24} \rho \left( \frac{\partial y_1 \partial z_2 - \partial y_2 \partial z_1}{2} \right)^2 \sum_{a=1,2} \partial_{y_a} v(y) v_a(z) \bigg|_{y,z \to x}
$$

(2D),

$$
K = \frac{(\theta_3)^2}{3!} \rho \epsilon_{v, \varphi_1, \varphi_2} \left( \left( \frac{1}{3!} \sum_{abc} \frac{\partial^3}{\partial y_a \partial z_b \partial u_c} \right)^3 v(y) \varphi_1(z) \varphi_2(u) \right) \bigg|_{y,z,u \to x}
$$

(3D).

The velocity in 3D is related to stream functions $\varphi_1$ and $\varphi_2$ as

$$
v^a = \frac{1}{2} \epsilon_{abc} \frac{\partial (\varphi_1, \varphi_2)}{\partial (x_b, x_c)}.
$$

4. Moyal product and path integral of a toy model

We have to understand the uncertainty relation, or the possibility of introducing by $\theta$ a minimum size to the element of the fluid. In case of two dimensions, the meaning of the Moyal product is clear. We know that the quantum mechanical operator algebra exists behind. We introduce two operators $\hat{A}(\hat{x})$ and $\hat{B}(\hat{x})$, and assume the following operator relation for the variables:

$$
[\hat{x}_a, \hat{y}_b] = i \theta \delta_{ab} \quad (a, b = 1, 2).
$$
Here we put a hat on the operators, and the commutator is the usual one in the operator algebra. We assume the following Fourier expansion:

\[
\hat{A}(\hat{x}) = \int \frac{dp}{(2\pi)^2} e^{-ip\hat{x}} A(p),
\]

\[
\hat{B}(\hat{y}) = \int \frac{dq}{(2\pi)^2} e^{-iq\hat{y}} B(q),
\]

which fixes the operator ordering of \(\hat{x}\) in \(\hat{A}(\hat{x})\) and \(\hat{B}(\hat{y})\). Then, we can prove that

\[
\hat{A}(\hat{x})\hat{B}(\hat{y}) = A(x) \ast B(x)|_{x \to \hat{x}}.
\]

Therefore, the Moyal bracket faithfully represents the commutation relation of the operator algebra, or

\[
[A(x), B(x)]_M |_{x \to \hat{x}} = [\hat{A}(\hat{x}), \hat{B}(\hat{x})].
\]

Now we can understand the uncertainty relation, which is also valid in the hydrodynamics of the non-commutative space,

\[
\langle (\Delta x)^2 \rangle^{1/2} \langle (\Delta y)^2 \rangle^{1/2} \geq \theta_2/2.
\]

Then, we may consider each element of the fluid to have a minimum area \(\theta/2\), or the fluid to consist of a granular material.

Next we compare the Moyal product and the expectation value in the path integral quantization of a toy model. The expectation value \(\langle O \rangle\) in terms of the path integral method of a toy model is given by

\[
\langle O(x) \rangle \propto \int DX DY O(X, Y) \exp \left( \frac{1}{\theta^2} \left[ iXY - \frac{1}{2} (X^2 + Y^2) \right] \right).
\]

If we consider \(X\) as a momentum and \(Y\) as a coordinate, this simplified model may represent the quantum mechanics, while if we consider both \(X\) and \(Y\) as coordinates, it may represent the non-commutative space. Here \(\frac{1}{2} (X^2 + Y^2)\) is a toy Hamiltonian. Notice that even after Wick rotation the phase factor remains a phase factor. The phase factor

\[
\exp \left( \frac{i}{\hbar} \int p \, dq \right)
\]

is the origin of quantum algebra, so that a phase factor

\[
\exp \left( \frac{i}{\theta^2} XY \right)
\]

in the Moyal product is the origin of non-commutativity in space. The expectation value can be calculated perturbatively as

\[
\langle O(X, Y) \rangle = O \left( \frac{\partial}{\partial J_X} \frac{\partial}{\partial J_Y} \right) \times \exp \left( \frac{i}{\theta^2} \left[ \frac{1}{i} \frac{\partial}{\partial J_X} \frac{1}{i} \frac{\partial}{\partial J_Y} - \frac{\theta_2}{2} (J_X^2 + J_Y^2) \right] \right) \bigg|_{J_X, J_Y \to 0}.
\]
This shows that \( X \) and \( Y \) in the operator \( O \) are contracted with \( X \) and \( Y \) in the phase factor with the propagator \( \langle XX \rangle = \langle YY \rangle = \theta_2 \), so that we may understand that

\[
\langle O(X, Y) \rangle = O(X, Y)_* = \exp \left( i\theta_2 \frac{\partial^2}{\partial X \partial Y} \right) O(X, Y). \tag{37}
\]

Here we have to comment on the relation between the ordering of factors in the Moyal product and the time ordering of them in the path integral. Consider the product \( A(X_+ +) \ast B(X_-) \); this corresponds to the time ordering in the path integral, or the path integral over \( A(X_+ = X(t_+)) B(X_- = X(t_-)) \) with \( t_+ > t_- \). Finally we have to take the limit \( t_+, t_- \rightarrow t \). The phase factor in this case is more precisely

\[
\exp \left( -\frac{i}{2!\theta_2} \epsilon_{abc} \int X^a dX^b \right) = \exp \left( \frac{i}{2!\theta_2} \epsilon_{abc} \frac{\partial^3}{\partial X \partial Y \partial Z} \right). \tag{38}
\]

Therefore, the Moyal product is understood to be equal to the path integral expectation value of the toy model. In general, the Moyal product and the quantum expectation value may differ, because of interactions other than the mass terms or the Gaussian damping factors.

Now we go to 3D hydrodynamics. How the uncertainty relation appears in this case is an interesting issue, but the discussion of it is postponed to the next section where the quantization of the Nambu dynamics will be discussed. Here, we simply compare the results of the Moyal product and the path integral, using a toy model. We consider

\[
\langle O(X, Y, Z) \rangle \propto \int DX \, DY \, DZ \, O(X, Y, Z) \times \exp \left( \frac{i}{\theta_3} XYZ - \frac{1}{2(\theta_3)^{2/3}} (X^2 + Y^2 + Z^2) \right). \tag{39}
\]

This includes the 3D phase space factor. The propagator in this case is \( (\theta_3)^{2/3} \), so that we have

\[
\langle O(X, Y, Z) \rangle = O(X, Y, Z)_* = \exp \left( i\theta_3 \frac{\partial^3}{\partial X \partial Y \partial Z} \right) O(X, Y, Z). \tag{40}
\]

About the ordering of the Moyal product, we have to examine the phase factor more explicitly:

\[
\exp \left( -\frac{i}{3!\theta_3} \epsilon_{abc} \int X^a \frac{\partial (X^b, X^c)}{\partial (\sigma, t)} d\sigma \, dt \right). \tag{41}
\]

If we restrict it to an infinitesimal rectangular region formed by four corners \( (A, B, C, D) \), the coordinates of which are

\[
\begin{bmatrix}
D(\sigma, t), & A(\sigma, t - \Delta t) \\
C(\sigma - \Delta \sigma, t), & B(\sigma - \Delta \sigma, t - \Delta t)
\end{bmatrix}, \tag{42}
\]

then the phase factor becomes

\[
\exp \left( \frac{i}{3!\theta_3} \epsilon_{abc} \left( X^a(A) X^b(B) X^c(C) + X^a(D) X^b(C) X^c(B) - X^a(C) X^b(B) X^c(A) \right) \right). \tag{43}
\]

In the next section we will understand that the quantum theory in 3D is a closed string theory. In this terminology, a closed string \( C \) develops in time by a deformation in which a portion \( BA \) of a closed
string $C$ is replaced by $\overrightarrow{BCDA}$ by a rectangular deformation $\delta C = \overrightarrow{ABCDA}$. The time evolution is done in this way, so that the “area” of the rectangular $\overrightarrow{ABCD}$ plays the role of “time”. Accordingly, the concept of time ordering in 2D should be changed in 3D. The ordering in 3D is the path ordering associated with the infinitesimal closed path $\delta C$, the boundary curve of the rectangular $\overrightarrow{ABCD}$. If we take the limit $\Delta t \Delta \sigma \rightarrow 0$, the phase factor becomes
\[ \exp \left( \frac{i}{3! \hbar^3} \epsilon_{abc} P(X^a X^b X^c) \right), \] (44)
where $P$ denotes the path ordering with respect to the closed path $\delta C$, or the boundary curve of the rectangular $\overrightarrow{ABCD}$. Now, the ordering of the Moyal product $A(X) \ast B(Y) \ast C(Z)$ means the path ordering of the three operators $(X, Y, Z)$ in this sense. So, the Moyal product may also give the expectation values in the path integral of the toy model in 3D, but it may not reproduce all of the quantum properties in more general cases, because of the possible existence of additional interactions. However, the Moyal product reproduces the essential part of the quantum, or non-commutative, properties.

5. Path integral quantization of Nambu dynamics and its uncertainty relation

The action of Nambu dynamics is given by Takhtajan in Theorem 7 of Ref. [17], but this action was already known by Nambu in the Hamilton–Jacobi formulation of the string theory [48]. The action is
\[ S_n = \int X_1 dX_2 \wedge \cdots \wedge dX_n - H_1 dH_2 \wedge \cdots \wedge dH_{n-1} \wedge dt, \] (45)
where $t$ is time. The fact that the minimum configuration of the action gives the equation of motion of Nambu dynamics is shown by Ref. [17]. Let us study the case of $n = 3$:
\[ S_3 = \int X dY \wedge dZ - H_1 dH_2 \wedge dt. \] (46)
As was pointed out in Refs. [17] and [48], this is not a point particle theory, but a closed string theory, the configuration of which is specified by a circle (2-cycle) $C(\sigma, t)$ on the 2D plane $(Y, Z) = (X_2, X_3)$, namely
\[ C(\sigma, t) = \{(Y(\sigma, t), Z(\sigma, t))\} \text{ with } (0 \leq \sigma \leq 2\pi, \ -\infty \leq t \leq +\infty), \] (47)
where the closed string means $C(0, t) = C(2\pi, t)$. Now, the path integral quantization of $n = 3$ Nambu dynamics is given by the following partition function:
\[ Z \propto \int DX(\sigma, t)DY(\sigma, t)DZ(\sigma, t) \exp \left( \frac{i}{\hbar^3} S_3[X(\sigma, t), Y(\sigma, t), Z(\sigma, t)] \right). \] (48)
Notice that this is the path integral in phase space $(X, Y, Z)$, and is not in configuration space. But, if the momentum $X$ is integrated out, then the usual path integral expression in configuration space is obtained. A path is specified by a configuration, $\{X(\sigma, t), C(\sigma, t)\} = \{X(\sigma, t), Y(\sigma, t), Z(\sigma, t)\}$ parameterized by two parameters, $\sigma$ and $t$.

Now we introduce the wave functional $\Psi[C(\sigma); t]$. Here we consider $\Psi$ to depend on the coordinates $Y$ and $Z$, but not on the momentum $X$. This is usually correct, since due to the uncertainty
relation, which will appear shortly, we are not able to specify all of these \((X, Y, Z)\), certainly at a given time \(t\). Then, \(\Psi_{\alpha, \beta}[C(\sigma); t]\) is given by

\[
\Psi_{\alpha, \beta}[C(\sigma); t] \propto \int_{C(\sigma), t_0} DX(\sigma, t)DY(\sigma, t)DZ(\sigma, t) \exp \left( \frac{i}{\theta_3} \left[ \int X dY \wedge dZ - H_1 dH_2 \wedge dt \right] \right) \times \Psi[C_{\alpha, \beta}(\sigma); t_0],
\]

(49)

where \(C_{\alpha, \beta}\) denote the initial configurations (shapes) of the closed strings at \(t_0\). The wave functional \(\Psi_1\) depends on the initial configurations that may label the state vectors \(|\Psi_1[C(\sigma); t]\rangle\).

The amplitude of an observable \(O(X(\sigma), C(\sigma); t)\) is given by

\[
\langle \alpha | \hat{O} | \beta \rangle \propto \int DX(\sigma, t)DY(\sigma, t)DZ(\sigma, t) \times \Psi_1[C(\sigma); t] | O(X(\sigma), C(\sigma); t) | \Psi_1[C(\sigma); t].
\]

(50)

Following Feynman [49], we can read off the operator algebra from the path integral expression. We introduce the area \(A(C)\) of the circle,

\[
A(C) = \oint_C Y \wedge dZ;
\]

(51)

the functional derivative \(\delta/\delta C(\sigma)\) corresponding to the path deformation at \(\sigma\), \(\delta C(\sigma)\), appeared in the last section. It is usually defined as

\[
\frac{\delta}{\delta C(\sigma)} = \lim_{\delta C(\sigma) \rightarrow 0} \frac{\Psi[C(\sigma) + \delta C(\sigma)] - \Psi[C(\sigma)]}{\text{area of } \delta C(\sigma)}.
\]

(52)

We understand

\[
\frac{\delta A(C)}{\delta C(\sigma)} = 1,
\]

(53)

so we have

\[
\frac{\delta}{\delta C(\sigma)} \Psi[C(\sigma); t] = \frac{i}{\theta_3} X(\sigma, t) \Psi[C(\sigma); t],
\]

(54)

\[
\frac{\partial}{\partial t} \Psi[C(\sigma); t] = -\frac{i}{\theta_3} \left( \oint_C H_1 dH_2 \right) \Psi[C(\sigma); t].
\]

(55)

If we choose \(O(X, Y, Z)\) in Eq. (50) as \(\hat{O}\) or \(\delta A(C)/\delta C(\sigma)\), and perform the partial path integrations, we have the following operator relations:

\[
i\theta_3 \hat{O} = \left[ O, \oint_C H_1 dH_2 \right] = \left[ O, \oint_C dV \right],
\]

(56)

\[
[X(\sigma, t), A(C)] = -i\theta_3,
\]

(57)

where the vector field \(V\) is that introduced by Nambu. It is also the Clebsch potential in hydrodynamics. The meaning of the operator relations can be understood from Eq. (50), namely

\[
\langle \alpha | \hat{O}_1 \hat{O}_2 | \beta \rangle = \sum_{\gamma} \langle \alpha | \hat{O}_1 | \gamma \rangle \langle \gamma | \hat{O}_2 | \beta \rangle.
\]

(58)

From the commutation relation Eq. (57), we have the following uncertainty relation using the standard method,

\[
\sqrt{\langle (\Delta X)^2 \rangle} \sqrt{\langle (\Delta A(C))^2 \rangle} \geq \frac{\theta_3}{2},
\]

(59)
where the expectation value means
\[ \langle \hat{O} \rangle \propto \sum_\alpha \langle \alpha | \hat{O} | \alpha \rangle. \] (60)

This is the uncertainty relation in the 3D case and is a generalization of the quantum mechanical uncertainty relation in the 2D case in Eq. (32).

Therefore, the 3D hydrodynamics on the non-commutative space gives the minimum volume of the space equal to \( \theta^3/2 \), so that the material comprising the fluid is not a point particle but a particle with a finite volume, or a granular material. In the general Nambu dynamics with \( n \)-dimensional phase space, the corresponding uncertainty relation yields
\[ \sqrt{\langle (\Delta X)^2 \rangle} \sqrt{\langle (\Delta V(C_{n-2}))^2 \rangle} \geq \theta_n/2, \] (61)

where \( V(C_{n-2}) \) is the volume of the \( (n-2) \)-cycle \( C_{n-2} \) on which the quantum theory is based.

To make clearer the connection of Nambu dynamics to strings (or more extended objects), we will write the action \( S_3 \) as follows:
\[ S_3 = \int \left[ X(\sigma, t) \frac{\partial (Y, Z)}{\partial (\sigma, t)} - \left( H_1 \frac{\partial}{\partial \sigma} H_2 \right) \right] d\sigma dt. \] (62)

Then, the Hamiltonian density of the string \( \mathcal{H} \) reads
\[ \mathcal{H} = H_1 \frac{\partial}{\partial \sigma} H_2. \] (63)

In the toy model in 3D,
\[ \mathcal{H} = \frac{1}{2} \left( X^2 + Y^2 + Z^2 \right), \] (64)

and so integration over \( X \) gives the Lagrangian density of the toy model as
\[ L = \left( \frac{\partial (Y, Z)}{\partial (\sigma, t)} \right)^2 - \frac{1}{2} \left( Y^2 + Z^2 \right). \] (65)

If we choose
\[ \mathcal{H} = \frac{1}{2} (X^2) + \left( \frac{\partial (Z, X)}{\partial (\sigma, t)} \right)^2 + \left( \frac{\partial (X, Y)}{\partial (\sigma, t)} \right)^2, \] (66)

then we have a string Lagrangian in the Shild gauge,
\[ L = \left( \frac{\partial (X^\mu, X^\nu)}{\partial (\sigma, t)} \right)^2. \] (67)

In the hydrodynamics, however, we have to clarify more explicitly the meaning of Hamiltonian density \( \mathcal{H} \), or of the Hamiltonian for the string field \( \Psi[C; t] \), which is written in terms of the Clebsh potential \( V \):
\[ \hat{H} = \frac{1}{\theta_3} \int d\sigma dt \mathcal{H} = \frac{1}{\theta_3} \oint_C dV. \] (68)

For this purpose, the fundamental relations (1) and (2) and the superposition of stream functions studied by Nambu in Ref. [38] will be important; this moves to incorporate the ensemble averaging and has an affinity with the string field theory as an example.
6. Conclusions and discussions

In this paper the hydrodynamics on non-commutative space has been explored, starting from the formulation of hydrodynamics in terms of the Poisson and Nambu brackets by Y. Nambu [38]. In particular, in order to introduce the finite size of the space point or the finite size of the fluid element, the Poisson and Nambu brackets are replaced by the corresponding Moyal brackets. In this process a parameter $\theta_2$ (dimension of area) or $\theta_3$ (dimension of volume) is introduced in 2D or 3D hydrodynamics, respectively. They represent the minimum size of area and volume that is acceptable in 2D and 3D spaces. The hydrodynamics so obtained has an additional term of $O(\theta^2)$, which does not exist in the usual Navier–Stokes equation. In order to examine whether our hydrodynamics represents the hydrodynamics of granular materials, we have to compare the computer simulation of our hydrodynamics with the motion of granular materials. We will do this in subsequent work.

To support the replacement of Poisson and Nambu brackets by Moyal brackets, we compare the Moyal product and the expectation value of the operator products in the path integral method. We adopt a toy model in which the most important phase factor, being related to 2D or 3D phase spaces, is kept definitely, but the Hamiltonian is a simple one consisting of the bilinear terms or the damping factors of the variables. Moyal products reproduce the path integral expectation values of the toy model. It is also recognized that the ordering of the Moyal product is related to a certain ordering in the path integral method. In the 2D case, this is the usual time ordering, but in the 3D case the ordering is related to the path ordering in $(\sigma, t)$ space. It is very important to recognize that the Nambu dynamics in 3D is a closed string theory in which temporal development is carried out by the deformation of the closed string $\delta C$. Moyal product ordering is related to the path ordering along this small closed string $\delta C$ in the path integral method.

To clarify the uncertainty relation when the Nambu dynamics is quantized, we study the path integral quantization. Using the action of the Nambu dynamics given by Takhtajan [17] and Nambu [48], we explicitly demonstrate the 3D case in terms of the closed string theory. Then, we can easily read the operator relations from the path integral expression, and clarify the uncertainty relations: in the 3D case, this is

$$\sqrt{\langle (\Delta X)^2 \rangle} \sqrt{\langle (\Delta A(C))^2 \rangle} \geq \frac{\theta_3}{2},$$

where $X$ is a coordinate, and $A(C)$ is the area surrounded by a closed string $C$ depicted on the $(Y, Z)$ plane, being perpendicular to the $X$-axis.

In the general Nambu dynamics with $n$-dimensional phase space, the uncertainty relation yields

$$\sqrt{\langle (\Delta X)^2 \rangle} \sqrt{\langle (\Delta V(C_{n-2}))^2 \rangle} \geq \frac{\theta_n}{2},$$

where $V(C_{n-2})$ is the volume of the $(n - 2)$-cycle $C_{n-2}$ on which the quantum theory is based.

It is very important to examine the various quantization methods of Nambu dynamics, or to examine the quantum analogs of Nambu brackets. Classical Nambu brackets satisfy a number of relations. It may be true that, depending on the ingredients of the granular materials, different quantization methods should be applied, and also all the relations satisfied by the Nambu brackets may not be required for some materials. Therefore, it is worthwhile to remind ourselves of some of the attempts so far made for Nambu brackets. For this purpose, there is a good summary of the studies before 2008. Please refer to footnote 2 of the paper by Chong-Sun Chu et al. [50]. In Ref. [17], the Nambu brackets are studied in detail and the Moyal product has also been studied. Modification of the Moyal brackets, the so-called Zariski quantization, has been observed in finite dimensions [18] Moyal brackets and the Zariski quantization are a kind of deformation of the Nambu–Poisson bracket. Furthermore,
there exists another way of generalizing the matrix commutator \([19]\) in finite dimensions. However, the relation between the algebraic structure and the Bagger–Lambert–Gustavsson (BLG) model \([51–54]\), which constructs a 3D \(N = 8\) superconformal field theory, is not clear at all because the triple commutator cannot meet the fundamental identity. In addition, in principle, it is possible to adopt the cubic matrix to describe the 3-algebra \([20,21]\), by which, unfortunately, the fundamental identity cannot be satisfied, and is available only for \(A_4\) algebra \([22]\). The Nambu–Poisson bracket with the cut-off representing the Lie 3-algebra in finite dimensions proposed in Ref. \([50]\) is considered to be the first attempt to meet the fundamental identity, so that it can be compatible with the BLG model.

After 2008, the M5-brane based on the Nambu–Poisson bracket \([55–59]\) has also been studied. Moreover, gauge theories constructed with the Nambu–Poisson bracket have also been studied in Ref. \([60]\) (for a recent review on the Nambu–Poisson bracket, see, e.g., Ref. \([37]\)). A complete independent basis for the structure constants of the volume-preserving diffeomorphism (VPD) has been examined \([61]\).

Finally, we will attempt to rewrite the Nambu dynamics as a matrix model. Matrix formulation of membrane theory was first carried out by Jens Hoppe in his Ph.D. thesis \([11,12]\). If the action \(S_3\) is invariant under the area-preserving diffeomorphisms in \((\sigma, t)\) space, then his method is applicable. We combine \(\sigma\) and \(t\) with \(\sigma_a (a = 1, 2)\) as \(\sigma_1 = \sigma\) and \(\sigma_2 = t\). Then, the infinitesimal area-preserving transformation reads
\[
\delta \sigma^a = \{\sigma^a, \xi(\sigma)\},
\]
and so it forms an algebra
\[
\delta \xi_1 \delta \xi_2 - \delta \xi_2 \delta \xi_1 = \delta [\xi_1, \xi_2].
\]
This algebra is shown to be equal to the \(N \to \infty\) limit of \(SU(N)\) in Refs. \([11,12]\), so that the \(X(\sigma)\), \(Y(\sigma)\), and \(Z(\sigma)\) as well as \(t\) can be replaced by \(N \times N\) Hermitian matrices with hats. Poisson brackets are replaced by the commutator of the corresponding matrices \([62,63]\),
\[
\{A, B\} \to \lim_{N \to \infty} \frac{N}{i} [\hat{A}, \hat{B}],
\]
and \(\int d\sigma dt\) becomes \((1/N)\) Tr of the matrices. In this way we may arrive at the action of a matrix model,
\[
S_3 = \frac{1}{i} \text{Tr} \left( \hat{X}[\hat{Y}, \hat{Z}] - \hat{H}_1[\hat{H}_2, \hat{I}_1] \right).
\]
This expression is, however, far from correct, since the area-preserving diffeomorphisms in \((\sigma, t)\) space do not exist or are obscure in non-relativistic hydrodynamics. However, apart from the symmetries in the treatment of \(X^\mu(\sigma_1, \sigma_2, \ldots, \sigma_D, t)\), if \(D = 2\), to consider \(\sigma_1\) and \(\sigma_2\) as indices of row and column is very natural, so that for \(D = 3\), the appearance of the cubic matrix is also natural. To consider what kind of symmetries may be crucial in studying the hydrodynamics of granular materials, since the symmetry of the ingredients such as a ball, cube, or tetrahedron may be partly considered in the symmetry of the variables describing the hydrodynamics.

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