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Radiation reaction in quantum vacuum

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Since the development of the radiating electron theory by P. A. M. Dirac in 1938 [P. A. M. Dirac, Proc. R. Soc. Lond. A **167**, 148 (1938)], many authors have tried to reformulate this model, called the "radiation reaction". Recently, this equation has become important for ultra-intense laser–electron (plasma) interactions. In our recent research, we found a stabilized model of the radiation reaction in quantum vacuum [K. Seto et al., Prog. Theor. Exp. Phys. **2014**, 043A01 (2014)]. It led us to an updated Fletcher–Millikan charge-to-mass ratio including radiation. In this paper, I will discuss the generalization of our previous model and the new equation of motion with the radiation reaction in quantum vacuum via photon–photon scatterings and also introduce the new tensor $d\mathfrak{E}^{\mu\nu\alpha\beta}/dm$, as the anisotropy of the charge-to-mass ratio.

Subject Index A00, A01

1. Introduction

In 1938, P. A. M. Dirac proposed the equation of an electron motion in classical-relativistic dynamics including the electron's self-interaction, the so-called Lorentz–Abraham–Dirac (LAD) equation [1]:

$$m_0 \frac{dw^\mu}{d\tau} = -e\left(F_{\rm ex}^{\mu\nu} + F_{\rm LAD}^{\mu\nu}\right) w_\nu. \tag{1}$$

Here, m_0, e, and τ are the rest mass, the charge, and the proper time of an electron. w is the 4-velocity defined by $w = \gamma(c, \mathbf{v})$. The Lorentz metric g has a signature of $(+,-,-,-)$, $g_{\mu\nu}a^\mu a^\nu = a^\nu a_\nu = a^0a^0 - a^1a^1 - a^2a^2 - a^3a^3$. $F_{\rm ex}$ is an arbitrary external field. The field $F_{\rm LAD}$ is the reaction field, which acts on the electron due to light emission. This field is defined by using the retarded field $F_{\rm ret}$ and the advanced field $F_{\rm adv}$:

$$F_{\rm LAD}{}^{\mu\nu}\big|_{x=x(\tau)} = \left.\frac{F_{\rm ret}{}^{\mu\nu} - F_{\rm adv}{}^{\mu\nu}}{2}\right|_{x=x(\tau)} = -\frac{m_0\tau_0}{ec^2}\left(\frac{d^2w^\mu}{d\tau^2}w^\nu - w^\mu\frac{d^2w^\nu}{d\tau^2}\right). \tag{2}$$

The constant τ_0 is $\tau_0 = e^2/6\pi\varepsilon_0 m_0 c^3 = \mathrm{O}(10^{-24})$. Following the considerations above, Dirac arrived at the relativistic force equation, departing from the non-relativistic equation of H. A. Lorentz [2] and M. Abraham [3]:

$$f_{\rm LAD}{}^\mu = -eF_{\rm LAD}{}^{\mu\nu}w_\nu = m_0\tau_0\frac{d^2w^\mu}{d\tau^2} + \frac{m_0\tau_0}{c^2}g_{\alpha\beta}\frac{dw^\alpha}{d\tau}\frac{dw^\beta}{d\tau}w^\mu. \tag{3}$$

This is called the LAD radiation reaction force. J. Schwinger derived the Larmor formula

$$\frac{dW}{dt} = -m_0\tau_0 g_{\alpha\beta}\frac{dw^\alpha}{d\tau}\frac{dw^\beta}{d\tau} = m_0c^2\tau_0\frac{\dot{\boldsymbol{\beta}}^2 - \left(\boldsymbol{\beta}\times\dot{\boldsymbol{\beta}}\right)^2}{\left(1-\boldsymbol{\beta}^2\right)^3} \tag{4}$$

by using this LAD field F_{LAD} [4]. We can find this Larmor formula as a coefficient in Eq. (3). The second term on the RHS of Eq. (3) is the so-called "direct radiation term"; therefore, this LAD equation has been considered the equation of an electron's motion with light emission. Consequently, the LAD equation is a standard model of a radiating electron under ultra-high intense lasers. With the rapid progress of ultra-short pulse laser technology, the maximum intensities reached by these lasers is of the order of 10^{22} W/cm^2 [5,6]. One laser facility that can achieve such ultra-high intensity is LFEX (Laser for Fast Ignition Experiment) at the Institute of Laser Engineering (ILE), Osaka University [7], and even higher intensities will be possible at the next-generation laser facility, proposed by the Extreme Light Infrastructure (ELI) project [8] in Europe. If the laser intensity is higher than 10^{22} W/cm^2, strong bremsstrahlung will occur. Accompanying this, the radiation reaction force (or damping force) can have a strong influence on the charged particle [9]. But the LAD equation has a very significant mathematical problem, as follows. The solution of the LAD equation has an exponential factor. Let f be the vector function, the solution of the LAD equation is

$$\frac{dw^{\mu}}{d\tau}(\tau) = f^{\mu}(\tau) \times \exp\frac{\tau}{\tau_0}. \tag{5}$$

This solution is derived by integration of the LAD equation, but it goes rapidly to infinity, since $\tau_0 = \mathrm{O}(10^{-24})$ is a very small value [10,11]. This run-away depends on the first term in Eq. (3), named the Schott term, and should be avoided in order to solve the equation stably.

For the avoidance of the run-away problem, we have considered in our previous paper [12] a radiating electron dressed by a field:

$$\frac{d}{d\tau}w^{\mu} = -\frac{e}{m_0\left(1-\eta\langle F_{\mathrm{LAD}} \mid F_{\mathrm{LAD}}\rangle\right)}\left(F_{\mathrm{ex}}{}^{\mu\nu} + F_{\mathrm{LAD}}{}^{\mu\nu}\right)w_{\nu}. \tag{6}$$

Here, I shall call the equation above the Seto–Zhang–Koga (SZK) equation. This dressed electron was described by vacuum polarization via the Heisenberg–Euler Lagrangian density [13,14]. The dress stabilizes run-away by changing the coupling constant $e/m_0 \times (1-\eta\langle F_{\mathrm{LAD}} \mid F_{\mathrm{LAD}}\rangle)^{-1}$. However, that model considered only the correction due to the radiation from an electron and the introduction of the external field was artificial (Eq. (24) in Ref. [12]).

To address these points, I introduce a new model of the radiation reaction, which incorporates a smooth installation of the external fields, including the interaction between radiation and external field described in this paper. To achieve this, we first consider, in Sect. 2, a more general equation of motion with the radiation reaction in quantum vacuum. In this phase, we will not investigate a more concrete dynamics of quantum vacuum beyond the Heisenberg–Euler vacuum, but we only assume that the Lagrangian density is a function of $\langle F \mid F\rangle = F_{\alpha\beta}F^{\alpha\beta}$ and $\langle F \mid {}^*F\rangle = F_{\alpha\beta}({}^*F)^{\alpha\beta}$. Next, in Sect. 3, I will proceed to a concrete model by using the lowest-order Heisenberg–Euler Lagrangian density as the model of quantum vacuum. I will present the stability of the new equation via analysis and numerical calculations. Finally, this will lead to an anisotropic correction for the charge-to-mass ratio by R. Fletcher and H. Millikan [15,16].

2. Derivation of a new radiation reaction model

The Heisenberg–Euler Lagrangian density includes the correction due to the dynamics of the quantum vacuum. However, this is only suitable for constant fields. In this section, let us consider the general Lagrangian density for quantum vacuum without a concrete definition, described by $\langle F \mid F\rangle$

and $\langle F \mid {}^*F\rangle$ like the Heisenberg–Euler Lagrangian density. Here, F is the electromagnetic tensor and ${}^*F$ is the dual tensor of F. Now, the Lagrangian density for propagating photons is,

$$L\left(\langle F \mid F\rangle, \langle F \mid {}^*F\rangle\right) = -\frac{1}{4\mu_0}\langle F \mid F\rangle + L_{\text{Quantum Vacuum}}\left(\langle F \mid F\rangle, \langle F \mid {}^*F\rangle\right). \tag{7}$$

Of course, this Lagrangian density $L_{\text{Quantum Vacuum}}$ needs to converge to the Heisenberg–Euler Lagrangian density when the field F is a constant field. For instance, we assume that L and $L_{\text{Quantum Vacuum}}$ are functions of C^∞. From this, the Maxwell equation is derived as follows:

$$\partial_\mu\left[F^{\mu\nu} - \eta f \times F^{\mu\nu} - \eta g \times {}^*F^{\mu\nu}\right] = 0 \tag{8}$$

$$\eta f\left(\langle F \mid F\rangle, \langle F \mid {}^*F\rangle\right) = 4\mu_0 \frac{\partial L_{\text{Quantum Vacuum}}}{\partial\langle F \mid F\rangle} \tag{9}$$

$$\eta g\left(\langle F \mid F\rangle, \langle F \mid {}^*F\rangle\right) = 4\mu_0 \frac{\partial L_{\text{Quantum Vacuum}}}{\partial\langle F \mid F\rangle}. \tag{10}$$

In these equations, $\eta = 4\alpha^2\hbar^3\varepsilon_0/45m_0^4c^3$. The field

$$\frac{1}{c\varepsilon_0}M^{\mu\nu} = -\eta f \times F^{\mu\nu} - \eta g \times {}^*F^{\mu\nu} \tag{11}$$

represents the vacuum "polarization"; therefore, $F - \eta f \times F - \eta g \times {}^*F$ refers to the dressed field set of $(\mathbf{D}, \mathbf{H})$. In addition, the following is satisfied: $\partial_\mu(F_{\text{ex}}{}^{\mu\nu} + F_{\text{LAD}}{}^{\mu\nu}) = 0$. Thus, Eq. (8) suggests a connection between $F - \eta f \times F - \eta g \times {}^*F$ and $(\mathbf{D}, \mathbf{H}) = F_{\text{ex}} + F_{\text{LAD}}$ with continuity and smoothness with C^∞ on all points in the Minkowski space-time. At a point far from an electron, the external fields are given and radiation can be observed (Fig. 1). At this point, Eq. (8) becomes

$$\boxed{F^{\mu\nu} - \eta f \times F^{\mu\nu} - \eta g \times {}^*F^{\mu\nu} = \mathfrak{F}^{\mu\nu}}. \tag{12}$$

Here, it is considered that $\mathfrak{F} = F_{\text{ex}} + F_{\text{LAD}}$. In our previous model [12], we assumed

$$F^{\mu\nu} - \eta f \times F^{\mu\nu} - \eta g \times {}^*F^{\mu\nu} = F_{\text{LAD}}{}^{\mu\nu}. \tag{13}$$

Therefore, we did not consider the correction of the external field, while here the external field can be naturally included. This is the most important difference between the new model and the old one.

By using the field's continuity and smoothness, Eq. (12) can be applied not only to points far from an electron, but also at the electron point itself. Our interest is in the bare (undressed) field $F = (\mathbf{E}, \mathbf{B})$ at the point of an electron for defining the electromagnetic force $-eF^{\mu\nu}w_\nu$. We consider the description of the tensor F from Eq. (12) as the way to obtain the solution:

$$\mathfrak{L}^{\mu\nu\alpha\beta}F_{\alpha\beta} = \mathfrak{F}^{\mu\nu} \tag{14}$$

$$\mathfrak{L}^{\mu\nu\alpha\beta} = (1 - \eta f)g^{\mu\alpha}g^{\nu\beta} - \eta g \times \frac{1}{2!}\varepsilon^{\mu\nu\alpha\beta}. \tag{15}$$

Here, $\mathfrak{L}$ is the permittivity tensor in Minkowski space-time. We then define a new tensor:

$$\begin{aligned}\bar{\mathfrak{K}}_{\rho\sigma\mu\nu} &= \frac{(1-\eta f)g_{\rho\mu}g_{\sigma\nu} + \eta g \times \frac{1}{2!}\varepsilon_{\rho\sigma\mu\nu}}{(1-\eta f)^2 + (\eta g)^2} \\ &= \frac{1}{1-\eta f} \times \frac{1}{1 + \frac{(\eta g)^2}{(1-\eta f)^2}}\left(g_{\rho\mu}g_{\sigma\nu} + \frac{\eta g}{1-\eta f} \times \frac{1}{2!}\varepsilon_{\rho\sigma\mu\nu}\right).\end{aligned} \tag{16}$$

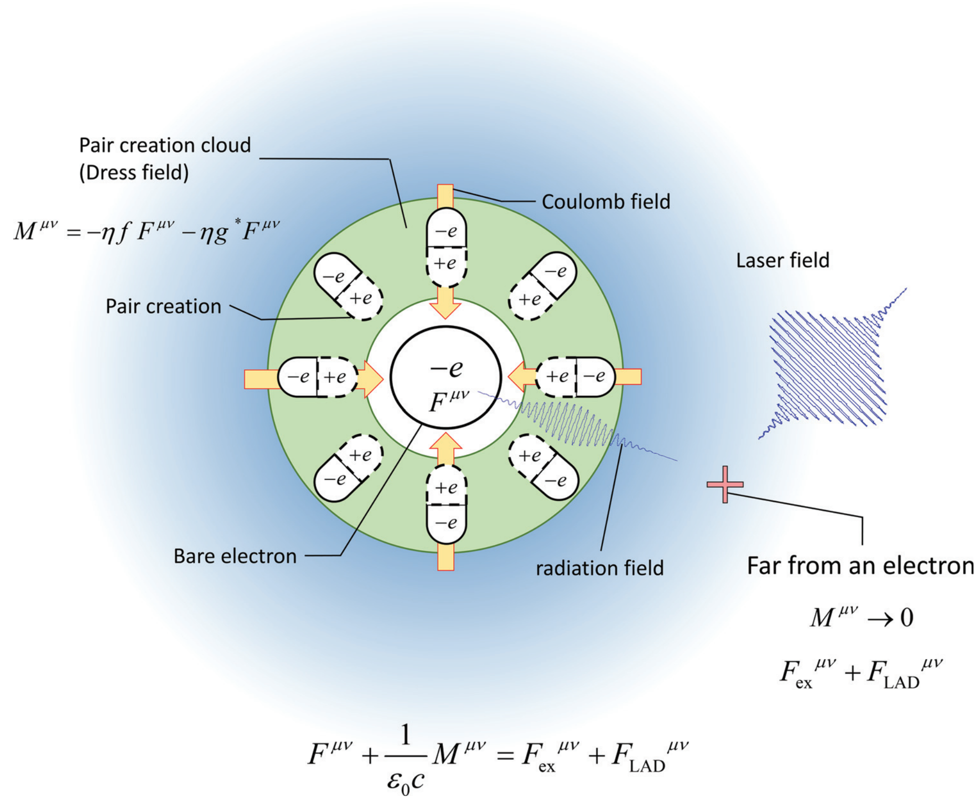

Fig. 1. The bare field and the dressed field.

From the relation in which $\varepsilon_{\rho\sigma\mu\nu}\varepsilon^{\mu\nu\alpha\beta} = -2\left(\delta^\alpha_\rho\delta^\beta_\sigma - \delta^\beta_\rho\delta^\alpha_\sigma\right)$ and considering the antisymmetry of F, it follows that $\bar{\mathfrak{K}}_{\rho\sigma\mu\nu}\mathfrak{L}^{\mu\nu\alpha\beta}F_{\alpha\beta} = F_{\rho\sigma}$. Therefore, the field F becomes

$$F^{\mu\nu} = \bar{\mathfrak{K}}^{\mu\nu\rho\sigma}\mathfrak{F}_{\rho\sigma} = \frac{1}{1-\eta f} \times \frac{1}{1+\frac{(\eta g)^2}{(1-\eta f)^2}}\left[\mathfrak{F}^{\mu\nu} + \frac{\eta g}{1-\eta f} \times {}^{*}\mathfrak{F}^{\mu\nu}\right]. \tag{17}$$

Since the form of the equation of motion is

$$m_0\frac{dw^\mu}{d\tau} = -eF^{\mu\nu}w_\nu, \tag{18}$$

by substitution of Eq. (17) into Eq. (18), we obtain

$$m_0(1-\eta f)\left[1+\frac{(\eta g)^2}{(1-\eta f)^2}\right]\frac{dw^\mu}{d\tau} = -e\mathfrak{F}^{\mu\nu}w_\nu - e\frac{\eta g}{1-\eta f}{}^{*}\mathfrak{F}^{\mu\nu}w_\nu$$
$$\Rightarrow m_0(1-\eta f_0)\frac{dw^\mu}{d\tau} = -e\mathfrak{F}^{\mu\nu}w_\nu - e\eta g_0\,{}^{*}\mathfrak{F}^{\mu\nu}w_\nu + \mathrm{O}\big(\eta^2\big). \tag{19}$$

Here, I have used Taylor's expansion of f and g near $\eta = 0$. Paying attention to the relation $F|_{\eta=0} = \mathfrak{F}$ and denoting that $f_0 = f(\langle\mathfrak{F} \mid \mathfrak{F}\rangle, \langle\mathfrak{F} \mid {}^{*}\mathfrak{F}\rangle)$ and $g_0 = g(\langle\mathfrak{F} \mid \mathfrak{F}\rangle, \langle\mathfrak{F} \mid {}^{*}\mathfrak{F}\rangle)$ for the simplification

$$f = f_0 + \eta\delta f + \mathrm{O}\big(\eta^2\big), \tag{20}$$
$$g = g_0 + \eta\delta g + \mathrm{O}\big(\eta^2\big), \tag{21}$$

by treating in the first order the quantum vacuum

$$m_0 \frac{dw^\mu}{d\tau} = -e \frac{\mathfrak{F}^{\mu\nu} + \eta g_0 \, {}^*\mathfrak{F}^{\mu\nu}}{1 - \eta f_0} w_\nu, \tag{22}$$

where, introducing the new tensor $\mathfrak{K}$ defined in Eq. (16),

$$\mathfrak{K}^{\mu\nu\alpha\beta} = \frac{g^{\mu\alpha} g^{\nu\beta} + \eta g_0 \times \frac{1}{2!} \varepsilon^{\mu\nu\alpha\beta}}{1 - \eta f_0}, \tag{23}$$

the field is modified as

$$F^{\mu\nu} = \mathfrak{K}^{\mu\nu\alpha\beta} \mathfrak{F}_{\alpha\beta}. \tag{24}$$

Finally, we need to pay attention to the fact that Eq. (24) is already included the radiation reaction field and quantum vacuum effects via the definition of Eq. (12). We can rewrite Eq. (22):

$$\boxed{m_0 \frac{dw^\mu}{d\tau} = -e \mathfrak{K}^{\mu\nu\alpha\beta} \mathfrak{F}_{\alpha\beta} \, w_\nu}. \tag{25}$$

This is the general formula of the radiation reaction in quantum vacuum. The limit of $\hbar \to 0$ leads to a smooth connection to the LAD equation, since $\eta = 4\alpha^2 \hbar^3 \varepsilon_0 / 45 m_0^4 c^3$ and $\mathfrak{K}^{\mu\nu\alpha\beta} \to g^{\mu\alpha} g^{\nu\beta}$.

3. First-order Heisenberg–Euler quantum vacuum

3.1. *Equation of motion*

In Sect. 2, the quantum vacuum was assumed to be a function of $\langle F|F\rangle$ and $\langle F|{}^*F\rangle$ without concrete formulations. The Heisenberg–Euler Lagrangian density expresses the dynamics of quantum vacuum, but can only be applied for constant fields. However, its lowest order should be contained in $L_{\text{Quantum Vacuum}}$ [12]. Therefore, in this section, I assume that,

$$L_{\text{Quantum Vacuum}} = L_{\substack{\text{The lowest order of}\\ \text{Heisenberg-Euler}}} = \frac{\alpha^2 \hbar^3 \varepsilon_0^2}{360 m_0^4 c} \left[4\langle F \mid F\rangle^2 + 7\langle F \mid {}^*F\rangle^2 \right]. \tag{26}$$

In this case, instead of Eq. (12), we write,

$$F^{\mu\nu} - \eta \langle F \mid F\rangle \times F^{\mu\nu} - \frac{7}{4} \eta \langle F \mid {}^*F\rangle \times {}^*F^{\mu\nu} = \mathfrak{F}^{\mu\nu}, \tag{27}$$

and, by using perturbations, f_0 and g_0 are

$$f_0 = \langle \mathfrak{F} \mid \mathfrak{F}\rangle = \langle F_{\text{LAD}} \mid F_{\text{LAD}}\rangle + 2\langle F_{\text{LAD}} \mid F_{\text{ex}}\rangle \tag{28}$$

$$g_0 = \frac{7}{4} \langle \mathfrak{F} \mid {}^*\mathfrak{F}\rangle = \frac{7}{2} \langle F_{\text{LAD}} \mid {}^*F_{\text{ex}}\rangle. \tag{29}$$

Here, I have used the relation that $\partial_\mu F_{\text{ex}}{}^{\mu\nu} = 0 \Rightarrow \langle F_{\text{ex}} \mid F_{\text{ex}}\rangle = 0$, $\langle F_{\text{ex}} \mid {}^*F_{\text{ex}}\rangle = 0$, and $\langle F_{\text{LAD}} \mid {}^*F_{\text{LAD}}\rangle \equiv 0$ [12]. Equation (24) becomes

$$F^{\mu\nu} = \mathfrak{K}^{\mu\nu\alpha\beta} \mathfrak{F}_{\alpha\beta} = \frac{1}{1 - \eta f_0} \mathfrak{F}^{\mu\nu} + \frac{\eta g_0}{1 - \eta f_0} {}^*\mathfrak{F}^{\mu\nu}. \tag{30}$$

When $1 - \eta f_0 = 0$, the field F becomes infinity and run-away occurs. It is required that $1 - \eta f_0 > 0$ for application. From the relations $\langle F_{\text{LAD}} \mid F_{\text{LAD}}\rangle = 2/e^2 c^2 \times g_{\mu\nu} f_{\text{LAD}}{}^\mu f_{\text{LAD}}{}^\nu = -2(m_0 \tau_0 / ec)^2 \ddot{\mathbf{v}}^2|_{\text{rest}} \leq 0$ and $\langle F_{\text{LAD}} \mid F_{\text{ex}}\rangle = 2 m_0 \tau_0 / ec^2 \times \ddot{\mathbf{v}} \cdot \mathbf{E}_{\text{ex}}|_{\text{rest}}$ in an electron's rest frame,

$$1 - \eta f_0 = \frac{2\eta}{ec} \times (m_0 \tau_0 \ddot{\mathbf{v}}|_{\text{rest}} - e\mathbf{E}_{\text{ex}}|_{\text{rest}})^2 + 1 - \frac{2\eta \mathbf{E}_{\text{ex}}{}^2|_{\text{rest}}}{c^2} > 1 - \frac{2\eta \mathbf{E}_{\text{ex}}{}^2|_{\text{rest}}}{c^2} \overset{\substack{\text{physical}\\ \text{requirements}}}{>} 0. \tag{31}$$

The stability only depends on the external field in the rest frame of an electron. By using the Schwinger limit field $E_{\text{Schwinger}} = m_0^2 c^3 / e\hbar$, it follows that $1 - \eta f_0 > 1 - (5.2 \times 10^{-5}) \times$

$(\mathbf{E}_{\text{ex}}|_{\text{rest}}/E_{\text{Schwinger}})^2$. The field $\mathbf{E}_{\text{ex}}|_{\text{rest}}$ should be treated below the Schwinger limit; therefore, $|\mathbf{E}_{\text{ex}}| \ll E_{\text{Schwinger}}$ is normally satisfied. Therefore, we require choices that satisfy Eq. (31) for $1 - \eta f_0 > 0$. Now, the stability depends on $\langle F_{\text{LAD}} \mid {}^*F_{\text{ex}}\rangle = 2m_0\tau_0/ec \times \ddot{\mathbf{v}} \cdot \mathbf{B}_{\text{ex}}|_{\text{rest}}$, or g_0 is not demonstrated. If the external fields are absent, this field converges to our previous model [12]:

$$F^{\mu\nu}\big|_{F_{\text{ex}}=0} = \frac{1}{1 - \eta\langle F_{\text{LAD}} \mid F_{\text{LAD}}\rangle} F_{\text{LAD}}^{\mu\nu}. \tag{32}$$

Therefore, this new model is a generalization of the previous one. It can adopt quantum vacuum not only via the radiation reaction, but also via external fields, such as those produced by lasers. The equation of motion is

$$\frac{dw^\mu}{d\tau} = -\frac{e}{m_0(1 - \eta\langle \mathfrak{F} \mid \mathfrak{F}\rangle)} \left(\mathfrak{F}^{\mu\nu} w_\nu + \frac{7}{4}\eta \langle \mathfrak{F} \mid {}^*\mathfrak{F}\rangle\, {}^*\mathfrak{F}^{\mu\nu} w_\nu \right). \tag{33}$$

3.2. Run-away avoidance

My previous model could avoid run-away (the effect of self-acceleration) [12]. In this section, I will show that this new equation can also avoid run-away by using a two-stage analysis. The first is the investigation of the radiation upper limit and the second is the asymptotic analysis proposed by F. Röhrlich [17]. The physical meaning of run-away is a time-continuous infinite light emission via stimulations by an electron's self-radiation. In other words, when we can limit the value of the radiation, we can say that the model avoids run-away. To check the stability of this equation, we consider the equation as follows, derived from Eq. (33):

$$g_{\mu\nu}\frac{dw^\mu}{d\tau}\frac{dw^\nu}{d\tau} = \frac{1}{m_0^2}\frac{\frac{e^2c^2}{2\eta}\eta f_0 + \frac{2}{7\eta}e^2c^2(\eta g_0)^2}{(1-\eta f_0)^2} + \frac{1}{m_0^2}\frac{g_{\mu\nu}\left[f_{\text{ex}}^\mu + \eta g_0({}^*f_{\text{ex}})^\mu\right]\left[f_{\text{ex}}^\nu + \eta g_0({}^*f_{\text{ex}})^\nu\right]}{(1-\eta f_0)^2}. \tag{34}$$

Here, I have defined the forces $f_{\text{ex}}{}^\mu = -eF_{\text{ex}}{}^{\mu\nu}w_\nu$ and ${}^*f_{\text{ex}}{}^\mu = -e\,({}^*F_{\text{ex}})^{\mu\nu}w_\nu$. In the rest frame, $f_0 = -2(m_0\tau_0/ec)^2\,\ddot{\mathbf{v}}^2|_{\text{rest}} + 4m_0\tau_0/ec^2 \times \ddot{\mathbf{v}} \cdot \mathbf{E}_{\text{ex}}|_{\text{rest}} = \mathrm{O}(\ddot{\mathbf{v}}_{\text{rest}}^2)$ and $g_0 = 7m_0\tau_0/ec \times \ddot{\mathbf{v}} \cdot \mathbf{B}_{\text{ex}}|_{\text{rest}} = \mathrm{O}(\ddot{\mathbf{v}}_{\text{rest}})$ are satisfied. When we face the run-away solution, then $|\ddot{\mathbf{v}}_{\text{rest}}| \to \infty$. Therefore, $\mathrm{O}(|g_0|) < \mathrm{O}(|f_0|)$ in the run-away case. Under the condition of Eq. (31),

$$\begin{aligned}
&\left| g_{\mu\nu}\frac{dw^\mu}{d\tau}\frac{dw^\nu}{d\tau} \right| \\
&\leq \frac{1}{m_0^2}\frac{e^2c^2}{2\eta}\frac{|\eta f_0|}{|1-\eta f_0|^2} + \frac{1}{m_0^2}\frac{2e^2c^2}{7\eta}\frac{|\eta g_0|^2}{|1-\eta f_0|^2} \\
&\quad + \frac{|g_{\mu\nu} f_{\text{ex}}{}^\mu f_{\text{ex}}{}^\nu|}{m_0{}^2}\frac{1}{|1-\eta f_0|^2} + 2\frac{|g_{\mu\nu} f_{\text{ex}}{}^\mu ({}^*f_{\text{ex}})^\nu|}{m_0{}^2}\frac{|\eta g_0|}{|1-\eta f_0|^2} + \frac{|g_{\mu\nu} ({}^*f_{\text{ex}})^\mu ({}^*f_{\text{ex}})^\nu|}{m_0{}^2}\frac{|\eta g_0|^2}{|1-\eta f_0|^2} \\
&\overset{\text{run-away}}{<} \frac{1}{m_0{}^2}\frac{e^2c^2}{2\eta}\frac{|\eta f_0|}{|1-\eta f_0|^2} + \frac{1}{m_0{}^2}\frac{2e^2c^2}{7\eta}\frac{|\eta f_0|^2}{|1-\eta f_0|^2} \\
&\quad + \frac{|g_{\mu\nu} f_{\text{ex}}{}^\mu f_{\text{ex}}{}^\nu|}{m_0{}^2}\frac{1}{|1-\eta f_0|^2} + 2\frac{|g_{\mu\nu} f_{\text{ex}}{}^\mu ({}^*f_{\text{ex}})^\nu|}{m_0{}^2}\frac{|\eta f_0|}{|1-\eta f_0|^2} + \frac{|g_{\mu\nu} ({}^*f_{\text{ex}})^\mu ({}^*f_{\text{ex}})^\nu|}{m_0{}^2}\frac{|\eta f_0|^2}{|1-\eta f_0|^2} \\
&< \infty,
\end{aligned} \tag{35}$$

since the functions $1/|1-x|^2$, $|x|/|1-x|^2$, and $|x|^2/|1-x|^2$ are finite in the domain $x \in (-\infty, 1)$. Now, $x = \eta f_0 \leq 2\eta \mathbf{E}_{\text{ex}}{}^2|_{\text{rest}}/c^2 < 1$ from Eq. (31). When we are in the case of run-away, $dw/d\tau$ also becomes infinite because it is the integral of $d^2w/d\tau^2$, and then $|g_{\mu\nu}(dw^\mu/d\tau)(dw^\nu/d\tau)| \to \infty$.

But this conflicts with Eq. (35). Therefore, the Larmor formula becomes

$$\frac{dW}{dt} = -m_0\tau_0 g_{\mu\nu}\frac{dw^\mu}{d\tau}\frac{dw^\nu}{d\tau} < \infty \tag{36}$$

for the whole time domain and run-away does not appear. Under the external field condition in Eq. (31), solving Eq. (33),

$$\frac{dw^\mu}{d\tau}(\tau) = \frac{e^{\frac{\tau}{\tau_0}}}{m_0\tau_0}\int_\tau^\infty d\tau'\left[f_{\mathrm{ex}}^\mu + \eta g_0\left({}^*f_{\mathrm{ex}}\right)^\mu + \frac{m_0\tau_0}{c^2}g_{\alpha\beta}\frac{dw^\alpha}{d\tau}\frac{dw^\beta}{d\tau}w^\mu\right]\times e^{-\frac{\tau'}{\tau_0}}\times e^{\int_\tau^{\tau'}\frac{d\tau''}{\tau_0}\eta f_0}. \tag{37}$$

If we choose $\eta = 0$, this solution becomes Eq. (I) in Röhrlich's article [17]. He derived the "asymptotic" boundary condition in $\tau \to \infty$ by using l'Hôpital's rule. This is a method based on our normal perception, "when ${f_{\mathrm{ex}}}^\mu$ vanishes in $\tau \to \infty$, $dw^\mu/d\tau(\infty)$ also vanishes". He suggested that, when run-away exists, $dw^\mu/d\tau$ is not zero because of the self-stimulation by radiation. Therefore, $dw^\mu/d\tau(\infty) = 0$ is required for the model stability. We apply this to Eq. (37), and, from l'Hôpital's rule, it becomes

$$m_0\frac{dw^\mu}{d\tau}(\infty) = {f_{\mathrm{ex}}}^\mu(\infty) + \eta g_0({}^*f_{\mathrm{ex}})^\mu(\infty) + \frac{m_0\tau_0}{c^2}g_{\alpha\beta}\frac{dw^\alpha}{d\tau}\frac{dw^\beta}{d\tau}w^\mu(\infty). \tag{38}$$

Here, I have used the signature of the limit by Röhrlich. When the given $f_{\mathrm{ex}}(\infty)$ and ${}^*f_{\mathrm{ex}}(\infty)$ become zero by following Röhrlich's method, then $m_0 dw^\mu/d\tau(\infty) = m_0\tau_0/c^2 \times g_{\alpha\beta}(dw^\alpha/d\tau)(dw^\beta/d\tau)\, w^\mu(\infty)$. We know only that the energy loss by radiation is finite, from Eq. (36). The square of this equation is

$$\frac{m_0}{\tau_0}\times\frac{m_0\tau_0}{c^2}g_{\alpha\beta}\frac{dw^\alpha}{d\tau}\frac{dw^\beta}{d\tau}(\infty) = \left(\frac{m_0\tau_0}{c^2}g_{\alpha\beta}\frac{dw^\alpha}{d\tau}\frac{dw^\beta}{d\tau}\right)^2(\infty). \tag{39}$$

Its solution is $m_0\tau_0/c^2 \times g_{\alpha\beta}(dw^\alpha/d\tau)(dw^\beta/d\tau)(\infty) = 0$, since $g_{\alpha\beta}(dw^\alpha/d\tau)(dw^\beta/d\tau) \le 0$. Therefore,

$$\lim_{\tau\to 0}\frac{dw^\mu}{d\tau} = 0. \tag{40}$$

Therefore, my equation (33) can satisfy Röhrlich's stability condition. The stability of Eq. (33) has been demonstrated in a two-stage analysis.

3.3. Calculations

As the final part of this section, I will present numerical calculation results showing the behavior of each model in a laser–electron interaction. The models are Eq. (33), the SZK equation Eq. (6), and the Landau–Lifshitz (LL) equation, which is the main method applied for simulations. The form of the LL equation is as follows [18]:

$$m_0\frac{dw^\mu}{d\tau} = -e{F_{\mathrm{ex}}}^{\mu\nu}w_\nu - \frac{e\tau_0}{m_0}g_{\theta\lambda}{F_{\mathrm{ex}}}^{\mu\theta}{f_{\mathrm{ex}}}^\lambda - e\tau_0\frac{d{F_{\mathrm{ex}}}^{\mu\theta}}{d\tau}w_\theta + \frac{\tau_0}{m_0c^2}g_{\theta\lambda}{f_{\mathrm{ex}}}^\theta{f_{\mathrm{ex}}}^\lambda w^\mu. \tag{41}$$

I chose the parameters of the Extreme Light Infrastructure—Nuclear Physics (ELI-NP) for calculations [19,20]. The characteristic point of Eq. (33) is the term $-e\,(\eta g_0)c\mathbf{B}_{\mathrm{ex}}$, which is derived from $-e\eta g_0\,({}^*F_{\mathrm{ex}})^{\mu\nu}w_\nu$. Therefore, we need to consider the condition in which an electron feels this force strongly. It will be the electron injection along the $\mathbf{B}_{\mathrm{ex}}$ field (Fig. 2).

The peak intensity of the laser is $1\times 10^{22}\,\mathrm{W/cm^2}$ in a Gaussian-shaped plane-wave like Eqs. (28,29) in Ref. [12]. The pulse width is 22 fsec and the laser wavelength is 0.82 μm. The electric field is set in the y direction; the magnetic field is in the z direction. The electron travels in the

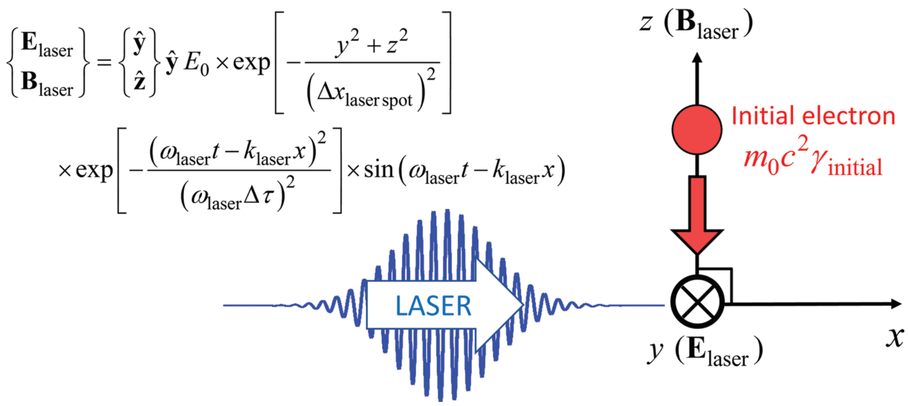

Fig. 2. Setup of laser–electron "90 degree collision". The laser propagates along the x axis. An electron travels in the negative z direction, which is the direction of the $\mathbf{B}_{\text{laser}}$ field.

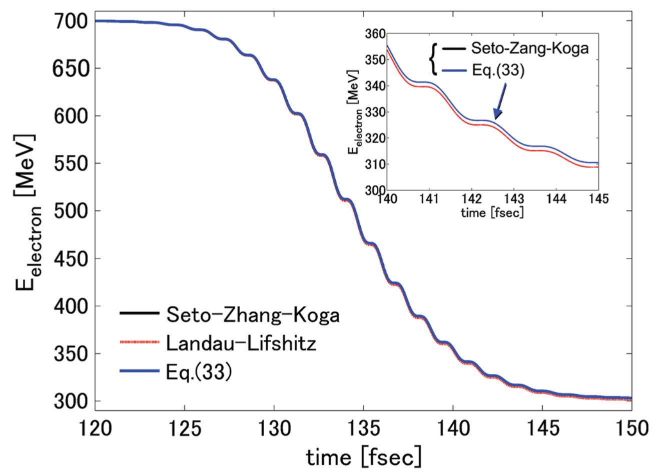

Fig. 3. The energy of the electron. All models converged. The final electron's energies are, Seto–Zhang–Koga: 302.8 MeV, the Landau–Lifshitz: 301.1 MeV and Eq. (33): 302.8 MeV. The inset is a close-up of the figure.

negative z direction, with an initial energy of 700 MeV. The numerical calculations were carried out by using the equations in the laboratory frame.

The radiation reaction appears directly in the time evolution of the electron's energy, as shown in Fig. 3. The energy drop refers to the radiation energy loss of the electron. This figure is most suited to understanding the behavior of the radiation reaction. We can say that the solutions are very similar to very high accuracy. In particular, Eq. (33) and the SZK equation overlap completely. Therefore, they cannot be distinguished in this figure and from the final energy of the electron. The final energy for Eq. (33) and the SZK equation is 302.8 MeV and that for the LL equation is 301.1 MeV, the energy difference being O(1 MeV). An explanation of the convergence between the SZK and the LL equations can be found in Ref. [12]. Therefore, I will present the convergence between Eq. (33) and the SZK equation.

The key parameters are ηf_0 and ηg_0; their plots are shown in Fig. 4. From these figures, we find that they are of the order of $\eta f_0 = \mathrm{O}(10^{-8})$ and $\eta g_0 = \mathrm{O}(10^{-10})$. I introduced the term $-e\eta g_0\,({}^*F_{\text{ex}})^{\mu\nu}w_\nu$ as a feature of Eq. (33). In the rest frame, this term becomes $|-e\,(\eta g_0)c\mathbf{B}_{\text{ex}}|_{\text{rest}} \sim 10^{-10}\times|-e\mathbf{E}_{\text{ex}}|_{\text{rest}}$. Thus, this new term is rounded into the external field like $-e\mathfrak{F}^{\mu\nu}w_\nu - e\eta g_0({}^*\mathfrak{F})^{\mu\nu}w_\nu \sim -e\mathfrak{F}^{\mu\nu}w_\nu$. For these reasons, Eq. (33) transforms to the SZK equation (6) as

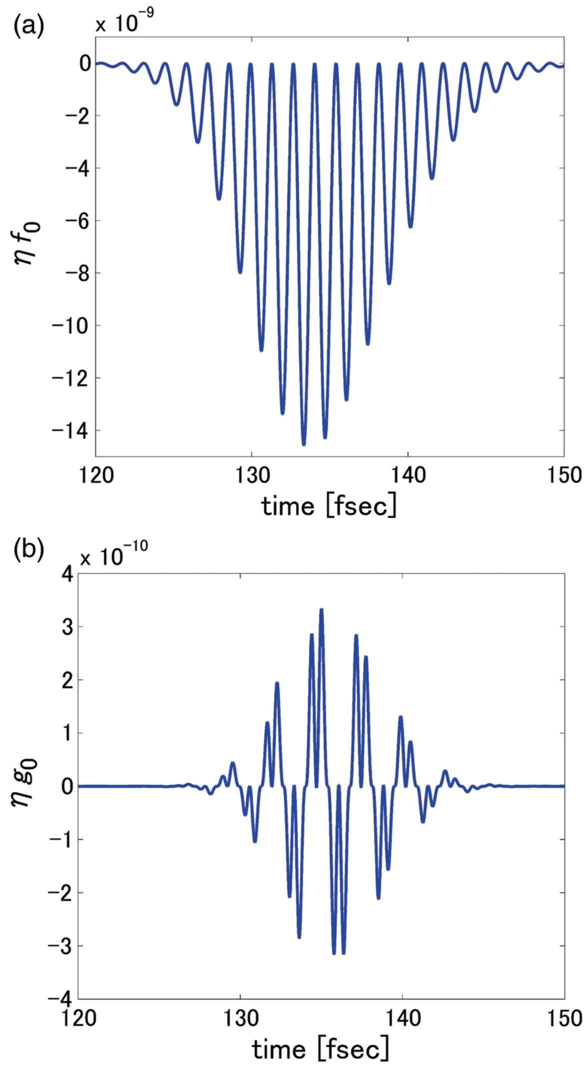

Fig. 4. Time evolution of factors: (a) ηf_0 and (b) ηg_0.

follows:

$$m_0 \frac{dw^\mu}{d\tau} = -\frac{e}{1-\eta f_0}[\mathfrak{F}^{\mu\nu} w_\nu + \eta g_0 ({}^*\mathfrak{F})^{\mu\nu} w_\nu]$$
$$= -\frac{e}{1-\eta \langle F_{\mathrm{LAD}} \mid F_{\mathrm{LAD}} \rangle} \mathfrak{F}^{\mu\nu} w_\nu + \mathrm{O}\left(\eta \langle F_{\mathrm{LAD}} \mid F_{\mathrm{ex}} \rangle, \eta \langle F_{\mathrm{LAD}} \mid {}^*F_{\mathrm{ex}} \rangle\right). \quad (42)$$

Here, $\mathrm{O}(\eta \langle F_{\mathrm{LAD}} \mid F_{\mathrm{ex}} \rangle) = \mathrm{O}(\eta \langle F_{\mathrm{LAD}} \mid {}^*F_{\mathrm{ex}} \rangle) = \mathrm{O}(\eta g_0)$ since the external field satisfies $\langle F_{\mathrm{ex}} \mid F_{\mathrm{ex}} \rangle = 0$ and $\langle F_{\mathrm{ex}} \mid {}^*F_{\mathrm{ex}} \rangle = 0$. My new equation (33) as an extension from our previous SZK equation has good properties for numerical calculations. We can say that the method using the LL equation, which is the first-order perturbation of the LAD, is nearly equal to the suppression due to the effects of quantum vacuum.

4. Conclusion

In summary, I have updated our previous equation of motion with the radiation reaction in quantum vacuum. The idea of the derivation of the new equation is the same as in our previous paper [12]; however, the biggest difference is the introduction of the external field effects by the following

replacement (Eqs. (12–13)):

$$F^{\mu\nu} - \eta f \times F^{\mu\nu} - \eta g \times {}^*F^{\mu\nu} = F_{\mathrm{LAD}}{}^{\mu\nu}$$
$$\Rightarrow F^{\mu\nu} - \eta f \times F^{\mu\nu} - \eta g \times {}^*F^{\mu\nu} = F_{\mathrm{ex}}{}^{\mu\nu} + F_{\mathrm{LAD}}{}^{\mu\nu}. \tag{43}$$

Via this replacement, the new model includes the interaction between radiation and the external field. Now we rewrite Eq. (25) as

$$\boxed{\frac{dw^\mu}{d\tau} = -\frac{e}{m_0}\mathfrak{K}^{\mu\nu\alpha\beta}\mathfrak{F}_{\alpha\beta}\, w_\nu} \tag{44}$$

or

$$\frac{dw^\mu}{d\tau} = -\frac{e}{m_0(1-\eta f_0)}\left(\mathfrak{F}^{\mu\nu} + \eta g_0 \,{}^*\mathfrak{F}^{\mu\nu}\right) w_\nu. \tag{45}$$

This equation is the main result of this paper. In my theoretical analysis, I was able to achieve the avoidance of run-away in the Heisenberg–Euler vacuum under Eq. (33), $1 - 2\eta \mathbf{E}_{\mathrm{ex}}^2|_{\mathrm{rest}}/c^2 > 0$. From the results of the numerical calculation, I showed that Eq. (45) agrees well with the LL equation (41). It follows that the first-order perturbation of the LAD equation is nearly equivalent to the run-away suppression by quantum vacuum. Focusing on the tensor $e/m_0 \times \mathfrak{K}$, it is a generalization of our previous charge-to-mass ratio [12]:

$$\frac{Q}{M} = \frac{e}{m_0(1-\eta\langle F_{\mathrm{LAD}} \mid F_{\mathrm{LAD}}\rangle)} = \frac{e}{m_0} + \frac{\delta e}{m_0} \in \mathbb{R}. \tag{46}$$

The charge–mass particle system is built on measure theory. Now the mass measure is denoted by $\mathfrak{m}$ and the general charge measure including anisotropy is defined as the tensor function $\mathfrak{E}^{\rho\sigma\mu\nu}$ in Minkowski space-time. The equation of motion should then be

$$\boxed{d\mathfrak{m}(x)\frac{dw^\mu}{d\tau} = -d\mathfrak{E}^{\mu\nu\alpha\beta}(x)\mathfrak{F}_{\alpha\beta}\, w_\nu}. \tag{47}$$

Since I considered a classical point particle, it will be based on the Dirac measure; however, I will not consider the concrete form of the measures because of the missing information on how mass and charge themselves are described. Nevertheless, the relation between $d\mathfrak{m}(x)$ and $d\mathfrak{E}^{\mu\nu\alpha\beta}(x)$ is very important. The measure can be connected to others via derivatives such as $d\mathfrak{E}^{\mu\nu\alpha\beta} = (d\mathfrak{E}^{\mu\nu\alpha\beta}/d\mathfrak{m})\, d\mathfrak{m}$. This $d\mathfrak{E}^{\mu\nu\alpha\beta}/d\mathfrak{m}$ is called the Radon–Nikodym derivative [21]:

$$d\mathfrak{m}(x)\left(\frac{dw^\mu}{d\tau} + \frac{d\mathfrak{E}^{\mu\nu\alpha\beta}}{d\mathfrak{m}}\mathfrak{F}_{\alpha\beta}\, w_\nu\right) = 0 \Rightarrow \frac{dw^\mu}{d\tau} + \frac{d\mathfrak{E}^{\mu\nu\alpha\beta}}{d\mathfrak{m}}\mathfrak{F}_{\alpha\beta}\, w_\nu = 0. \tag{48}$$

This equation must become equivalent to Eq. (44). Therefore, the Radon–Nikodym derivative becomes

$$\boxed{\frac{d\mathfrak{E}^{\mu\nu\alpha\beta}}{d\mathfrak{m}} = \frac{e}{m_0}\mathfrak{K}^{\mu\nu\alpha\beta} = \frac{e}{m_0}\frac{g^{\mu\alpha}g^{\nu\beta} + \eta g_0 \times \frac{1}{2!}\varepsilon^{\mu\nu\alpha\beta}}{1-\eta f_0}}. \tag{49}$$

This is a generalization of the charge-to-mass ratio by Fletcher and Millikan [15,16] including the anisotropy of quantum vacuum.

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