Dipole splitting algorithm: A practical algorithm to use the dipole subtraction procedure

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Received August 28, 2015; Accepted October 7, 2015; Published November 25, 2015

The Catani–Seymour dipole subtraction is a general and powerful procedure to calculate the QCD next-to-leading order corrections for collider observables. We clearly define a practical algorithm to use the dipole subtraction. The algorithm is called the dipole splitting algorithm (DSA). The DSA is applied to an arbitrary process by following well defined steps. The subtraction terms created by the DSA can be summarized in a compact form by tables. We present a template for the summary tables. One advantage of the DSA is to allow a straightforward algorithm to prove the consistency relation of all the subtraction terms. The proof algorithm is presented in the following paper [K. Hasegawa, arXiv:1409.4174]. We demonstrate the DSA in two collider processes, \( pp \rightarrow \mu^- \mu^+ \) and 2 jets. Further, as a confirmation of the DSA, it is shown that the analytical results obtained by the DSA in the Drell–Yan process exactly agree with the well known results obtained by the traditional method.

1. Introduction

The operating CERN Large Hadron Collider (LHC) discovered a new boson whose mass is around 125 GeV in 2012. The new boson is identified as the Higgs boson in the standard model. In order to specify the property of the field and the interactions precisely, we need more results from the LHC experiments and the various examinations of the results must be carried by comparing them to the theoretical predictions. In the present article, we study the theoretical prediction for an arbitrary process at a hadron collider like LHC. The calculation for the prediction consists of two ingredients, the parton distribution function (PDF) and the partonic cross section. The PDF is a process-independent quantity and is determined as a numerical function from the experimental data. Recent reviews on the PDF can be found in Refs. [1–4]. The partonic cross section is calculated by the perturbative expansion of the strong coupling constants of quantum chromodynamics (QCD). The prediction, which includes only the leading order (LO), has a large dependence on the renormalization scale \( \mu_R \) in the strong coupling constants \( \alpha_s(\mu_R) \), and the factorization scale \( \mu_F \) in the PDF \( f(x, \mu_F) \). The large dependence on \( \mu_R \) and \( \mu_F \) leads to the large uncertainty of the prediction. The QCD next-to-leading order (NLO) corrections reduce the \( \mu_R \) and \( \mu_F \) dependence and makes the prediction more precise. At next-to-next-to-leading order (NNLO), the \( \mu_R \) and \( \mu_F \) dependence is more reduced. Actually, the cutting edge of the calculation technique is now at NNLO. The general environment for the NLO calculation is still under development and has much room for improvement and sophistication. In the present paper, we deal with the QCD NLO corrections only.
There are two main prediction schemes: the matrix element prediction and the showered prediction, as mentioned in Ref. [5]. The matrix element prediction is as follows. The partonic cross section includes the matrix element that represents the transition amplitude from the initial partons to the final partons. A jet algorithm defines the jet observables. Typical jet algorithms are constructed in Refs. [6–10]. A jet algorithm is directly applied to the partons in the final state of a matrix element and the distributions of the jet observables are compared to the experimental results. The showered prediction is as follows. The final-state partons of a matrix element are showered by a shower algorithm. Then a jet algorithm is applied to the partons after showering and finally the distributions of the jet observables are compared to the experimental results. The hadronization effect may also be included for better simulation. The merit of the matrix element prediction is that it is less involved and simpler than the showered prediction. The merit of the showered prediction is that it simulates the phenomena happening after the collision better than the matrix element prediction. In the present article, we deal with QCD NLO corrections only in the matrix element prediction scheme for simplicity.

The tradition of the calculation of QCD NLO corrections at hadron colliders probably started with the Drell–Yan process, \( pp/\bar{p}p \to \mu^-\mu^+ \), in the pioneering work of Refs. [11–19]; this process might be the simplest of the hadron collider processes. In the pioneering works, the analytical expression of the NLO corrections is obtained by using dimensional regularization in \( d \) dimensions throughout. All the ultraviolet (UV) and infrared, or, more precisely, soft and collinear, divergences are regulated as the poles of \( 1/\epsilon \) and \( 1/\epsilon^2 \). The NLO corrections to the cross section are generally written as

\[
\sigma_{\text{NLO}} = \sigma_R + \sigma_V + \sigma_C, \tag{1.1}
\]

where the symbols \( \sigma_R \), \( \sigma_V \), and \( \sigma_C \) represent the real emission correction, the virtual 1-loop correction, and the collinear subtraction term, respectively. The 1-loop matrix element in the virtual correction includes the UV divergences that are subtracted by the renormalization program. Each of the three terms \( \sigma_R \), \( \sigma_V \), and \( \sigma_C \) includes the soft/collinear divergences. Complete cancellation of the soft and collinear divergences from the three terms can be realized only after analytical integration of the \( d \)-dimensional phase space (PS). Then we obtain the finite physical cross section. For more complicated multiparton processes, this method mainly encounters two difficulties: the evaluation of the 1-loop matrix element and the analytical integration of the \( d \)-dimensional PS. This is because the matrix elements for the multiparton leg processes become complex and long expressions. The difficulty with the evaluation of the 1-loop matrix element has been solved by technical developments. Reviews on the recent developments can be found in Refs. [20–22].

During the 1990s, methods were invented to overcome the difficulty with the analytical integration of the \( d \)-dimensional PS: the phase-space slicing method [23–26] and the subtraction method [27–29]. Among these, the Catani–Seymour dipole subtraction procedure [27,28] in the subtraction method is quite successful and has been widely used. In the dipole subtraction procedure, subtraction terms are introduced and the NLO corrections are reconstructed as

\[
\sigma_{\text{NLO}} = (\sigma_R - \sigma_D) + (\sigma_V + \sigma_I) + \sigma_P + \sigma_K, \tag{1.2}
\]

where the symbols \( \sigma_D \), \( \sigma_I \), \( \sigma_P \), and \( \sigma_K \) represent the the dipole (D), I, P, and K terms, respectively. The terms \( \sigma_D \) and \( \sigma_I \) subtract all the soft/collinear divergences from the terms \( \sigma_R \) and \( \sigma_V \), respectively, at the integrand level on each phase-space point. Then the PS integration of the subtracted cross sections, \( (\sigma_R - \sigma_D) \) and \( (\sigma_V + \sigma_I) \), can be executed in 4 dimensions to be finite. \( \sigma_P \) and \( \sigma_K \) are separately finite under the 4D integration. In this way, in the dipole subtraction, all the PS integrations are done in 4 dimensions and the procedure can avoid the difficulty of \( d \)-dimensional analytical
integration. Thus, the procedure makes possible the calculation of the NLO corrections for multiparton processes. The dipole subtraction has already been used for many processes at LHC. The complete list of achievements is too long to show here, so we select only the most impressive ones: processes with massless quarks [30–48]; processes with massive quarks [49–63]; and processes done by projects, the Monte Carlo for FeMtobarn [64–74], the Vector boson fusion at next-to-leading order [75–83], and the BlackHat [84–89]. Of these, the highest achievements up to now may be those in the following processes: \( pp \rightarrow 5 \) jets [48], \( pp \rightarrow W + 5 \) jets [88], \( pp \rightarrow t\bar{t}b\bar{b} \) [54], and \( pp \rightarrow t\bar{t} + 2 \) jets [57]. The large number of successful calculations proves the generality and the strong power of the dipole subtraction. The price that we must pay to employ the procedure is the construction of the subtraction terms, the D, I, P, and K terms. For multiparton leg processes, the total number of all the subtraction terms sometimes exceeds one hundred and the expression of each subtraction term is not so simple. Construction by hand requires too much working time and we usually make some mistakes. Thus, automation as a computer program is really desirable. Fortunately, construction of all the subtraction terms is given in a general algorithm and hence automation as a computer program is possible. Several computer packages have already been made [90–97]. Among these, the publicly available packages are those in Refs. [91,93–95,97].

Now that the dipole subtraction has been applied for so many processes, we can see some drawbacks. Among them, we would like to point out the following difficulties. The person who has obtained the results for the NLO corrections by dipole subtraction sometimes has difficulty confirming the results, because many subtraction terms are involved and a large amount of calculations are executed as numerical evaluation for the Monte Carlo integration. For the other person who does not do the calculations him- or herself, confirmation of the results shown is more difficult. All the soft and collinear divergences from the D and I terms must exactly cancel the divergences from the real correction \( \sigma_R \) and the virtual correction \( \sigma_V \), respectively. If we use any wrong collection or any wrong expression for the D and I terms, the cancellation is spoiled. A successful cancellation provides one check on the singular parts of the D and I terms. The P and K terms are separately finite by themselves and a check by the cancellation is impossible. In this sense, the uses of the P and K terms are the place where we can easily make mistakes. In many articles, it is not clear which and how many subtraction terms are used. Of course, this explanation would be too long and it is unreasonable to expect that explicit expressions of all subtraction terms are given in the articles. However, it is possible that minimal information specifying the subtraction terms can be shown. At least the total number of subtraction terms used should be clearly mentioned. In many works, the dipole subtraction is applied by using automated programs. In this case, similar criticisms should be made about the implementation, especially the algorithm to create the subtraction terms. In the packages in Refs. [95] and [97], the creation algorithms are not so strictly documented, and, on the execution of the packages, information on the terms created is not printed out. We cannot know which and how many D, I, and P/K terms are created under a given input process. In the package in Ref. [93], the creation algorithm is presented in the article. In the output of the Mathematica program, information on all the subtraction terms created and the total number of terms are printed out separately. The output codes of the D, I, and P/K terms are saved separately in the corresponding folders and can be easily identified. In all three packages, no algorithm to check the consistency of all the subtraction terms created is provided.

In order to solve one part of the difficulties and the criticisms, we need a clear definition of a practical algorithm to use the dipole subtraction. Although the general algorithm is given in the original articles [27,28], we need a more practical algorithm that can be directly used step by step.
Also, to automatize the procedure into a computer program, we need such a practical algorithm to be applied to an arbitrary process. In consequence, we desire an algorithm that provides clear definitions of the following items:

1. Input, output, creation order, and all formulae in the document,
2. Necessary information to specify each subtraction term,
3. Summary table of all subtraction terms created,
4. Associated proof algorithm.

In the last entry in the wishlist, an associated proof algorithm means the following. When the NLO cross sections in Eqs. (1.1) and (1.2) are equated, we obtain the relation of the cancellation as

$$\sigma_{\text{subt}} = \sigma_D + \sigma_C - \sigma_I - \sigma_P - \sigma_K = 0. \quad (1.3)$$

We call this relation the consistency relation of the subtraction terms. The consistency relation is an identity that means that the relation stands in an arbitrary process. For the factorization scale $\mu_F$ in Eq. (1.3), apart from the PDFs, only the two terms $\sigma_C$ and $\sigma_P$ have the same $\mu_F$ dependence, which cancel each other. An associated proof algorithm is an algorithm to prove the consistency relation in Eq. (1.3) for an arbitrary process.

The purpose of this article is to present a practical algorithm that satisfies all the requirements in the above wishlist. We actually present such an algorithm, called the dipole splitting algorithm (DSA).

In the DSA, the input is all real emission processes that contribute to a collider process like $pp \rightarrow n$ jets. Each real emission process, such as like $u\bar{u} \rightarrow u\bar{u}g$, creates the output of the D, I, P/K terms, and all of them have the same initial states as the input real process. In this sense, the subtraction terms are sorted by the initial-state partons. All the subtraction terms are also sorted by the kind of splittings that each subtraction term possesses as a part. The sorting by splittings is equivalent with the sorting by the reduced Born process that each subtraction term possesses as a remaining part when the splittings part is removed. In order to specify each subtraction term uniquely, we introduce a bijection mapping, called field mapping. Each field mapping is made for each subtraction term. The field mapping exactly specifies the connection between the legs of the Born process reduced from an input real process and the fixed legs of the same reduced Born process. By using the field mapping, we can specify each term in a compact form without confusion. Thanks to this well defined compact form, we can summarize the subtraction terms in tables. We will present a template for the summary tables. We intend that the person who does not create the subtraction terms him- or herself can specify and write down them only by reading the tables in a document without any direct communication with the author of the tables. For the last entry in the wishlist, we have constructed a straightforward algorithm to prove the consistency relation of the subtraction terms created by the DSA. We present the proof algorithm (PRA) in the following paper [98].

We mention here the relation between the DSA and the algorithm implemented in the AutoDipole package [93]. The creation algorithm of the D and I terms in the DSA is essentially the same as the algorithm in AutoDipole. In this sense, the DSA originates in the algorithm of AutoDipole. In the DSA, the concrete expressions are more clearly defined by using the field mapping and the necessary information to specify each subtraction is given in a compact form. The creation algorithm for the P and K terms in the DSA is different from the algorithm in AutoDipole. In the DSA, the initial states of all the created subtraction terms under one input real process are the same as the initial state of the input real process. In the algorithm of AutoDipole, the initial states of the P and K terms with the nondiagonal splittings are different from the initial state of the input real process. The difference in
the creation of the P and K terms makes possible the construction of the proof algorithm associated with the DSA.

The present article is organized as follows. The DSA is defined in Sect. 2. The formulae for the subtraction terms are collected in Appendix A. The DSA is demonstrated in the collider processes \( pp \rightarrow \mu^- \mu^+ \) and 2 jets in Sects. 3 and 4, respectively. The summary tables for the dijet process are shown in Appendix B. We give a confirmation that the analytical results obtained by the DSA in the Drell–Yan process coincide with the results by the traditional method in Sect. 5. We give a summary in Sect. 6.

2. Dipole splitting algorithm

In this section, the DSA is defined. In Sect. 2.1, all the steps of the DSA and the master formulae are presented. In Sects. 2.2, 2.3, and 2.4, the creation algorithms of the D, I, and P/K terms are explained, respectively. The advantages of the DSA are clarified in Sect. 2.5. The formulae for all the subtraction terms are collected in Appendix A.

2.1. Definition and Step 1: List of \( R_i \)

The DSA consists of the following steps:

**Step 1.** List of real emission processes \( \{ R_i \} \),

2. \( D(R_i) \),

3. \( I(R_i) \),

4. \( P(R_i) \) and \( K(R_i) \),

5. \( \sigma_{NLO} = \sum_i \sigma(R_i) \).  \hspace{1cm} (2.1)

We assume that we wish to make a prediction for the observables in a collider process. Once a collider process is selected, the contributing partonic real emission processes are specified. **Step 1** is to specify all the real processes denoted as \( R_i \) and to write down the list. The number of real processes is written as \( n_{\text{real}} \) and the number of fields in the final states is denoted as \( (n+1) \). For example, the collider process \( pp \rightarrow 2 \) jets has real emission processes as

\[
R_1 = u\bar{u} \rightarrow u\bar{u}g, \\
R_2 = uu \rightarrow uuug, \\
\vdots \\
R_{n_{\text{real}}} = gg \rightarrow ggg.
\]  \hspace{1cm} (2.2)

**Steps 2, 3, and 4**, are explained in Sects. 2.2, 2.3, and 2.4, respectively. The last step of the DSA, **Step 5**, is to obtain the NLO corrections as

\[
\sigma_{NLO} = \sum_{i=1}^{n_{\text{real}}} \sigma(R_i),
\]  \hspace{1cm} (2.3)

where each cross section \( \sigma(R_i) \) is defined as

\[
\sigma(R_i) = \int dx_1 \int dx_2 \ f_{F_{(x)}}(x_1)f_{F_{(x)}}(x_2) \\
\times \left[ (\hat{\sigma}_R(R_i) - \hat{\sigma}_D(R_i)) + (\hat{\sigma}_V(B1(R_i)) + \hat{\sigma}_I(R_i)) + \hat{\sigma}_P(R_i) + \hat{\sigma}_K(R_i) \right],
\]  \hspace{1cm} (2.4)
where \( f_{F(x_{a/b})}(x_{1/2}) \) is the PDF and the subscript \( F(x_{a/b}) \) represents the field species of the initial-state parton of the leg \( a/b \), which is again defined in the next section. The symbols \( \hat{\sigma}(R_i) \) with subscripts \( R, D, V, I, P, \) and \( K \) represent the partonic cross sections of the real correction, the dipole (D) term, the virtual correction, the I term, the P term, and the K term, respectively. Each partonic cross section is defined as

\[
\hat{\sigma}_R(R_i) = \frac{1}{S_{R_i}} \Phi(R_i)_4 \cdot |M(R_i)|^2_4, \tag{2.5}
\]

\[
\hat{\sigma}_D(R_i) = \frac{1}{S_{R_i}} \Phi(R_i)_4 \cdot \frac{1}{n_s(a)n_s(b)} \cdot D(R_i), \tag{2.6}
\]

\[
\hat{\sigma}_V(B_1) = \frac{1}{S_{B_1}} \Phi(B_1)_d \cdot |M_{\text{virt}}(B_1)|^2_d, \tag{2.7}
\]

\[
\hat{\sigma}_I(R_i) = \frac{1}{S_{B_1}} \Phi(B_1)_d \cdot I(R_i), \tag{2.8}
\]

\[
\hat{\sigma}_P(R_i) = \int_0^1 dx \sum_{B_j} \frac{1}{S_{B_j}} \Phi_a(R_i : B_j, x)_4 \cdot P(R_i, x_a : B_j, x_{p_a}) + (a \leftrightarrow b), \tag{2.9}
\]

\[
\hat{\sigma}_K(R_i) = \int_0^1 dx \sum_{B_j} \frac{1}{S_{B_j}} \Phi_a(R_i : B_j, x)_4 \cdot K(R_i, x_a : B_j, x_{p_a}) + (a \leftrightarrow b). \tag{2.10}
\]

In Eqs. (2.5) and (2.6), the symbol \( S_{R_i} \) is the symmetric factor of the real process \( R_i \) and the symbol \( \Phi(R_i)_4 \) is the 4D \((n+1)\)-body PS including the flux factor as

\[
\Phi(R_i)_4 = \frac{1}{\mathcal{F}(p_a, p_b)} \prod_{i=1}^{n+1} \int \frac{d^2p_i}{(2\pi)^2} \frac{1}{2E_i} \cdot (2\pi)^4 \delta^{(4)} \left( p_a + p_b - \sum_{i=1}^{n+1} p_i \right). \tag{2.11}
\]

The energy is denoted as \( E_I \) for \( I = a, b, 1, \ldots, n+1 \). The flux factor is written as \( \mathcal{F}(p_a, p_b) = 4E_aE_b \). \(|M(R_i)|^2_4 \) is the square of the matrix element of the real emission process \( R_i \) after the average over spin and color in 4 dimensions. \( n_s(a/b) \) represents the spin degree of freedom of the leg \( a/b \) in \( R_i \) in 4 dimensions. In Eqs. (2.7) and (2.8), the process \( B_1 \) is an abbreviation of \( B_1(R_i) \), which is made by removing one gluon from the final states of \( R_i \), denoted as \( B_1(R_i) = R_i - g \). \( S_{B_1} \) is the symmetric factor of process \( B_1 \), and the symbol \( \Phi(B_1)_d \) represents the \( d \)-dimensional \( n \)-body PS with the flux factor as

\[
\Phi(B_1)_d = \frac{1}{\mathcal{F}(p_a, p_b)} \prod_{i=1}^{n} \int \frac{d^{d-1}p_i}{(2\pi)^{d-1}} \frac{1}{2E_i} \cdot (2\pi)^d \delta^{(d)} \left( p_a + p_b - \sum_{i=1}^{n} p_i \right). \tag{2.12}
\]

The symbol \(|M_{\text{virt}}(B_1)|^2_d \) is the abbreviation of the quantity \((M_{\text{LO}}(B_1) M_{1-\text{loop}}(B_1) + M_{\text{LO}}(B_1)^* M_{1-\text{loop}}(B_1)) \) after the average over spin and color in \( d \) dimensions, where \( M_{\text{LO}}(B_1) \) and \( M_{1-\text{loop}}(B_1) \) are the matrix elements of the LO and the 1-loop correction of process \( B_1 \), respectively. In Eqs. (2.9) and (2.10), \( B_j \) is an abbreviation of \( B_j(R_i) \) and is a Born process reduced from \( R_i \), which is precisely defined in Sect. 2.2. \( S_{B_j} \) is the symmetric factor of \( B_j \) and \( \Phi_a(R_i : B_j, x)_4 \) is the 4D \( n \)-body PS with the scaled momentum \( x_{p_a} \) and the flux factor \( \mathcal{F}(x_{p_a}, p_b) = 4x E_aE_b \) as

\[
\Phi_a(R_i : B_j, x)_4 = \frac{1}{\mathcal{F}(x_{p_a}, p_b)} \prod_{i=1}^{n} \int \frac{d^3p_i}{(2\pi)^3} \frac{1}{2E_i} \cdot (2\pi)^4 \delta^{(4)} \left( x_{p_a} + p_b - \sum_{i=1}^{n} p_i \right). \tag{2.13}
\]

In Eqs. (2.6), (2.8), (2.9), and (2.10), concrete expressions of the subtraction terms \( D(R_i), I(R_i), \) and \( P/K(R_i : B_j, x_{p_a}) \) are presented in Sects. 2.2, 2.3, and 2.4, respectively. The jet functions
\( F_{j}^{(n+1)}(p_1, \ldots, p_{n+1}) \) must be multiplied to the partonic cross sections in Eqs. (2.5)–(2.10). For the real correction in Eq. (2.5), the jet function with \((n+1)\) fields is multiplied as

\[
\hat{\sigma}_R(R_i) = \frac{1}{S_{R_i}} \Phi(R_i)_d \cdot |M(R_i)|_d^2 \cdot F_{j}^{(n+1)}(p_1, \ldots, p_{n+1}). \tag{2.14}
\]

For the cross sections in Eqs. (2.7)–(2.10), the jet function with \(n\) fields \(F_{j}^{(n)}(p_1, \ldots, p_n)\) is multiplied. For the dipole term in Eq. (2.6), the jet function \(F_{j}^{(n)}\) is multiplied and the \(n\) momenta of the arguments are identified with the reduced momenta \((p(y_1), \ldots, p(y_n))\). The details of the use of the jet functions in the dipole subtraction are explained in Ref. [27]. For compact notation, we do not show the jet functions explicitly hereafter.

The PS integration in Eq. (2.4) is finite in 4 dimensions and we see here the finite parts separately. The real correction \(\hat{\sigma}_R(R_i)\) has soft and collinear divergences, which are subtracted by the dipole terms \(\hat{\sigma}_D(R_i)\). The subtracted cross section \((\hat{\sigma}_R(R_i) - \hat{\sigma}_D(R_i))\) is finite in 4 dimensions as

\[
\hat{\sigma}_R(R_i) - \hat{\sigma}_D(R_i) = \frac{1}{S_{R_i}} \Phi(R_i)_d \cdot \left[ |M_{\text{real}}(R_i)|_d^2 - \frac{1}{n_s(a)n_s(b)} D(R_i) \right] < \infty. \tag{2.15}
\]

The virtual correction \(\hat{\sigma}_V(B1)\) includes the poles of the soft and collinear divergences \(1/\epsilon\) and \(1/\epsilon^2\) after the subtraction of the UV divergences \(1/\epsilon_{\text{UV}}\) by the renormalization program. The I term \(\hat{\sigma}_I(R_i)\) cancels all the soft and collinear poles in \(d\) dimensions as

\[
\hat{\sigma}_V(B1) + \hat{\sigma}_I(R_i) = \frac{1}{S_{B1}} \Phi(B1)_d \cdot \left[ |M_{\text{virt}}(B1)|_d^2 + I(R_i) \right]. \tag{2.16}
\]

After the cancellation of the poles, we can integrate the PS in 4 dimensions as

\[
\hat{\sigma}_V(B1) + \hat{\sigma}_I(R_i) = \frac{1}{S_{B1}} \Phi(B1)_d \cdot \left[ |M_{\text{virt}}(B1)|^2 + I(R_i) \right] < \infty. \tag{2.17}
\]

The cross sections of the P and K terms, \(\hat{\sigma}_P(R_i)\) and \(\hat{\sigma}_K(R_i)\), are themselves finite separately in 4 dimensions. In this way, all the PS integrations in Eq. (2.4) are executed in 4 dimensions.

To complete the master formulae, we add the LO contribution as

\[
\sigma_{\text{LO}} = \sum_{i=1}^{n_{\text{LO}}} \sigma(L_i), \tag{2.18}
\]

\[
\sigma(L_i) = \int dx_1 \int dx_2 f_{F(x_1)}(x_1) f_{F(x_2)}(x_2) \hat{\sigma}(L_i). \tag{2.19}
\]

\[
\hat{\sigma}(L_i) = \frac{1}{S_{L_i}} \Phi(L_i)_d \cdot |M(L_i)|^2_d, \tag{2.20}
\]

where the subpartonic LO processes that contribute to the selected collider process are denoted as \(L_i\) and the number of LO processes is denoted as \(n_{\text{LO}}\). \(\Phi(L_i)_d\) is the 4D \(n\)-body PS including the flux factor. Then the prediction at NLO accuracy is written as

\[
\sigma_{\text{prediction}} = \sigma_{\text{LO}} + \sigma_{\text{NLO}}, \tag{2.21}
\]

where \(\sigma_{\text{LO}}\) and \(\sigma_{\text{NLO}}\) are defined in Eqs. (2.18) and (2.3), respectively.

### 2.2. Step 2: D term creation

In this section, ‘Step 2. D(R_i)’ is explained. The input and output of the step are written as

- **Input:** \(R_i\)
- **Output:** \(D(R_i)\)
The input process is each real emission process \( R_i \) among all real processes, \( R_1, \ldots, R_{n_{\text{real}}} \), which are specified in **Step 1**. The input defines set \( \{ x \} \) with the field species \( F(x_I) \) and the momenta \( p_I \) for the indices \( I = a, b, 1, \ldots, n + 1 \) as

\[
\{ x \} = \{ x_a, x_b; x_1, \ldots, x_{n+1} \},
\]

\[
F(\{ x \}) = \{ F(x_a), F(x_b); F(x_1), \ldots, F(x_{n+1}) \},
\]

\[
\text{Momenta : } \{ p_a, p_b; p_1, \ldots, p_{n+1} \}.
\]

The output \( D(R_i) \) is the summation of all the created dipole terms. The creation is repeated over all the input processes \( R_i \) with \( i = 1, \ldots, n_{\text{real}} \), and the outputs are the corresponding dipole terms \( D(R_i) \) with \( i = 1, \ldots, n_{\text{real}} \).

In the original article by Catani and Seymour [27], each dipole term is specified by three legs \((I, J, K)\), where we call the pair of legs \((I, J)\) and the combined leg \( \bar{IJ} \) the emitter pair and the emitter, respectively. Leg \( K \) is called the spectator. All possible combinations of \((I, J, K)\) are chosen without duplicate from all \((n + 3)\) legs of \( R_i \). In the DSA, the creation algorithm of the dipole terms is divided into substeps as follows:

1. Choose all the possible emitter pairs \((x_I, x_J)\) from set \( \{ x \} \) in the order of the splittings from (1) to (7) in Fig. A1 in Appendix A1.
2. Choose all the possible spectators \( x_K \) from \( \{ x \} \) for each choice of the pair \((x_I, x_J)\).
3. Make one field mapping for each combination \((x_I, x_J, x_K)\).
4. Write down the concrete expressions of all the dipole terms.

Substeps 1 and 2 are explained in Sect. 2.2.1. Substep 3 is explained in Sect. 2.2.2, and substep 4 in Sect. 2.2.3. Some concrete examples are shown in Sect. 2.2.4. Finally, we give a summary in Sect. 2.2.5. The formulae for the dipole terms are collected in Appendix A1.

### 2.2.1. Creation order

In the DSA, creation of the dipole terms is sorted by the kind of splittings of the emitter pairs \((x_I, x_J)\). The sorting order is shown in Fig. A1. The created dipoles are grouped into categories, **Dipoles** 1, 2, 3, 4, and subcategories, (1)–(7), as follows:

- **Dipole 1**: (1), (2), (3), (4),
- **Dipole 2**: (5),
- **Dipole 3**: (6),
- **Dipole 4**: (7).

**Dipole 1** includes the splitting of a gluon emission, which is sometimes called diagonal splitting. **Dipoles** 2, 3, 4 have further subcategories for the quark flavors in the five-massless-flavor scheme as follows:

- **Dipole 2**: \( u, c, d, s, b, \)
- **Dipole 3**: \( u, c, d, s, \bar{u}, \bar{c}, \bar{d}, \bar{s}, \bar{b}, \)
- **Dipole 4**: \( u, c, d, s, \bar{u}, \bar{c}, \bar{d}, \bar{s}, \bar{b}, \)

where, e.g., Dipole2u means that the species of fields \( x_i \) and \( x_j \) are up- and anti-up-quarks as \( (F(x_i), F(x_j)) = (u, \bar{u}) \), and Dipole3\( \bar{u} \) means \( (F(x_a), F(x_j)) = (\bar{u}, \bar{u}) \). The spectator \( x_K \) can be a quark or gluon in either the final or the initial state. If the spectator is in the final/initial state, we denote the case as subcategory \(-1/2\). Then the category of the dipole terms are further divided into
subcategories as

\[
\begin{align*}
\text{Dipole 1} & \quad (1)-1/2, \quad (2)-1/2, \quad (3)-1/2, \quad (4)-1/2, \\
\text{Dipole 2} & \quad (5)-1/2, \\
\text{Dipole 3} & \quad (6)-1/2, \\
\text{Dipole 4} & \quad (7)-1/2.
\end{align*}
\]

(2.25)

The summation of the dipole terms that belong to the same category is written, respectively, as

\[
D(R_i, \text{dip}1, (1)), \quad D(R_i, \text{dip}1, (2)), \quad D(R_i, \text{dip}1, (3)), \quad D(R_i, \text{dip}1, (4)), \\
D(R_i, \text{dip}2), \\
D(R_i, \text{dip}3), \\
D(R_i, \text{dip}4),
\]

where the subcategories for the quark flavors and for the spectator in the final or initial states are all summed. The dipole terms belonging to Dipole 1 are summed as

\[
D(R_i, \text{dip}1) = \sum_{j=1}^{4} D(R_i, \text{dip}1, (j)), \quad (2.26)
\]

and the summation of all the dipole terms as output is written as

\[
D(R_i) = \sum_{j=1}^{4} D(R_i, \text{dip}j). \quad (2.27)
\]

The concrete expression for each dipole is given in the original article [27] in the form

\[
D(R_i, \text{dip}j)_{IJ,K} = -\frac{1}{s_{IJ}} \frac{1}{x_{IJ}} \left( B_j \left| \frac{T_{IJ} \cdot T_K}{T_{IJ}^2} V_{IJ,K} \right| B_j \right), \quad (2.28)
\]

where \(s_{IJ}\) is defined as \(s_{IJ} = 2p_I \cdot p_J\), and \(x_{IJ,K}\) is specified in Sect. 2.2.3. \(B_j\) is a Born process that is reduced from the input process \(R_i\) by removing the splitting part. Then the dipole terms \(D(R_i, \text{dip}j)\) in the category Dipole \(j\) have the reduced Born process \(B_j\), which is made from \(R_i\) with the following rules:

\[
\begin{align*}
\text{Dipole 1 : } B1 & = R_i - g_f, \\
\text{Dipole 2u : } B2u & = R_i - u_f - \bar{u}_f + g_f, \\
\text{Dipole 3u : } B3u & = R_i - u_f - u_i + g_i, \\
\text{Dipole 4u : } B4u & = R_i - u_f - g_i + \bar{u}_i.
\end{align*}
\]

(2.29)

where the symbols \(g_{f/i}\) and \(u_{f/i}\) represent a gluon and an up-quark in the final/initial state. The operation \(\pm g_f\) means to add/remove a gluon to/from the final state. The other operations, such as \(\pm u_f\), are similarly defined. For \(B2u, B3u,\) and \(B4u\), other subcategories with other quark flavors also exist. The symbol \((T_{IJ} \cdot T_K)/T_{IJ}^2\) represents the operators of the color factor insertions and \(V_{IJ,K}\) is the dipole splitting function with the helicity correlation. The actions of the color and helicity operators on the amplitude of the reduced Born process are clearly defined by using the field mapping, explained in the next section.
2.2.2. *Field mapping*

Each dipole term includes the square of a reduced Born amplitude, shown in Eq. (2.28). The original \((n + 3)\) legs of the input process \(R_i\) are connected to the \((n + 2)\) legs of the reduced Born process. In order to specify the connection clearly in a compact form we introduce a bijection mapping for each dipole, called the *field mapping*. For each combination choice \((x_I, x_J, x_K)\), we can make a new set \(\{\tilde{x}\}\) by the unification of the elements \((x_I, x_J) \rightarrow x_{IJ}\) and the replacement \(x_K \rightarrow x_{\tilde{K}}\). To explain the definition of set \(\{\tilde{x}\}\) precisely, we separate the dipole terms into four categories, \((IJ, K) = (ij, k), (ij, a), (ai, k),\) and \((ai, b)\), where the indices \(i, j, k\) represent the legs in the final state and the indices \(a\) and \(b\) represent the legs in the initial state. The four categories are called the final-final, final-initial, initial-final, and initial-initial dipole terms, respectively. The relations between the four categories and the categories defined in Sect. 2.2.1 are shown as

\[
\begin{align*}
\text{Final-final : } (ij, k) & \supset \text{ Dipole 1 (1)-1, (2)-1, } \\
& \quad \text{ Dipole 2 (5)-1, } \\
\text{Final-initial : } (ij, a) & \supset \text{ Dipole 1 (1)-2, (2)-2, } \\
& \quad \text{ Dipole 2 (5)-2, } \\
\text{Initial-final : } (ai, k) & \supset \text{ Dipole 1 (3)-1, (4)-1, } \\
& \quad \text{ Dipole 3 (6)-1, } \\
& \quad \text{ Dipole 4 (7)-1, } \\
\text{Initial-initial : } (ai, b) & \supset \text{ Dipole 1 (3)-2, (4)-2, } \\
& \quad \text{ Dipole 3 (6)-2, } \\
& \quad \text{ Dipole 4 (7)-2. } \\
\end{align*}
\] (2.30)

- **Final-final: \((ij, k)\)**

In this category, set \(\{\tilde{x}\}\) is defined as

\[
\{\tilde{x}\} = \{x_a, x_b; x_1, \ldots, x_{n-2}, x_{n-1}, x_n\} \\
= \{x_a, x_b; x_1, \ldots, x_{n+1}, x_{ij}, x_{\tilde{K}}\}, \\
\] (2.31)

\[
F(\{\tilde{x}\}) = \{F(x_a), F(x_b); F(x_1), \ldots, F(x_{n-2}), F(x_{n-1}), F(x_n)\} \\
= \{F(x_a), F(x_b); F(x_1), \ldots, F(x_{n+1}), F(x_{ij}), F(x_{\tilde{K}})\}, \\
\] (2.32)

\[
P(\{\tilde{x}\}) = \{P(x_a), P(x_b); P(x_1), \ldots, P(x_{n-2}), P(x_{n-1}), P(x_n)\} \\
= \{P(x_a), P(x_b); P(x_1), \ldots, P(x_{n+1}), P(x_{ij}), P(x_{\tilde{K}})\}, \\
\] (2.33)

where the symbols \(F(x_a)\) and \(P(x_a)\) represent the field species and the momenta of the elements \(x_a\) with the indices \(a = a, b, 1, \ldots, n\). The field species are determined as

\[
F(x_{n-1}) = F(x_{ij}), \\
F(x_n) = F(x_{\tilde{K}}) = F(x_k), \\
F(x_a) = F(x_L) \quad \text{for } \alpha = a, b, 1, \ldots, (n - 2). \\
\] (2.34-2.36)

In Eq. (2.34), the field species of the element \(x_{ij}\), \(F(x_{ij})\), is defined as the field species of the root of the splitting \(x_{ij} \rightarrow x_i + x_j\), where the legs \(x_i\) and \(x_j\) are external legs in the final state, and
$x_{ij}$ is the internal line that is attached to the gray circle at the center of Fig. A1. The relation in Eq. (2.35) means that the field species of the spectator $x_k$ is identical to the species of the original leg $x_k$ in set $\{x\}$. The relation in Eq. (2.36) means that the field species of the other elements $\tilde{x}_\alpha$, with $\alpha = a, b, 1, \ldots, n-2$, are the same as the element $x_L$ in $F(\tilde{x}_\alpha) = F(x_L)$, where the element $x_L$ is specified as $\tilde{x}_\alpha = x_L$, in Eq. (2.31) for the indices $L = a, b, 1, \ldots, n+1$, skipping the indices $i, j, k$. The momenta of the elements in set $\{\tilde{x}\}$ are determined as

\begin{align}
P(\tilde{x}_{n-1}) &= P(x_{ij}) = \tilde{p}_{ij}, \quad (2.37) \\
P(\tilde{x}_n) &= P(x_k) = \tilde{p}_k, \quad (2.38) \\
P(\tilde{x}_\alpha) &= P(x_L) = p_L, \quad \text{for } \alpha = a, b, 1, \ldots, (n-2), \quad (2.39)
\end{align}

where the reduced momenta $\tilde{p}_{ij}$ and $\tilde{p}_k$ are defined in Eqs. (A31) and (A32) in Appendix A1. Similarly to the relation in Eq. (2.36), the momenta of the other elements $\tilde{x}_\alpha$, with $\alpha = a, b, 1, \ldots, n-2$, are the same as the corresponding original legs $x_L$. It is noted that the order of the elements in set $\{\tilde{x}\}$ in Eq. (2.31) is not strict and other orders are also possible as long as the elements in the initial and final states are not mixed. What is necessary is that the field species and the momenta of all the elements in $\{\tilde{x}\}$ are properly determined. To demonstrate the other possibilities, we here take one example with $n = 3$. Set $\{x\}$ is defined as

\begin{align}
\{x\} &= \{x_a, x_b; x_1, x_2, x_3, x_4\}. \quad (2.40)
\end{align}

Momenta : $\{p_a, p_b; p_1, p_2, p_3, p_4\}$

\begin{align}
F(\{x\}) &= \{F(x_a), F(x_b); F(x_1), F(x_2), F(x_3), F(x_4))\}. \quad (2.41)
\end{align}

According to the relations in Eqs. (2.31), (2.32), and (2.33), set $\{\tilde{x}\}$ is made on the choice $(I, J, K) = (1, 2, 3)$, for instance, as

\begin{align}
\{\tilde{x}\} &= \{\tilde{x}_a, \tilde{x}_b; \tilde{x}_1, \tilde{x}_2, \tilde{x}_3\} \\
&= \{x_a, x_b; x_1, x_2, x_3\}, \quad (2.43)
\end{align}

\begin{align}
F(\{\tilde{x}\}) &= \{F(\tilde{x}_a), F(\tilde{x}_b); F(\tilde{x}_1), F(\tilde{x}_2), F(\tilde{x}_3)\} \\
&= \{F(x_a), F(x_b); F(x_1), F(x_2), F(x_3)\}. \quad (2.44)
\end{align}

\begin{align}
P(\{\tilde{x}\}) &= \{P(\tilde{x}_a), P(\tilde{x}_b); P(\tilde{x}_1), P(\tilde{x}_2), P(\tilde{x}_3)\} \\
&= \{P(x_a), P(x_b); P(x_1), P(x_2), P(x_3)\}. \quad (2.45)
\end{align}

The following choice of set $\{\tilde{x}\}$ is also possible:

\begin{align}
\{\tilde{x}\} &= \{\tilde{x}_a, \tilde{x}_b; \tilde{x}_1, \tilde{x}_2, \tilde{x}_3\} \\
&= \{x_a, x_b; x_1, x_2, x_3\}, \quad (2.46)
\end{align}

\begin{align}
F(\{\tilde{x}\}) &= \{F(\tilde{x}_a), F(\tilde{x}_b); F(\tilde{x}_1), F(\tilde{x}_2), F(\tilde{x}_3)\} \\
&= \{F(x_a), F(x_b); F(x_1), F(x_2), F(x_3)\}. \quad (2.47)
\end{align}

\begin{align}
P(\{\tilde{x}\}) &= \{P(\tilde{x}_a), P(\tilde{x}_b); P(\tilde{x}_1), P(\tilde{x}_2), P(\tilde{x}_3)\} \\
&= \{P(x_a), P(x_b); P(x_1), P(x_2), P(x_3)\}. \quad (2.48)
\end{align}

The field species and the momenta in this case are similarly determined. The note about the freedom of the order of the elements in $\{\tilde{x}\}$ is also valid for cases $(i, j, a)$, $(ai, k)$, and $(ai, b)$. 

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Final-initial: \((ij, a)\)

For the category of the final-initial dipoles \((ij, a)\), set \(\{\tilde{x}\}\) is made as

\[
\{\tilde{x}\} = \{\tilde{x}_a, \tilde{x}_b; \tilde{x}_1, \ldots, \tilde{x}_{n-1}, \tilde{x}_a\} = \{x_{\tilde{a}}, x_b; x_1, \ldots, x_{n+1}, x_{\tilde{a}}\},
\]

\[
F(\{\tilde{x}\}) = \{F(x_{\tilde{a}}), F(x_b); F(x_1), \ldots, F(x_{n+1}), F(x_{\tilde{a}})\}.
\]

\[
P(\{\tilde{x}\}) = \{P(x_{\tilde{a}}), P(x_b); P(x_1), \ldots, P(x_{n+1}), P(x_{\tilde{a}})\}.
\]

The field species are defined in a similar way to the previous case in Eqs. (2.34), (2.35), and (2.36) as

\[
F(\tilde{x}_n) = F(x_{\tilde{a}}),
\]

\[
F(\tilde{x}_a) = F(x_{\tilde{a}}) = F(x_a),
\]

\[
F(\tilde{x}_a) = F(x_{\tilde{a}}) \quad \text{for} \quad \alpha = b, 1, \ldots, (n - 1),
\]

where the \(F(x_{\tilde{a}})\) is the same as in Eq. (2.34). For the relation in Eq. (2.54), the elements \(\tilde{x}_a\), with \(\alpha = b, 1, \ldots, n - 1\), are identified with the elements \(x_L\) as \(\tilde{x}_a = x_L\) in Eq. (2.49) for \(L = b, 1, \ldots, n + 1\), skipping the indices \(i\) and \(j\). The momenta are also similarly defined as

\[
P(\tilde{x}_n) = P(x_{\tilde{a}}) = \tilde{p}_{ij},
\]

\[
P(\tilde{x}_a) = P(x_a) = \tilde{p}_a,
\]

\[
P(\tilde{x}_a) = P(x_L) = p_L \quad \text{for} \quad \alpha = b, 1, \ldots, (n - 1),
\]

where the reduced momenta \(\tilde{p}_{ij}\) and \(\tilde{p}_a\) are defined in Eqs. (A37) and (A38).

Initial-final: \((ai, k)\)

For the initial-final dipoles \((ai, k)\), set \(\{\tilde{x}\}\) is made as

\[
\{\tilde{x}\} = \{\tilde{x}_a, \tilde{x}_b; \tilde{x}_1, \ldots, \tilde{x}_{n-1}, \tilde{x}_a\} = \{x_{\tilde{a}}, x_b; x_1, \ldots, x_{n+1}, x_{\tilde{a}}\},
\]

\[
P(\{\tilde{x}\}) = \{P(x_{\tilde{a}}), P(x_b); P(x_1), \ldots, P(x_{n+1}), P(x_{\tilde{a}})\},
\]

\[
F(\{\tilde{x}\}) = \{F(x_{\tilde{a}}), F(x_b); F(x_1), \ldots, F(x_{n+1}), F(x_{\tilde{a}})\}.
\]

The field species are defined as

\[
F(\tilde{x}_a) = F(x_{\tilde{a}}),
\]

\[
F(\tilde{x}_n) = F(x_{\tilde{a}}) = F(x_k),
\]

\[
F(\tilde{x}_a) = F(x_L) \quad \text{for} \quad \alpha = b, 1, \ldots, (n - 1).
\]

For the relation in Eq. (2.61), the field species of the element \(x_{\tilde{a}}\), \(F(x_{\tilde{a}})\), is defined as the field species of the root of the splitting \(x_a \to x_{\tilde{a}} + x_i\), where the legs \(x_a\) and \(x_i\) are external legs, and \(x_{\tilde{a}}\) is the internal line that is attached to the gray circle at the center of Fig. A1. Similarly to the previous cases, the field species of the spectator and the other elements are the same as those of
the original legs. The momenta are defined as
\[ P(\tilde{x}_a) = P(x_{ai}) = \tilde{p}_{ai}, \quad \text{(2.64)} \]
\[ P(\tilde{x}_n) = P(x_{i}) = \tilde{p}_k, \quad \text{(2.65)} \]
\[ P(\tilde{x}_a) = P(x_L) = p_L, \quad \text{for } \alpha = b, 1, \ldots, (n - 1), \quad \text{(2.66)} \]
where the reduced momenta \( \tilde{p}_{ai} \) and \( \tilde{p}_k \) are defined in Eqs. (A42) and (A43).

\( \text{Initial-initial: (} ai, b \text{)} \)

For the initial-initial dipoles \( (ai, b) \), set \( \{\tilde{x}\} \) is made as
\[ \{\tilde{x}\} = \{\tilde{x}_a, \tilde{x}_b, \tilde{x}_1, \ldots, \tilde{x}_n\} \]
\[ = \{x_{ai}, x_{bi}, x_1, \ldots, x_{n+1}\}, \quad \text{(2.67)} \]
\[ P(\tilde{x}) = \{P(x_{ai}), P(x_{bi}); P(x_1), \ldots, P(x_{n+1})\}, \quad \text{(2.68)} \]
\[ F(\tilde{x}) = \{F(x_{ai}), F(x_{bi}); F(x_1), \ldots, F(x_{n+1})\}. \quad \text{(2.69)} \]

The field species are defined as
\[ F(\tilde{x}_a) = F(x_{ai}), \quad \text{(2.70)} \]
\[ F(\tilde{x}_b) = F(x_{bi}) = F(x_b), \quad \text{(2.71)} \]
\[ F(\tilde{x}_a) = F(x_L) \quad \text{for } \alpha = 1, \ldots, n. \quad \text{(2.72)} \]

In Eq. (2.70), the definition of \( F(x_{ai}) \) is the same as the previous case of \( (ai, k) \). \( \tilde{x}_a \) represent the other elements with \( \alpha = 1, \ldots, n \). The momenta are defined as
\[ P(\tilde{x}_a) = P(x_{ai}) = \tilde{p}_{ai}, \quad \text{(2.73)} \]
\[ P(\tilde{x}_n) = P(x_{i}) = \tilde{p}_b, \quad \text{(2.74)} \]
\[ P(\tilde{x}_a) = P(x_L) = \tilde{k}_L \quad \text{for } \alpha = 1, \ldots, n, \quad \text{(2.75)} \]
where the reduced momenta \( \tilde{p}_{ai} \) and \( \tilde{k}_L \) are defined in Eqs. (A46) and (A47). It is noted that in this case the momentum of the spectator is not changed, as shown in Eq. (2.74), and the momenta of all the other elements \( \tilde{x}_a \) with \( \alpha = 1, \ldots, n \) are changed into \( \tilde{k}_L \) with \( L = 1, 2, \ldots, (n + 1) \), skipping the index \( i \), as shown in Eq. (2.75).

Next we deal with the definition of set \( \{y\} \), which is made from a reduced Born process. Under the input \( R_i \) the reduced Born process \( B_j \) that belongs to Dipole \( j \) is made by the rules shown in Eq. (2.29). One Born process \( B_j \) determines set \( \{y\} \) with the field species and the momenta as
\[ \{y\} = \{y_a, y_b; y_1, \ldots, y_n\}, \quad \text{(2.76)} \]
\[ F(\{y\}) = \{F(y_a), F(y_b); F(y_1), \ldots, F(y_n)\}, \quad \text{(2.77)} \]
\[ P(\{y\}) = \{P(y_a), P(y_b); P(y_1), \ldots, P(y_n)\}. \quad \text{(2.78)} \]

The number of elements of set \( \{y\} \) is \( (n + 2) \), which is the same as set \( \{\tilde{x}\} \). We can always find a bijection (one-to-one correspondence) mapping from set \( \{\tilde{x}\} \) to set \( \{y\} \) as
\[ y_{\beta} = f(\tilde{x}_a), \quad \text{(2.79)} \]
with the indices $\alpha, \beta = a, b, 1, \ldots, n$, which satisfies two conditions:

- $F(y_\beta) = F(\tilde{x}_\alpha)$,
- The argument $\tilde{x}_\alpha$ and the image $y_\beta$ are both in either the final or the initial state.

We call the mapping, $f$, the field mapping. The two conditions mean that the mapping, $f$, connects the elements whose species are identical, and it does not mix the elements in the initial and final states. The inverse mapping is denoted as $\tilde{x}_\alpha = f^{-1}(y_\beta)$. After the construction of a mapping, the element $\tilde{x}_\alpha$ is identified with the image $y_\beta$. Using the inverse mapping, the identification of the elements is generally written as

$$B_j : (y_a, y_b; y_1, \ldots, y_n) = \left( f^{-1}(y_a), f^{-1}(y_b); f^{-1}(y_1), \ldots, f^{-1}(y_n) \right), \quad (2.80)$$

where the elements are sorted by the order of elements of set $\{y\}$. The momenta of set $\{y\}$ are defined as

$$P(y_\beta) = P(f^{-1}(y_\beta)) = P(\tilde{x}_\alpha). \quad (2.81)$$

For convenience, we introduce the notations $y_{emi}$ and $y_{spe}$, defined as $y_{emi} = f(x_{Ij})$, and $y_{spe} = f(x_{I})$. One example of the field mapping is shown in Fig. 1, where the number of final states of an input process $R_i$ is assumed to be $(n + 3) = 6$. The dipole term with the combination $(I, J, K) = (x_1, x_3, x_4)$ is selected.

Using the field mapping, the concrete form of each dipole term can be clearly expressed as

$$D(\text{dip})_{IJ,K} = \left\{ y \right\}_{\text{dip}} \left| \begin{array}{c} B_j = \{y_a, y_b; y_1, \ldots, y_n\} | T_f(x_{Ij}) \cdot T_f(x_{I}) V^{f(x_{Ij})}_{IJ,K} | B_j \end{array} \right\}. \quad (2.82)$$

Using the notation $y_{emi}$ and $y_{spe}$, the concrete form of each dipole term is slightly simplified as

$$D(\text{dip})_{IJ,K} = \left\{ y \right\}_{\text{dip}} \left| \begin{array}{c} B_j = \{y\} | T_{y_{emi}} \cdot T_{y_{spe}} V^{y_{emi}}_{IJ,K} | B_j \end{array} \right\}. \quad (2.83)$$

The subscript indices $I$, $J$, and $K$ in the quantities $s_{IJ}$, $x_{IJ,K}$, and $V^{y_{emi}}_{IJ,K}$ refer to the elements of set $\{x\}$ and the momenta $\{p_a, p_b, p_1, \ldots, p_{n+1}\}$ in Eq. (2.24). The indices $y_{emi}$ and $y_{spe}$ in the operators $T_{y_{emi}}$, $T_{y_{spe}}$, and $V^{y_{emi}}_{IJ,K}$ refer to the elements of set $\{y\}$. The Casimir operator $T^2_{F(y_{emi})}$ is defined as
Fig. 2. The structure of the square of the reduced Born process with the helicity and color correlation, \( (B_j | T_{\text{yemi}} \cdot T_{\text{yspe}} V_{\text{yemi}}^{\text{yemi}} | B_j) \), is shown. The inner product of two color operators is denoted as \( T \cdot T = \sum_c T^c \cdot T^c \).

the constants \( C_F = 4/3 \) in the case \( F(y_{\text{emi}}) = \) quark, and \( C_A = 3 \) in the case \( F(y_{\text{emi}}) = \) gluon. The momenta input into the reduced Born amplitude \( B_j \) are the momenta \( P(y_{\beta}) \), defined in Eq. (2.81).

The legs of the Born process, on which the color and helicity operators act, are clearly specified on the basis of set \( \{ y \} \). The action of the color and helicity operators in the square of the reduced Born process is illustrated in Fig. 2. In the DSA, the field species \( F(\{ y \}) \) of the reduced Born process \( B_j \) are fixed in one category \( \text{Dipole}_j \); namely, an identical set of field species, \( \{ F(y_a), F(y_b); F(y_1), \ldots, F(y_n) \} \), is used for all the dipole terms that belong to the category \( \text{Dipole}_j \). The momenta of \( B_j \), \( P(\{ y \}) = \{ P(y_a), P(y_b); P(y_1), \ldots, P(y_n) \} \), are generally different functions of the original momenta \( \{ p_a, p_b, p_1, \ldots, p_{n+1} \} \) associated with set \( \{ x \} \) depending on the choice \((IJ, K)\).

2.2.3. Concrete formulae

In this section, we specify the full expressions of the dipole terms. The expressions are separated into cases in which the field species of the emitter, \( F(y_{\text{emi}}) \), is a quark or a gluon. In the case in which \( F(y_{\text{emi}}) = \) quark, the expression in Eq. (2.83) is written as

\[
D(\text{dip}_{ij}, (1)_{ij,k}) = -\frac{1}{s_{IJ}} \frac{1}{x_{IJK}} \frac{1}{C_F} V_{ij,k}^y (B_j | T_{y_{\text{emi}}} \cdot T_{y_{\text{spe}}} V_{y_{\text{emi}}^{\text{yemi}}} | B_j),
\]

where the helicity correlation of the dipole splitting function \( V_{ij,k}^y \) disappears and the function is fully factorized to the reduced Born amplitude. In the case in which \( F(y_{\text{emi}}) = \) gluon, the expression is written as

\[
D(\text{dip}_{ij})_{IJ,K} = -\frac{1}{s_{IJ}} \frac{1}{x_{IJK}} \frac{1}{C_A} (B_j | T_{y_{\text{emi}}} \cdot T_{y_{\text{spe}}}, V_{ij,k}^y_{\text{yemi}} | B_j).
\]

The formulae for the dipole terms in all categories in Eq. (2.25) are collected in Appendix A1. Here we select two examples.

The first example is in the category \( \text{Dipole} 1 (1)-1 \). In this category, the dipole terms are given in Eq. (A3) as

\[
D(\text{dip} 1, (1)-1)_{ij,k} = -\frac{1}{s_{ij}} \frac{1}{C_F} V_{ij,k} (B_1 | T_{y_{\text{emi}}} \cdot T_{y_{\text{spe}}} | B_1),
\]

and the constants \( C_F = 4/3 \) in the case \( F(y_{\text{emi}}) = \) quark, and \( C_A = 3 \) in the case \( F(y_{\text{emi}}) = \) gluon.
where the dipole splitting function is given in Eq. (A4) as

\[
V_{ij,k} = 8\pi\alpha_s C_F \left[ \frac{2}{1 - z_i(1 - y_{ij,k})} - 1 - z_i \right].
\]  

(2.87)

The reduced momenta are given in Eqs. (A31) and (A32). The scalars \(y_{ij,k}\) and \(z_i\) appearing in the above formulae are defined in Eqs. (A34) and (A35).

The second example is the category Dipole 3 (6)-2, which is given in Eq. (A25) as

\[
D(\text{dip3}, (6)-2)_{ai,b} = -\frac{1}{s_{ai}} \frac{1}{x_{i,ab}} \frac{1}{C_A} \left( B^3 | T_{\text{emi}} T_{\text{spe}} V_{\text{emi},k}^{ai} B^3 \right),
\]  

(2.88)

where the dipole splitting function is given in Eq. (A26) as

\[
V_{\text{emi}, i}^{\mu\nu} = 8\pi\alpha_s C_F \left[ -g^{\mu\nu} x_{i,ab} + \frac{1}{2} \frac{p_a \cdot p_b}{p_i \cdot p_a p_i \cdot p_b} \left( p_i^\mu - \frac{p_i p_a}{p_b p_a} p_b^\mu \right) \left( p_i^\nu - \frac{p_i p_a}{p_b p_a} p_b^\nu \right) \right].
\]  

(2.89)

The reduced momenta \(\vec{p}_{ai}\) and \(\vec{k}_L\) are defined in Eqs. (A46) and (A47). The scalar \(x_{i,ab}\) is defined in Eq. (A48). Sometimes it is convenient that the gluon polarization vector in the matrix element is taken in the basis of the helicity eigenstate as the circular polarization vector. Formulae for the dipole terms with the helicity correlation in the helicity basis are constructed in Refs. [99] and [93].

### 2.2.4. Examples

In order to demonstrate the creation algorithm, we take the dijet process \(pp \rightarrow 2\) jets, and select one input real process as

\[
\text{Input} : R_1 = u\bar{u} \rightarrow u\bar{u}g.
\]  

(2.90)

The input defines set \(\{x\}\) with the field species and the momenta as

\[
\{x\} = \{x_a, x_b; x_1, x_2, x_3\},
\]  

(2.91)

\[
F(\{x\}) = \{u, \bar{u}; u, \bar{u}, g\},
\]  

(2.92)

\[
\text{Momenta} : \{p_a, p_b; p_1, p_2, p_3\}.
\]  

(2.93)

Following the creation order in Eq. (2.25) and in Fig. A1, we create all the dipole terms from the input \(R_1\) by choosing the three legs \((x_I x_J, x_K)\) as

\[
\text{Dipole1 (1) - 1} : (13, 2), (23, 1),
\]

\[
- 2 : (13, a), (13, b), (23, a), (23, b),
\]

\[
(3) - 1 : (a3, 1), (a3, 2), (b3, 1), (b3, 2),
\]

\[
- 2 : (a3, b), (b3, a),
\]

\[
\text{Dipole2u (5) - 1} : (12, 3),
\]

\[
- 2 : (12, a), (12, b),
\]

\[
\text{Dipole3u (6) - 1} : (a1, 2), (a1, 3),
\]

\[
- 2 : (a1, b),
\]

\[
\text{Dipole3\bar{u} (6) - 1} : (b2, 1), (b2, 3),
\]

\[
- 2 : (b2, a),
\]  

(2.94)

where the notation \((IJ, K)\) is an abbreviation of \((x_I x_J, x_K)\). We select four dipole terms and write down the concrete expressions.
Example 1: Dipole1(1)-1:(13,2)  
Set \{\tilde{x}\} is defined with the field species and the momenta as
\[
\{\tilde{x}\} = \{\tilde{x}_a, \tilde{x}_b; \tilde{x}_1, \tilde{x}_2\} = \{x_a, x_b; x_{\tilde{1}}, x_{\tilde{2}}\},
\]
(2.95)
\[
F(\{\tilde{x}\}) = \{F(\tilde{x}_a), F(\tilde{x}_b); F(\tilde{x}_1), F(\tilde{x}_2)\}
\]
(2.96)
\[
P(\{\tilde{x}\}) = \{P(\tilde{x}_a), P(\tilde{x}_b); P(\tilde{x}_1), P(\tilde{x}_2)\}
\]
(2.97)
where the reduced momenta \(\tilde{p}_{13}\) and \(\tilde{p}_2\) are defined in Eqs. (A31) and (A32). Next we fix the reduced Born process as \(B1 = u\tilde{u} \rightarrow u\tilde{u}\), which defines set \{y\} with the field species and the momenta as
\[
\{y\} = \{y_a, y_b; y_1, y_2\},
\]
(2.98)
\[
F(\{y\}) = \{u, \bar{u}; u, \bar{u}\},
\]
(2.99)
\[
P(\{y\}) = \{P(y_a), P(y_b); P(y_1), P(y_2)\}.
\]
(2.100)
A field mapping \(y_\beta = f(\tilde{x}_a)\) is found as
\[
f(\tilde{x}_a) = y_a,
\]
\[
f(\tilde{x}_b) = y_b,
\]
\[
f(\tilde{x}_1) = y_1,
\]
\[
f(\tilde{x}_2) = y_2.
\]
(2.101)
The field mapping is unique in this case. The field mapping is interpreted as the identification between the elements in sets \{\tilde{x}\} and \{y\} as
\[
(y_a, y_b; y_1, y_2) = (\tilde{x}_a, \tilde{x}_b; \tilde{x}_1, \tilde{x}_2) = (x_a, x_b; x_{\tilde{1}}, x_{\tilde{2}}).
\]
(2.102)
Since the field species of set \{y\} are fixed in the category Dipole1, the expression in Eq. (2.102) can be abbreviated without confusion as
\[
(a, b ; \tilde{1}_3, \tilde{2}).
\]
(2.103)
The momenta \(P(\{y\})\) are determined as
\[
P(\{y\}) = \{P(y_a), P(y_b); P(y_1), P(y_2)\}
\]
\[
= \{P(x_a), P(x_b); P(x_{\tilde{1}}), P(x_{\tilde{2}})\}
\]
(2.104)
\[
P(\{y\}) = \{p_a, p_b; \tilde{p}_{13}, \tilde{p}_2\}.
\]
Then the dipole term is given in Eq. (A3) and is written down as
\[
D(\text{dipole}-(1)-1)_{13,2} = -\frac{1}{s_{13}} \frac{1}{T_{F(x_{\tilde{1}})}^2} \langle B1 = \{y_a, y_b; y_1, y_2\} |T_{f(x_{\tilde{1}})} \cdot T_{f(x_{\tilde{2}})}| B1 \rangle
\]
\[
= -\frac{1}{s_{13}} \frac{1}{C_F} \langle B1 = \{y\} |T_{y_1} \cdot T_{y_2} | B1 \rangle.
\]
(2.105)
where the dipole splitting function $V_{13,2}$ is written in Eq. (A4) and the reduced momenta are determined in Eq. (2.104). The legs on which two color insertion operators act are clearly specified by referring to the elements $y_1$ and $y_2$ of set $\{y\}$. It is again noted that, in the same category, $\text{Dipole}^j$, in the present case, $\text{Dipole}^1$, the same field species, $\text{F}(\{y\})$, is used. Under the agreements in the DSA, we can abbreviate the expression in Eq. (2.105) as

$$D_{13,2} = -\frac{1}{s_{13}} \frac{1}{C_F} V_{13,2} \langle 1, 2 \rangle, \tag{2.106}$$

with the definition $\langle 1, 2 \rangle = \langle B_1 | T_{y_1} \cdot T_{y_2} | B_1 \rangle$. While the indices 1, 2, and 3 in the quantities $s_{13}$ and $V_{13,2}$ refer to the momenta of the original legs of $R_1$, $\{p_a, p_b; p_1, p_2, p_3\}$, the arguments ‘1,2’ inside the brackets $\langle \rangle$ refer to the elements of set $\{y\}$.

- **Example 2:** Dipole$^1(1)$-1: (23,1)

Set $\{\tilde{x}\}$ is defined as

$$\{\tilde{x}\} = \{\tilde{x}_a, \tilde{x}_b; \tilde{x}_1, \tilde{x}_2\} = \{x_a, x_b; x_{\tilde{2}3}, x_{\tilde{1}}\}, \tag{2.107}$$

$$\text{F}(\{\tilde{x}\}) = \{u, \tilde{u}; \tilde{u}, u\}, \tag{2.108}$$

$$\text{P}(\{\tilde{x}\}) = \{p_a, p_b; \tilde{p}_1, \tilde{p}_23\}. \tag{2.109}$$

The definition of set $\{y\}$ is the same as in Eqs. (2.98), (2.99), and (2.100). The field mapping is found as

$$f(\{\tilde{x}\}) = f(x_a, x_b; x_{\tilde{2}3}, x_{\tilde{1}}) = (y_a, y_b; y_2, y_1). \tag{2.110}$$

The mapping determines the identification of the elements as

$$(y_a, y_b; y_1, y_2) = (x_a, x_b; x_{\tilde{1}}, x_{\tilde{2}3}). \tag{2.111}$$

which can be abbreviated as $(a, b; \tilde{1}, \tilde{2}3)$. The momenta of set $\{y\}$ are determined as

$$\text{P}(\{y\}) = \{\text{P}(y_a), \text{P}(y_b); \text{P}(y_1), \text{P}(y_2)\} = \{\text{P}(x_a), \text{P}(x_b); \text{P}(x_{\tilde{1}}), \text{P}(x_{\tilde{2}3})\} = \{p_a, p_b; \tilde{p}_1, \tilde{p}_23\}. \tag{2.112}$$

The dipole term is written as

$$D_{23,1} = -\frac{1}{s_{23}} \frac{1}{C_F} V_{23,1} \langle 2, 1 \rangle. \tag{2.113}$$

- **Example 3:** Dipole$^2u(5)$-2: (12,a)

Set $\{\tilde{x}\}$ is defined with the field species and the momenta as

$$\{\tilde{x}\} = \{\tilde{x}_a, \tilde{x}_b; \tilde{x}_1, \tilde{x}_2\} = \{x_a, x_b; x_{\tilde{2}3}, x_{\tilde{1}}\}, \tag{2.114}$$

$$\text{F}(\{\tilde{x}\}) = \{u, \tilde{u}; g, g\}, \tag{2.115}$$

$$\text{P}(\{\tilde{x}\}) = \{p_a, p_b; \tilde{p}_{12}, p_3\}. \tag{2.116}$$
The reduced Born process is fixed as $B2 = u\bar{u} \rightarrow gg$, which determines set \{y\} with the field species and the momenta as

\[
\{y\} = \{y_a, y_b; y_1, y_2\},
\]
\[
F(\{y\}) = \{u, \bar{u}; g, g\},
\]
\[
P(\{y\}) = \{P(y_a), P(y_b); P(y_1), P(y_2)\}.
\]

In this case, the field mapping has two possibilities due to the two identical fields in the final state, two gluons, as

\[
\{x_\alpha, x_\beta; x_1, x_2\} = \{y_a, y_b; y_1, y_2\} \quad \text{or} \quad \{y_a, y_b; y_1, y_1\}.
\]

Both field mappings are equally allowed. To write down the concrete expression, either of the two possibilities must be chosen. Here we choose the first case in Eq. (2.120). The identification of the elements is written as

\[
\{y_a, y_b; y_1, y_2\} = \{x_\alpha, x_\beta; x_1, x_2\},
\]
which is abbreviated as $(\alpha, \beta; \tilde{1}2, 3)$. The momenta is determined as

\[
P(\{y\}) = \{P(y_a), P(y_b); P(y_1), P(y_2)\} = \{\tilde{p}_a, p_b; \tilde{p}_{12}, p_3\}.
\]

Referring to the formula in Eq. (A21), the dipole term is written as

\[
D(\text{dip}2u, (5)−2)_{12,a} = \frac{1}{s_{12}} \frac{1}{x_{12,a}} \left\{ \frac{B2u = \{y\} | T_f(x_{12}) \cdot T_f(x_{12}) | V_{12,a}^{f(x_{12})} | B2u} \right. \biggr| B2u \biggr| \frac{1}{T_f(x_{12})} \right\}
\]
\[
= \frac{1}{s_{12}} \frac{1}{x_{12,a}} \frac{1}{C_A} \left\{ \frac{B2u | T_{y_1} \cdot T_{y_2} | V_{12,a}^{y_1} | B2u} \right. \biggr| \left\{ 1, a, V_{12,a}^{y_1} \right\}.
\]

Keeping in mind the reduced Born process $B2u$, the expression is abbreviated as

\[
D_{12,a} = \frac{1}{s_{12}} \frac{1}{x_{12,a}} \frac{1}{C_A} \left\{ 1, a, V_{12,a}^{y_1} \right\}.
\]

- **Example 4:** Dipole3u (6)-2: (a1,b)

Set $\{	ilde{x}\}$ is defined as

\[
\{\tilde{x}\} = \{\tilde{x}_\alpha, \tilde{x}_\beta; \tilde{x}_1, \tilde{x}_2\}
\]
\[
= \{x_{\tilde{a}_1}, x_{\tilde{b}_1}; x_2, x_3\},
\]
\[
F(\{\tilde{x}\}) = \{g, \tilde{u}; \tilde{u}, g\},
\]
\[
P(\{\tilde{x}\}) = \{P(\tilde{x}_a), P(\tilde{x}_b); P(\tilde{x}_1), P(\tilde{x}_2)\}
\]
\[
= \{\tilde{p}_{a1}, p_{b1}; \tilde{k}_2, \tilde{k}_3\}.
\]

Next we fix the reduced Born process as

\[
B3u = g\tilde{u} \rightarrow \tilde{u}g,
\]
which determines set \{y\} as

\[
\{y\} = \{y_a, y_b; y_1, y_2\},
\]
\[
F(\{y\}) = \{g, \tilde{u}; \tilde{u}, g\},
\]
\[
P(\{y\}) = \{P(y_a), P(y_b); P(y_1), P(y_2)\}.
\]
The field mapping is uniquely found as
\[ f([\tilde{\mathbf{x}}]) = f(x_{a1}, x_{b}; x_2, x_3) = (y_a, y_b; y_1, y_2), \] (2.132)
which is interpreted as the identification as
\[ (y_a, y_b; y_1, y_2) = (x_{a1}, x_{b}; x_2, x_3). \] (2.133)
The expression is abbreviated as \((a, \tilde{b}; 2, 3)\). The momenta are also determined as
\[ P(\{y\}) = \{P(x_{a1}), P(x_{b}); P(x_2), P(x_3)\}, \]
\[ = \{p_{a1}, p_b; \tilde{k}_2, \tilde{k}_3\}. \] (2.134)
The dipole term is written as
\[ D(\text{dip}3u, (6)\!-\!2)_{a1,b} = -\frac{1}{s_{a1}} \frac{1}{x_{1,ab}} \frac{1}{T_{\bar{F}(x_{a1})}} \left\{ B3u^{'T_{\bar{F}(x_{a1})}} : T_{f(x_{a1})} \cdot T_{f(x_{b})} V_{a1,b}^{f(x_{a1})} \right\} B3u \]
\[ = -\frac{1}{s_{a1}} \frac{1}{x_{1,ab}} \frac{1}{C_A} \left\{ B3u^{T_{y_{a}} : T_{y_{b}} V_{a1,b}^{y_{a}} \right\} B3u' \}, \] (2.135)
which is abbreviated as
\[ D_{a1,b} = -\frac{1}{s_{a1}} \frac{1}{x_{1,ab}} \frac{1}{C_A} \{a, b, V_{a1,b}^{y_{a}}\}. \] (2.136)
For this dipole term, we have another possibility to fix the reduced Born process as
\[ B3u' = \tilde{u} g \rightarrow \tilde{u} g, \] (2.137)
which defines set \(\{y\}\) as
\[ \{y\} = \{y_{a}, y_{b}; y_1, y_2\}. \] (2.138)
\[ F(\{y\}) = \{\tilde{u}, g; \tilde{u}, g\}, \] (2.139)
\[ P(\{y\}) = \{P(y_{a}), P(y_{b}); P(y_1), P(y_2)\}. \] (2.140)
In this case, the field mapping is found as
\[ f([\tilde{\mathbf{x}}]) = f(x_{a1}, x_{b}; x_2, x_3) = (y_b, y_a; y_1, y_2), \] (2.141)
which is interpreted as
\[ (y_a, y_b; y_1, y_2) = (x_{a1}, x_{b}; x_2, x_3). \] (2.142)
This is abbreviated as \((b, \tilde{a}; 2, 3)\). The momenta are determined as
\[ P(\{y\}) = \{P(x_{b}), P(x_{a1}); P(x_2), P(x_3)\}, \]
\[ = \{p_b, \tilde{p}_{a1}; \tilde{k}_2, \tilde{k}_3\}. \] (2.143)
The dipole term is written as
\[ D_{a1,b} = -\frac{1}{s_{a1}} \frac{1}{x_{1,ab}} \frac{1}{T_{\bar{F}(x_{a1})}} \left\{ B3u'^{T_{f(x_{a1})}} : T_{f(x_{b})} \cdot T_{f(x_{a1})} V_{a1,b}^{f(x_{a1})} \right\} B3u' \]
\[ = -\frac{1}{s_{a1}} \frac{1}{x_{1,ab}} \frac{1}{C_A} \left\{ B3u'^{T_{y_{b}} : T_{y_{a}} V_{a1,b}^{y_{b}} \right\} B3u' \}, \] (2.144)
which is abbreviated, keeping B3u’ in mind, as
\[ D_{a1,b} = -\frac{1}{s_{a1}} \frac{1}{x_{1,ab}} \frac{1}{C_A} \{b, a, V_{a1,b}^{b}\}. \] (2.145)
In the first case of B3u, the element x_{a1} is identified with the element y_{a} and, in the second case of B3u’, x_{a1} is identified with y_{b}. The first case appears to be a simpler expression in the sense
that the leg \(a\) of the input process \(R_1\) is connected to the leg \(a\) of the reduced Born process \(B3u\).

For this reason, the first case may be more favored than the second.

2.2.5. Summary

The hadronic cross section of a real correction subtracted by the dipole terms is written as

\[
\sigma_R(R_i) - \sigma_D(R_i) = \int dx_1 \int dx_2 \ f_{F(x_1)}(x_1) f_{F(x_2)}(x_2) \ (\hat{\sigma}_R(R_i) - \hat{\sigma}_D(R_i)),
\]

(2.146)

which is part of Eq. (2.4). The partonic cross section is written as

\[
\hat{\sigma}_R(R_i) - \hat{\sigma}_D(R_i) = \frac{1}{\mathcal{S}_{R_i}} \Phi(R_i) \cdot \left[ |M(R_i)|^2 - \frac{1}{n_s(a)n_s(b)} D(R_i) \right].
\]

(2.147)

It is sufficient that the real correction \( |M(R_i)|^2 \) and the dipole term \( D(R_i) \) are obtained in 4 dimensions. The real correction \( |M(R_i)|^2 \) is summed and averaged over both the spin and the color. The dipole term \( D(R_i) \) is the summation of all the dipole terms under the input \( R_i \) and is separated into subcategories as

\[
D(R_i) = \sum_{j=1}^{4} D(R_i, \text{dip} j).
\]

(2.148)

In each category \( \text{dip} \ j \), the reduced Born process \( B j \) is fixed, which defines set \( \{y\} \) with the field species as

\[
B j \rightarrow \{y\} = \{y_1, y_2; \ldots, y_n\},
\]

\[
F(\{y\}) = \{F(y_1), \ldots, F(y_n)\}.
\]

(2.149)

Once the reduced Born process and set \( \{y\} \) are fixed, the necessary information to specify each dipole is the three elements of set \( \{x\} \) and the field mapping as

1. \((x_Ix_J, x_K), \)

(2.150)

2. \((y_a, y_b; y_1, \ldots, y_n) = (f^{-1}(y_a), f^{-1}(y_b); f^{-1}(y_1), \ldots, f^{-1}(y_n)) = (x_a, x_b; x_1, \ldots, x_{IJ}, \ldots, x_{K}, \ldots, x_{n+1}). \)

(2.151)

The form to specify the information can be abbreviated as

1. \((IJ, K), \)

2. \((a, b; 1, \ldots, IJ, \ldots, K, \ldots, n + 1). \)

(2.152)

Using the notation \( y_{\text{emi}} = f(x_{IJ}) \) and \( y_{\text{spe}} = f(x_{K}) \), each dipole term is simply written down as

\[
D(R_i, \text{dip} j)_{IJ, K} = \frac{1}{s_{IJ} x_{IK}} \frac{1}{T_{F(y_{\text{emi})}}} (B j \left| T_{y_{\text{emi}}} \cdot T_{y_{\text{spe}}} \right| y_{\text{emi}}_{IJ, K} \left| B j \right). \]

(2.153)

which is abbreviated as

\[
D(\text{dip} j)_{IJ, K} = \frac{1}{s_{IJ} x_{IK}} \frac{1}{T_{y_{\text{emi}}}^{2}} \left( y_{\text{emi}} \cdot y_{\text{spe}} \right) \left| y_{\text{emi}}_{IJ, K} \right| \).
\]

(2.154)

The square of the reduced Born amplitude with color and spin correlations, \( \langle B j | T \cdot T V | B j \rangle \), is summed and averaged over the color degree of freedom. It is also summed over the spin configurations, but not averaged. Instead, the dipole terms are divided by the spin average factor of the input real process, as shown in Eq. (2.147). It is also noted that the symmetric factor by which the dipole terms are divided is not the symmetric factor of the reduced Born processes \( S_{B j} \), but the symmetric factor of the input process \( S_{R_i} \), as shown in Eq. (2.147).
2.3. Step 3: I term creation

In this section, ‘Step 3: \( I(R_i) \)’ is explained. The input and output of this step are:

- **Input:** \( B_1(R_i) \)
- **Output:** \( I(R_i) \).

\( B_1(R_i) \) is the reduced Born process of the category Dipole 1, which is made from the input process \( R_i \) on the rule \( B_1 = R_i - g_f \), as shown in Eq. (2.29).

The creation algorithm given in the original article [27] is to choose all combinations of two elements \( (y_I, y_K) \) without duplicate from the set \( \{y\} \) of the process \( B_1(R_i) \), where the elements to be chosen are quark or gluon as \( F(y_I/K) = \text{quark or gluon} \). In the DSA, the creation algorithm of the I terms is divided into substeps as follows:

1. Choose all possible elements \( y_I \) from set \( \{y\} \) in the order in Fig. A2 in Appendix A2.
2. Choose all the possible elements \( y_K \) from set \( \{y\} \) for each choice of the element \( y_I \).
3. Write down the concrete expressions of all the I terms.

Substeps 1 and 2 are explained in Sect. 2.3.1. Substep 3 is explained in Sects. 2.3.2 and 2.3.3. Some examples are shown in Sect. 2.3.4. Finally, we give a summary in Sect. 2.3.5. The formulae for the I terms are collected in Appendix A2.

2.3.1. Creation order

In the DSA, the order to choose the first element \( y_I \) is determined as

\[
(1) \text{f}_\text{fin}, \quad (2) \text{g}_\text{fin}, \quad (3) \text{f}_\text{ini}, \quad (4) \text{g}_\text{ini},
\]

(2.155)

where \( f_{\text{fin/ini}} \) represents a quark in the final/initial state and \( g_{\text{fin/ini}} \) represents a gluon in the final/initial state, as shown in Fig. A2. Each choice of \( y_I \) is followed by the choice of the second element \( y_K \). The choice of \( y_K \) in the final state is first and the choice in the initial state is second, which are denoted as subcategories 1 and 2 respectively. Then the creation order is written as

\[
(1) -1/2, \quad (2) -1/2, \quad (3) -1/2, \quad (4) -1/2.
\]

(2.156)

Each pair \( (y_I, y_K) \) specifies each I term, which is denoted as \( I(R_i)_{I, K} \). The summation of all the created I terms is the output I \( (R_i) \), which is written as

\[
I(R_i) = \sum_{(I, K)} I(R_i)_{I, K}.
\]

(2.157)

When all \( (n+2) \) legs of \( B_1 \), and, equally, all \( (n+2) \) elements of set \( \{y\} \), are quark or gluon, the indices \( I \) and \( K \) run over \( I, K = a, b, 1, \ldots, n \), with the condition \( I \neq K \), and the total number of I terms is \( (n+2)(n+1) \).

2.3.2. Concrete formulae

The concrete expression of each I term is given in the universal form as

\[
I_{I, K} = -A_d \frac{1}{T_{F(y_I)}} \mathcal{V}_{F(y_I)} s_{IK}^{\varepsilon} \cdot \langle \{y_a, y_b; y_1, \ldots, y_n\} | T_{y_I} \cdot T_{y_K} | B_1 \rangle,
\]

(2.158)

where the common factor \( A_d \) is defined as

\[
A_d = \frac{\alpha_s}{2\pi} \frac{(4\pi \mu^2)^\varepsilon}{\Gamma(1-\varepsilon)}.
\]

(2.159)

with the free parameter \( \mu \) introduced in the dimensional regularization with \( d = 4 - 2\varepsilon \). The definition of the Casimir operator \( T_{F(y_I)}^2 \) is the same as in Eq. (2.83). The universal singular function
\[ \mathcal{V}_{\text{F}(y)} \] is defined as
\[ \mathcal{V}_f = \mathcal{V}_{fg}(\epsilon) \]
\[ \mathcal{V}_g = \frac{1}{2} \mathcal{V}_{gg}(\epsilon) + N_f \mathcal{V}_{f}(\epsilon), \]
where the singular functions \( \mathcal{V}_{fg}(\epsilon) \), \( \mathcal{V}_{gg}(\epsilon) \), and \( \mathcal{V}_{f}(\epsilon) \) are written in Eqs. (A56), (A57), and (A58), respectively. The symbol \( N_f \) represents the number of massless quark flavors. The momenta of the reduced Born process B1 are written as
\[ P((y)) = \{P(y_a), P(y_b); P(y_1), \ldots, P(y_n)\}, \]
which are defined as the momenta of the \( n \)-body PS. The scalar \( s_{IK} \) is denoted as \( s_{IK} = 2P(y_I) \cdot P(y_K) \). The partonic cross section of the I term is written in Eq. (2.8) as
\[ \hat{\sigma}_I = \frac{1}{S_{\text{B1}}} \Phi(B1) d \cdot I(R_d). \]
where the n-body PS integration is defined in Eq. (2.12) as
\[ \Phi(B1)d = \frac{1}{\mathcal{F}(P(y_a), P(y_b))} \prod_{i=1}^{n} \int d^{d-1}P(y_i) \frac{1}{(2\pi)^{d-1}} (2\pi)^d \delta(d) \left( P(y_a) + P(y_b) - \sum_{i=1}^{n} P(y_i) \right). \]
The square of the color-correlated Born amplitude, \( \langle B1|T_{y_I} \cdot T_{y_K} | B1 \rangle \), is obtained in \( d \) dimensions, and summed and averaged over the spin and color including the spin average factor, which is in contrast to the case of the dipole terms in Eq. (2.6). Again, in contrast to ‘Step 2. D(R_d)’, in Step 3 only one process, B1, and only one set, \( \{y\} \), appear. Then we can drop the specification of B1 in Eq. (2.158) as
\[ I_{I,K} = -A_d \prod_{I} \mathcal{V}_{I}^{(I)} [ I, K ], \]
where we introduce the following notation for convenience:
\[ [ I, K ] = s_{IK}^{\epsilon} \langle B1|T_{y_I} \cdot T_{y_K} | B1 \rangle. \]

2.3.3. Complete set
Each I term \( I_{I,K} \) includes the square of a reduced Born amplitude with color correlations as \( (T_I \cdot T_K) \). The summation of I terms \( I = \sum_{(I,K)} I_{I,K} \) includes the square of a Born amplitude with all combinations of the pairs \( (I, K) \). We call the set that consists of the elements \( (T_I \cdot T_K) \) with all the combinations of \( (I, K) \) ‘the complete set of the square of the color-correlated Born amplitude B1’. The name is sometimes abbreviated as ‘the complete set of B1’, which is explicitly written down as
\[ \langle \langle B1|T_I \cdot T_K | B1 \rangle \rangle_{\text{comp}} = \{ \langle a, b \rangle, \langle a, 1 \rangle, \langle a, 2 \rangle, \ldots, \langle a, n \rangle, \langle b, a \rangle, \langle b, 1 \rangle, \langle b, 2 \rangle, \ldots, \langle b, n \rangle, \langle 1, a \rangle, \langle 1, b \rangle, \langle 1, 2 \rangle, \ldots, \langle 1, n \rangle, \langle 2, a \rangle, \langle 2, b \rangle, \langle 2, 1 \rangle, \ldots, \langle 2, n \rangle, \ldots, \langle n, a \rangle, \langle n, b \rangle, \langle n, 1 \rangle, \ldots, \langle n, n - 1 \rangle \}, \]
where all the legs of the reduced Born process B1 are assumed to be quark or gluon. The number of elements is \( (n + 2)(n + 1) \), which is the same as the number of I terms. The complete set
of B1 is always included in the dipole terms in the category Dipole1, because the dipole terms D(Ri, dipl) include the square of the reduced Born amplitude B1 with all combinations of the pair (yemi, yspe) shown in Eq. (2.83) as ⟨B1 | T_{yemi} · T_{yspe} | B1⟩. Once we have obtained analytical or numerical expressions for the complete set of B1 as a function of the arbitrary input momenta {P(ya), P(yb); P(y1), . . . , P(yn)} for the calculation of the dipole terms, the expressions can be used again for the calculation of the I terms as well. Such a reuse of expressions can save a certain amount of work in constructing the subtraction terms.

### 2.3.4. Examples

We show some examples in the same process in Eq. (2.90), R1 = uū → uūg. The input for Step 3 is

\[
B1(R_1) = uū → uū,
\]

which determines set \{y\} with the field species in Eqs. (2.98), (2.99), and (2.100) as

\[
\{y\} = \{y_a, y_b; y_1, y_2\},
\]

\[
F(\{y\}) = \{u, \bar{u}; u, \bar{u}\},
\]

\[
P(\{y\}) = \{P(y_a), P(y_b); P(y_1), P(y_2)\}.
\]

The input momenta into the Born amplitude B1, \{P(y_a), P(y_b); P(y_1), P(y_2)\}, are the momenta in the 2-body phase space in Eq. (2.164). The output I(R1) is simply written as

\[
I(R_1) = \sum_{(I,K)} I(R_1)_{I,K}
\]

\[
= -A_d \frac{\mathcal{V}_f}{C_F} \cdot \{[1, 2] + [2, 1] + [1, a] + [1, b] + [2, a] + [2, b] + [a, 1] + [a, 2] + [b, 1] + [b, 2] + [a, b] + [b, a]\}. \tag{2.174}
\]

I(R1) is calculated in terms of the complete set of B1 = uū → uūg, \{(B1 | T_I · T_K | B1)\}_{comp}. The number of elements of the complete set is twelve, which exactly corresponds to the twelve, ⟨B1 | T_{yemi} · T_{yspe} | B1⟩, included in the dipole terms in category Dipole1 in Eq. (2.94).
2.3.5. Summary

The contributions of the virtual correction and the I term to the hadronic cross section are written as

\[ \sigma_v(B1) + \sigma_I(R_i) = \int dx_1 \int dx_2 \, f_{v(x_1)}(x_1) \, f_{v(x_2)}(x_2) \left( \hat{\sigma}_v(B1) + \hat{\sigma}_I(R_i) \right), \]  

(2.175)

which is part of Eq. (2.4). The partonic cross section is written as

\[ \hat{\sigma}_v(B1) + \hat{\sigma}_I(R_i) = \frac{1}{S_{B1}} \Phi(B1) \cdot \left[ |M_{\text{virt}}(B1)|^2 + I(R_i) \right], \]  

(2.176)

where the virtual correction, \( |M_{\text{virt}}(B1)|^2 \), is obtained in \( d \) dimensions, and summed and averaged over the spin and color. The I term, \( I(R_i) \), is also obtained in \( d \) dimensions, and summed and averaged over the spin and color including the spin average factor. The output \( I(R_i) \) is the summation of all the created I terms as

\[ I(R_i) = \sum_{(I,K)} I(R_i)_{I,K}. \]  

(2.177)

Once we determine the reduced Born process \( B1 \) and the associated set \( \{y\} \), each I term \( I(R_i)_{I,K} \) is specified by the information of the pair \( (I,K) \).

The concrete expression of each I term is written in the universal form as

\[ I(R_i)_{I,K} = -A_d \frac{1}{\hat{\tau}_{F(I)}} \mathcal{V}_{F(I)}[I,K], \]  

(2.179)

with the notation in Eq. (2.166). The universal singular functions are defined in Eqs. (A54) and (A55). The virtual correction has soft and collinear singularities in the form \( 1/\epsilon^2 \) and \( 1/\epsilon \), which are subtracted by the same poles with opposite signs in the I term. After the cancellation of the poles, PS integration is carried out, in 4 dimensions to be finite, as follows:

\[ \hat{\sigma}_v(B1) + \hat{\sigma}_I(R_i) = \frac{1}{S_{B1}} \Phi(B1) \cdot \left[ |M_{\text{virt}}(B1)|^2 + I(R_i) \right]_{4}. \]  

(2.180)

2.4. Step 4: P and K terms creation

In this section, ‘Step 4. P(R_i) and K(R_i)’ is explained. The input and output are written as

Input: \( R_i \) and \( B j(R_i) \),

Output: \( P(R_i) \) and \( K(R_i) \).

The symbol \( B j(R_i) \) represents the reduced Born process of the category Dipole \( j \), which is made from the input process \( R_i \) by the rules in Eq. (2.29). The creation algorithm in the DSA is divided into the following substeps:

1. Take set \( \{x\} \) of the process \( R_i \) and choose all possible pairs \( (x_a, x_i) \) of the splittings in the order in Fig. A3 in Appendix A3.

2. For each pair \( (x_a, x_i) \), take set \( \{y\} \) of the corresponding Born process \( B j \), which is determined in ‘Step 2. D(R_i)’.

3. If \( F(x_{a1}) = F(y_a) \), create the pairs \( (y_a, y_K) \) with \( K = 0, 1, \ldots, n, b \).

   If \( F(x_{a2}) = F(y_b) \), create the pairs \( (y_b, y_K) \) with \( K = 0, 1, \ldots, n, a \).

4. Write down concrete expressions for all the P and K terms.

Substeps 1, 2, and 3 are explained in Sect. 2.4.1. Substep 4 is explained in Sects. 2.4.2 and 2.4.3. Some examples are shown in Sect. 2.4.4. Finally, we give a summary in Sect. 2.4.5. The formulae for the P and K terms are collected in Appendix A3.
2.4.1. Creation order

We take set \( \{ x \} \) of the process \( R_i \) in Eq. (2.22). The creation order is divided into cases with leg-\( a \) \((x_\alpha)\) and leg-\( b \) \((x_\beta)\). We start with the leg-\( a \) case. We choose the possible pairs \((x_\alpha, x_i)\) from set \( \{ x \} \) in the order of the splittings (3), (4), (6), and (7) shown in Fig. A3. The possible splittings are the same as those chosen with leg-\( a \) in ‘Step 2. D\( (R_i) \)’. For each choice of one pair \((x_\alpha, x_i)\), we can always find the corresponding reduced Born process \( B\) and set \( \{ y \} \), which have already been fixed in Step 2. Then, for each choice of one pair \((x_\alpha, x_i)\), we take the corresponding set \( \{ y \} \) and check which of the two relations \( F(x_{\alpha i}^{-}) = F(y_\alpha) \), \( F(x_{\alpha i}^{-}) = F(y_\beta) \) stands. Here \( F(x_{\alpha i}^{-}) \) represents the field species of the element \( x_{\alpha i}^{-} \), defined in Eqs. (2.61) and (2.70). \( F(y_{\alpha/b}) \) are defined in Eq. (2.77). Then we create the P and K terms in the following way:

\[
\text{If } F(x_{\alpha i}^{-}) = F(y_\alpha) \rightarrow \text{Create pairs } : (y_\alpha, y_K) \quad \text{for } K = 0, 1, 2, \ldots, n, b, \tag{2.181}
\]

where each pair creates each P and K terms as

\[
\begin{align*}
\{ P(R_i, x_\alpha : B\, j, y_\alpha, y_K) \} & \quad \text{for } K = 1, 2, \ldots, n, b, \\
\{ K(R_i, x_\alpha : B\, j, y_\alpha, y_K) \} & \quad \text{for } K = 0, 1, 2, \ldots, n, b,
\end{align*}
\tag{2.182}
\]

or

\[
\text{If } F(x_{\alpha i}^{-}) = F(y_\beta) \rightarrow \text{Create pairs } : (y_\beta, y_K) \quad \text{for } K = 0, 1, 2, \ldots, n, a, \tag{2.183}
\]

where each pair creates the P and K terms as

\[
\begin{align*}
\{ P(R_i, x_\alpha : B\, j, y_\beta, y_K) \} & \quad \text{for } K = 1, 2, \ldots, n, a, \\
\{ K(R_i, x_\alpha : B\, j, y_\beta, y_K) \} & \quad \text{for } K = 0, 1, 2, \ldots, n, a.
\end{align*}
\tag{2.184}
\]

Among the elements \( y_K \) with \( K = 1, 2, \ldots, n, a, \) and \( b, \) only the colored fields are taken, namely, \( F(y_K) = \text{quark or gluon}. \) For convenience, we categorize the P and K terms by the type of second leg \( y_K. \) The P and K terms with \( y_K = y_0, y_k \) for \( k = 1, \ldots, n \) and \( y_{\alpha/b}, \) are categorized with the labels \(-0, -1, \) and \(-2), \) respectively. It is noted that, when the final state of \( R_i \) includes identical fields, only one pair of \((x_\alpha, x_i)\) is taken and the other pairs must be discarded. For example, we take the process \( R_i = u\bar{u} \rightarrow ggg \) and the set \( \{ x \} = \{ x_\alpha, x_\beta; x_1, x_2, x_3 \}. \) From the input, we can find three pairs of splitting (3) as

\[
(x_\alpha, x_1), \quad (x_\alpha, x_2), \quad \text{and } (x_\alpha, x_3). \tag{2.185}
\]

Among these three pairs, we can take only one pair, for instance, \((x_\alpha, x_1)\), and must discard the other two pairs \((x_\alpha, x_2)\) and \((x_\alpha, x_3)\). The discard rule is in contrast to Step 2, where all three pairs must be taken for the creation of the dipole terms. The creation order with leg-\( b \) is completely analogous to the case with leg-\( a \).

2.4.2. Concrete formula for the P term

The concrete formula for the P term with leg-\( a \) \((x_\alpha)\) in the case of \( F(x_{\alpha i}^{-}) = F(y_\alpha) \) is written in the universal form as

\[
P(R_i, x_\alpha : B\, j, y_\alpha, y_K) = \frac{\alpha_s}{2\pi} \frac{1}{T_{F(y_\alpha)}^2} \int_{x, x_\alpha y_K} p_{F(y_\alpha)}^a(x) \ln \frac{\mu_F^2}{x_\alpha y_K} \times \langle B\, j = (y_\alpha, y_\beta; y_1, \ldots, y_n) | T_{y_\alpha} T_{y_K} | B\, j \rangle, \tag{2.186}
\]

where the definition of the Casimir operator \( T_{F(y_\alpha)}^2 \) is the same as in Eq. (2.83) and the symbol \( p_{ab}^a(x) \) represents the 4D Altarelli–Parisi splitting function shown in Eqs. (A62), (A63), (A64), and (A65).
The Lorentz scalar $s_{xa,yK}$ is defined as $s_{xa,yK} = 2 p_a \cdot P(y_K)$ with $p_a$ in Eq. (2.24). The square of the reduced Born amplitude with color correlation, $\langle B j | T_{xa} \cdot T_{yK} | B j \rangle$, is obtained in 4 dimensions, which is the same function of the momenta, $P(\{y\})$, in the dipole term in Eq. (2.84), except for the spin average factor. In the P term, the squared amplitude $\langle T \cdot T \rangle$ is summed and averaged over the spin and color including the spin average factor. The input momenta into the Born amplitude are written as

$$P(\{y\}) = \{P(y_a), P(y_b); P(y_1), \ldots, P(y_n)\} \tag{2.187}$$

which are defined in the contribution to the cross section as

$$\hat{\sigma}_p(R_i, x_a : B j, y_a, y_K) = \int_0^1 dx \frac{1}{S_{Bj}} \Phi_a(P(y_a), P(y_b) \rightarrow P(y_1), \ldots, P(y_n))_4 \times P(R_i, x_a : B j, y_a, y_K) \tag{2.188}$$

The $n$-body phase space including the flux factor is defined as

$$\Phi_a(P(y_a), P(y_b) \rightarrow P(y_1), \ldots, P(y_n))_4 = \frac{1}{\mathcal{F}(P(y_a), P(y_b))} \prod_{i=1}^n \int d^3P(y_i) \frac{1}{(2\pi)^3} \frac{1}{2E_{y_i}} \left(\frac{1}{4!} \delta^{(4)}(P(y_a) + P(y_b) - \sum_{i=1}^n P(y_i))\right), \tag{2.189}$$

with the initial momenta $(P(y_a), P(y_b)) = (xpa, pb)$ and the energy $E_{y_i} = P(y_i)^\mu_\mu = 0$. The phase space is the same as in Eq. (2.13) with the identification $p_i = P(y_i)$ for $i = 1, 2, \ldots, n$. The expression for the P term in the case $F(x_{a\,\mu}) = F(y_b)$ is similarly written as

$$P(R_i, x_a : B j, y_b, y_K) = \frac{\alpha_s}{2\pi} \frac{1}{T_{F(y_b)}^2} \rho_{F(x_a)F(y_b)}(x) \ln \frac{\mu_F^2}{s_{xa,yK}} \times \langle B j = \{y_a, y_b; y_1, \ldots, y_n\} | T_{y_b} \cdot T_{yK} | B j \rangle. \tag{2.190}$$

The phase space in this case, $\Phi_a(P(y_a), P(y_b) \rightarrow P(y_1), \ldots, P(y_n))_4$, is the same expression as in Eq. (2.189) and the initial momenta are defined as $P(y_a), P(y_b) = (p_xa, xpa)$. The concrete formulae for the P term with leg-$b$, $P(R_i, x_b : B j, y_b/a, y_K)$, and the phase space $\Phi(b)(P(y_a), P(y_b) \rightarrow P(y_1), \ldots, P(y_n))_4$ are completely analogous to the case with leg-$a$.

We define the output $P(R_i)$ as the set that consists of all the created P terms,

$$P(R_i) = \{P(R_i, x_a), P(R_i, x_b)\} \tag{2.191}$$

where the subset $P(R_i, x_a)$ is defined as

$$P(R_i, x_a) = \{P(R_i, x_a : B1), P(R_i, x_a : B3), P(R_i, x_a : B4)\}. \tag{2.192}$$

Each element $P(R_i, x_a : B j)$ is the summation over the P terms including $B j$ as

$$P(R_i, x_a : B j) = \sum_{k=1}^n P(R_i, x_a : B j, y_a/b, y_k) + P(R_i, x_a : B j, y_a/b, y_b/a). \tag{2.193}$$

In the case with leg-$b$, the set $P(R_i, x_b)$ and the summation $P(R_i, x_b : B j)$ are similarly defined.
2.4.3. Concrete formula for the K term

The concrete formula for the K term with leg-\(a\) (\(x_a\)), in the case of \(F(x_{\bar{a}i}) = F(y_{\bar{a}})\), is separated into the three categories, -0, -1, and -2, introduced above. The formulae are written as

\[
K(R_i, x_a : B_j, y_a, y_0) = \frac{\alpha_s}{2\pi} \tilde{K}^{F(x_a)F(y_a)}(x) \cdot \langle B_j | B_j \rangle,
\]

\[
K(R_i, x_a : B1, y_a, y_k) = \frac{\alpha_s}{2\pi} \frac{\gamma F(y_k)}{T^2_{F(y_k)}} h(x) \cdot \langle B1 | T_{y_a} \cdot T_{y_k} | B1 \rangle,
\]

\[
K(R_i, x_a : B_j, y_a, y_k) = -\frac{\alpha_s}{2\pi} \frac{1}{T^2_{F(y_a)}} \tilde{K}^{F(x_a)F(y_a)}(x) \cdot \langle B_j | T_{y_a} \cdot T_{y_k} | B_j \rangle.
\]

The symbol \(\langle B_j | B_j \rangle\) is the abbreviation of \(\langle B_j = \{y_a, y_b; y_1, \ldots, y_n\} | B_j \rangle\), which is the usual squared amplitudes in the LO process. The symbol \(\langle B_j | T_{y_a} \cdot T_{y_k} | B_j \rangle\) is the abbreviation of \(\langle B_j = \{y_a, y_b; y_1, \ldots, y_n\} | T_{y_a} \cdot T_{y_k} | B_j \rangle\), which is the same quantity as in Eq. (2.186). The functions of the argument \(x\), \(\tilde{K}^{F(x_a)F(y_a)}(x)\), \(h(x)\), and \(\tilde{K}^{F(x_a)F(y_a)}(x)\), and the symbol \(\gamma F(y_k)\) are defined in Appendix A3. It is noted that the K terms with pair \((y_a, y_k)\) with \(k = 1, 2, \ldots, n\) exist only in the case of diagonal splittings, namely, the case including process B1 shown in Eq. (2.195).

In the same way as the P term, the input momenta into the Born amplitude, \(P(|y\rangle)\), are given in the contribution to the cross section as

\[
\hat{\sigma}_K(R_i, x_a : B_j, y_a, y_K) = \int_0^1 dx \frac{1}{S_{Bj}} \Phi_a(P(y_a), P(y_b) \rightarrow P(y_1), \ldots, P(y_n)) \times K(R_i, x_a : B_j, y_a, y_K),
\]

where the \(n\)-body PS, \(\Phi_a\), is the same as in Eq. (2.189). The formulae in the case of \(F(x_{\bar{a}i}) = F(y_{\bar{a}})\) are similarly given as

\[
K(R_i, x_a : B_j, y_b, y_0) = \frac{\alpha_s}{2\pi} \tilde{K}^{F(x_a)F(y_b)}(x) \cdot \langle B_j | B_j \rangle,
\]

\[
K(R_i, x_a : B1, y_b, y_k) = \frac{\alpha_s}{2\pi} \frac{\gamma F(y_k)}{T^2_{F(y_k)}} h(x) \cdot \langle B1 | T_{y_b} \cdot T_{y_k} | B1 \rangle,
\]

\[
K(R_i, x_a : B_j, y_b, y_k) = -\frac{\alpha_s}{2\pi} \frac{1}{T^2_{F(y_b)}} \tilde{K}^{F(x_a)F(y_b)}(x) \cdot \langle B_j | T_{y_b} \cdot T_{y_k} | B_j \rangle.
\]

The formulae for the K term with leg-\(b\), \(K(R_i, x_b : B_j, y_{b/a}, y_K)\), and the phase space \(\Phi_b\) are completely analogous to the leg-\(a\) case.

Again, similar to the case of the P term, we define the output \(K(R_i)\) as the set that consists of all the created K terms,

\[
K(R_i) = \{K(R_i, x_a), K(R_i, x_b)\},
\]

where the subset \(K(R_i, x_a)\) is defined as

\[
K(R_i, x_a) = \{K(R_i, x_a : B1), K(R_i, x_a : B3), K(R_i, x_a : B4)\},
\]

where the elements \(K(R_i, x_a : B_j)\) are the summation of the K terms with the process \(B_j\) as

\[
K(R_i, x_a : B_j) = \sum_{K=0}^{n} K(R_i, x_a : B_j, y_{a/b}, y_K) + K(R_i, x_a : B_j, y_{a/b}, y_{b/a}).
\]

The set \(K(R_i, x_b)\) and the summation \(K(R_i, x_b : B_j)\) are similarly defined.
To demonstrate the creation of the P and K terms, we take the same input process used for the dipole term creation in Eq. (2.90) as

\[
\text{Input : } R_1 = u\bar{u} \rightarrow u\bar{u}g, \tag{2.204}
\]

which defines set \( \{x\} \) with the field species and the momenta in Eqs. (2.91), (2.92), and (2.93) as

\[
\{x\} = \{x_a, x_b; x_1, x_2, x_3\}, \tag{2.205}
\]

\[
F(\{x\}) = \{u, \bar{u}; u, \bar{u}, g\}, \tag{2.206}
\]

\[
\text{Moments : } \{p_a, p_b; p_1, p_2, p_3\}. \tag{2.207}
\]

The possible reduced Born processes \( B_j \) and the associated set \( \{y\} = \{y_a, y_b; y_1, y_2\} \) are specified in **Step 2** as

\[
B_1 : F(\{y\}) = \{u, \bar{u}; u, \bar{u}\}, \tag{2.208}
\]

\[
B_3u : F(\{y\}) = \{g, \bar{u}; \bar{u}, g\}, \tag{2.209}
\]

\[
B_3\bar{u} : F(\{y\}) = \{u, g; u, g\}. \tag{2.210}
\]

where \( B_1 \) and \( B_3u \) are explicitly defined in Eqs. (2.96) and (2.130), respectively. Then we start the creation of the P and K terms with leg-\( a \) \( (x_a) \) as

Dipole1 (3): \( B_1 = \{u, \bar{u}; u, \bar{u}\} \)

\[
(x_a, x_3) \rightarrow F(x_{a3}) = u = F(y_a) \rightarrow 1. (y_a, y_0), 2. (y_a, y_1), 3. (y_a, y_2), 4 \ (y_a, y_b),
\]

Dipole3u (6): \( B_3u = \{g, \bar{u}; \bar{u}, g\} \)

\[
(x_a, x_1) \rightarrow F(x_{a1}) = g = F(y_a) \rightarrow 5. (y_a, y_0), 6. (y_a, y_1), 7. (y_a, y_2), 8. (y_a, y_b).
\]

Next we proceed to the creation with leg-\( b \) as

Dipole1 (3): \( B_1 = \{u, \bar{u}; u, \bar{u}\} \)

\[
(x_b, x_3) \rightarrow F(x_{b3}) = \bar{u} = F(y_b) \rightarrow 9. (y_b, y_0), 10. (y_b, y_1), 11. (y_b, y_2), 12. (y_b, y_a),
\]

Dipole3\( \bar{u} \) (6): \( B_3\bar{u} = \{u, g; u, g\} \)

\[
(x_b, x_2) \rightarrow F(x_{b2}) = g = F(y_b) \rightarrow 13. (y_b, y_0), 14. (y_b, y_1), 15. (y_b, y_2), 16. (y_b, y_a).
\]

Sixteen pairs are created; each pair corresponds to one P and one K term. Two kinds of exceptions have already been noted above. The first exception is the pairs of type \( (y_{a/b}, y_0) \), in the present example, 1, 5, 9, and 13, which produce only a K term. The second exception is the following. As noted, the K terms with the pair \( (y_{a/b}, y_k) \) with \( k = 1, 2, \ldots, n \) exist only for diagonal splittings. Then the K terms with the pairs with nondiagonal splittings, pairs, 6, 7, 14, and 15, do not exist. We select three pairs, 1. \( (y_a, y_0) \), 2. \( (y_a, y_1) \) and 16. \( (y_b, y_a) \), for instance, and show their concrete expressions.

\[
\circ \text{Example: } (x_a, x_3), B_1 = \{u, \bar{u}; u, \bar{u}\}, 1. (y_a, y_0)
\]

\[
K(R_1, x_a; B_1, y_a, y_0) = \frac{\alpha_s}{2\pi} \tilde{K}^{uu}(x) \cdot \langle B_1 | B_1\rangle, \tag{2.211}
\]

where the concrete expression of \( \tilde{K}^{uu}(x) = \tilde{K}^{ff}(x) \) is given in Eq. (A72)\(^1\).

---

\(^1\) The P and K terms with the Born process \( B_1 \) include the \( +\)-distribution like \( 1/(1-x)_+ \). This is defined in Ref. [27] and also in Eq. (5.16) in the present paper. When we use the definition directly for the Monte Carlo simulation, the multiplied matrix element \( \langle B_1 | B_1\rangle \) must be evaluated twice at points \( x \) and \( x = 1 \). An excellent technique to avoid this might be available, where the matrix element is evaluated only once at \( x = 1 \). The technique is briefly explained in Sect. 4 of Ref. [96]. We suggest that interested readers consult that reference.
Example: \((x_a, x_3), B1 = \{u, \bar{u}; u, \bar{u}\}\), 2. \((y_a, y_1)\)

\[
P(R_1, x_a : B1, y_a, y_1) = \frac{\alpha_s}{2\pi} \frac{1}{C_F} P^{uu}(x) \ln \frac{\mu_F^2}{x s_{x_a y_1}} \cdot \langle B1 | T_{y_a} \cdot T_{y_1} | B1 \rangle, \tag{2.212}
\]

\[
K(R_1, x_a : B1, y_a, y_1) = \frac{\alpha_s}{2\pi} \frac{\gamma_a}{C_F} h(x) \cdot \langle B1 | T_{y_a} \cdot T_{y_1} | B1 \rangle, \tag{2.213}
\]

where the quantities \(P^{uu}(x) = P^{ff}(x)\), \(\gamma_a = \gamma_f\), and \(h(x)\) are defined in Eqs. (A62), (A69), and (A71), respectively. The Lorentz scalar \(s_{x_a y_1}\) is defined as \(s_{x_a y_1} = 2p_a \cdot P(y_1)\).

Example: \((x_b, x_2), B3\bar{u} = \{u, g; u, g\}\), 16. \((y_b, y_a)\)

\[
P(R_1, x_b : B3\bar{u}, y_b, y_a) = \frac{\alpha_s}{2\pi} \frac{1}{C_A} P^{\bar{u}g}(x) \ln \frac{\mu_F^2}{x s_{x_b y_a}} \cdot \langle B3\bar{u} | T_{y_b} \cdot T_{y_a} | B3\bar{u} \rangle, \tag{2.214}
\]

\[
K(R_1, x_b : B3\bar{u}, y_b, y_a) = -\frac{\alpha_s}{2\pi} \frac{1}{C_A} \tilde{K}^{\bar{u}g}(x) \cdot \langle B3\bar{u} | T_{y_b} \cdot T_{y_a} | B3\bar{u} \rangle, \tag{2.215}
\]

where the functions \(P^{\bar{u}g}(x) = P^{fg}(x)\) and \(\tilde{K}^{\bar{u}g}(x) = \tilde{K}^{fg}(x)\) are given in Eqs. (A64) and (A77). The Lorentz scalar \(s_{x_b y_a}\) is defined as \(s_{x_b y_a} = 2p_b \cdot P(y_a) = 2p_b \cdot p_a\).

### 2.4.5. Summary

The contributions of the P and K terms to the hadronic cross section are written as

\[
\sigma_P(R_i) + \sigma_K(R_i) = \int dx_1 \int dx_2 \int f_{F(x_a)}(x_1) f_{F(x_b)}(x_2) \left( \hat{\sigma}_P(R_i) + \hat{\sigma}_K(R_i) \right). \tag{2.216}
\]

The partonic cross sections are written as

\[
\hat{\sigma}_P(R_i) + \hat{\sigma}_K(R_i) = \int_0^1 dx \sum_{B_j} \frac{1}{S_{B_j}} \Phi_a(R_i : B_j, x_4)

\times \left( P(R_i, x_a : B_j) + K(R_i, x_a : B_j) \right) + (a \leftrightarrow b), \tag{2.217}
\]

where the PS, \(\Phi_a\), is defined in Eq. (2.189). The PS integrations of the P and K terms are separately finite in 4 dimensions. The outputs, \(P(R_i)\) and \(K(R_i)\), are the sets defined in Eqs. (2.191) and (2.201), respectively. Once we select an input process \(R_i\), a leg-\(a\) or -\(b\), and a reduced Born process, \(B_j\), each P and K term, \(P/K(R_i, x_{a/b} : B_j, y_{a/b}, y_K)\), is specified by information on the pair \((y_{a/b}, y_K)\), which is abbreviated as

\[
(a/b, K). \tag{2.218}
\]

Concrete formulae for the P and K terms are collected in Appendix A3.

### 2.5. Advantages of the DSA

In the present section, the advantages of the DSA are clarified. For this purpose, we first point out the special features of the DSA presented in Sects. 2.1, 2.2, 2.3, and 2.4. The master formula of the DSA is shown in Eq. (2.4) as

\[
\sigma(R_i) = \int dx_1 \int dx_2 \int f_{F(x_a)}(x_1) f_{F(x_b)}(x_2)

\times \left[ (\hat{\sigma}_R(R_i) - \hat{\sigma}_D(R_i)) + (\hat{\sigma}_V(B1(R_i)) + \hat{\sigma}_L(R_i)) + \hat{\sigma}_P(R_i) + \hat{\sigma}_K(R_i) \right]. \tag{2.219}
\]

This formula shows that the real correction \(\hat{\sigma}_R(R_i)\), the virtual correction \(\hat{\sigma}_V(B1(R_i))\), and all the subtraction terms \(\hat{\sigma}_D(R_i), \hat{\sigma}_L(R_i), \hat{\sigma}_P(R_i), \) and \(\hat{\sigma}_K(R_i)\), which are created from one input process
the summation of all the introduced subtraction terms must vanish as in Eq. (1.3),

\[ \sum_{R_i} \hat{\sigma}_{\text{subt}}(R_i) = \sum_{R_i} \left[ \hat{\sigma}_{D}(R_i) + \hat{\sigma}_{C}(R_i) - \hat{\sigma}_{I}(R_i) - \hat{\sigma}_{P}(R_i) - \hat{\sigma}_{K}(R_i) \right] = 0, \quad (2.220) \]

which we call the consistency relation of the subtraction terms. The first advantage is that a straightforward proof of the consistency relation in Eq. (2.220) is possible. The cancellation among the subtraction terms can be realized among subtraction terms with the same initial parton states and the same reduced Born processes. According to the first two features of the DSA, we can systematically identify the categories of the subtraction terms that cancel each other. Then a systematic proof of the consistency relation becomes possible. We have succeeded in constructing a straightforward proof algorithm (PRA), which is presented in the following article [98]. The second advantage is the following. In order to construct the computer codes for the Monte Carlo integration, we must collect subtraction terms with the same initial parton states to be multiplied by the same PDFs. Such collection is realized in the DSA thanks to the first feature that the created subtraction terms are sorted by the initial parton states. The third advantage is the following. According to the third feature, we can specify all the subtraction terms in a compact form. For example, the compact forms for the D, I, P, and K terms are shown in Eqs. (2.106), (2.173), (2.214), and (2.215), respectively. Furthermore, on the fixed reduced Born processes, the minimal information to specify the subtraction terms is defined for the D, I, and P/K terms in Eqs. (2.152), (2.178), and (2.218), respectively. With agreement on the form of expression, everyone can understand the summary tables of all the created subtraction terms written in a template form. Summary tables for the Drell–Yan and the dijet processes are explicitly shown in Sect. 3 and Appendix B, respectively.

Finally, we compare the DSA against the algorithm implemented in the AutoDipole package, because the comparison makes the advantages of the DSA clearer. The creation algorithm of the D and I terms in AutoDipole is essentially the same as the DSA. The creation algorithm of the P and K terms is different from the DSA. In order to demonstrate the difference, we use the same example process as in Eq. (2.204), \( R_1 = u\bar{u} \to u\bar{u}g \). The creation algorithm of the P and K terms in AutoDipole takes only the reduced Born process, \( B_1 = R_1 - g_f = u\bar{u} \to u\bar{u} \), as the input. Then the P and K terms are created by adding to process \( B_1 \) the possible splittings in Fig. A3. Splitting (3) can be added to \( y_a \) of \( B_1 \) and the elements \( y_K \) are chosen. The choice creates the P and K terms as \( P/K(R_1, x_a : B_1, y_a, y_K) \), which are the same as the DSA. As the next choice, splitting (7) can be
added and $y_K$ are chosen. The choice creates the P and K terms, which are written in the notation defined in the DSA in Eq. (2.182) as

$$P/K(R_i = ug \rightarrow u\bar{u}u, x_a : B4u, y_a, y_K).$$

(2.221)

As the notation of the DSA shows, these P and K terms are created from the input $R_i = ug \rightarrow u\bar{u}u$, when splitting (7) is applied. In this way, the creation places of the P and K terms with the nondiagonal splittings (6) and (7) are different between the AutoDipole algorithm and the DSA. The advantage of the AutoDipole algorithm is that all the P and K terms include only one kind of reduced Born process $B1$. In this sense, the collected expressions of the P and K terms, which are created from the input $R_i$, are simpler than the case of the DSA. The disadvantage of the AutoDipole algorithm is that the P and K terms with nondiagonal splittings, created by the AutoDipole algorithm, have different initial parton states from the other subtraction terms. The involvement of different initial states spoils the first feature of the DSA and hence causes the loss of the first and second advantages of the DSA. Namely, in the AutoDipole algorithm, the proof of the consistency relation becomes more complex, and re-collection of the P and K terms with nondiagonal splittings for the Monte Carlo integration is required as extra work for the users. The third advantage of the DSA, the expressions and the summary tables in a compact form, also holds for the AutoDipole algorithm, because the subtraction terms are also sorted by the reduced Born processes in the AutoDipole algorithm.

3. Drell–Yan: $pp \rightarrow \mu^+\mu^- + X$

In the present section, we apply the DSA to the Drell–Yan process. The five steps in Eq. (2.1) are executed in Sects. 3.1, 3.2, 3.3, 3.4, and 3.5, respectively.

3.1. List of $R_i$

In Step 1 we make a list of the contributing real emission processes $\{R_i\}$ as follows:

$$R_1 = u\bar{u} \rightarrow \mu^-\mu^+g,$$

$$R_2 = ug \rightarrow \mu^-\mu^+u,$$

$$R_3 = \bar{u}g \rightarrow \mu^-\mu^+\bar{u}.\quad(3.1)$$

There are three independent processes, which sets $n_{\text{real}} = 3$. The number of final states is three, which sets $(n + 1) = 3$. In order to exhaust all independent partonic processes in the Drell–Yan event, it is sufficient to take into account one quark flavor, $u$, for instance.

3.2. $D$ term

In Step 2, we create the dipole terms $D(R_i)$ from the inputs $\{R_1, R_2, R_3\}$ in Eq. (3.1).

$D(R_1)$ creation

The input process $R_1 = u\bar{u} \rightarrow \mu^-\mu^+g$ determines set $\{x\}$ with the field species and the momenta as

$$\{x\} = \{x_a, x_b; x_1, x_2, x_3\},$$

(3.2)

$$F(\{x\}) = \{u, \bar{u}; \mu^-, \mu^+, g\},$$

(3.3)

Momenta : $\{p_a, p_b; p_1, p_2, p_3\}.$

(3.4)
We create the dipole terms in the order shown in Fig. 1 as
\[ \text{Dipole}_1 \text{(3)} - 2 : \ 1. (a_3, b), \ 2. (b_3, a). \] (3.5)

Only two dipole terms are created. The reduced Born process of the category Dipole_1 is fixed as \( B_1(R_1) = u\bar{u} \rightarrow \mu^-\mu^+ \), which determines set \( \{ y \} \) with the field species and the momenta as
\[ \{ y \} = \{ y_a, y_b ; y_1, y_2 \}, \] (3.6)

\[ F(\{ y \}) = \{ u, \bar{u} ; \mu^-, \mu^+ \}, \] (3.7)

\[ P(\{ y \}) = \{ P(y_a), P(y_b) ; P(y_1), P(y_2) \}. \] (3.8)

Then we specify the field mapping for each dipole term and write down the concrete expression.

1. (a3,b) Set \( \{ \tilde{x} \} \) is defined with the field species and the momenta as
\[ \{ \tilde{x} \} = \{ \tilde{x}_a, \tilde{x}_b ; \tilde{x}_1, \tilde{x}_2 \}, \] (3.9)

\[ F(\{ \tilde{x} \}) = \{ u, \bar{u} ; \mu^-, \mu^+ \}, \] (3.10)

\[ P(\{ \tilde{x} \}) = \{ P_{a3}, P_b ; \tilde{k}_1, \tilde{k}_2 \}, \] (3.11)

where the reduced momenta, \( \tilde{p}_{a3} \) and \( \tilde{k}_{1,2} \), are defined in Eqs. (A46) and (A47). Then we construct the field mapping as
\[ f(\{ \tilde{x} \}) = f(x_{\tilde{a}3}, x_{\tilde{b}}; x_1, x_2) = (y_a, y_b ; y_1, y_2), \] (3.12)

which is interpreted as the identification of the elements as
\[ (y_a, y_b ; y_1, y_2) = (x_{\tilde{a}3}, x_{\tilde{b}}; x_1, x_2). \] (3.13)

The expression is abbreviated as \( (\tilde{a}, \tilde{b} ; 1, 2) \). The field mapping determines the momenta as
\[ P(\{ y \}) = \{ P(x_{\tilde{a}3}), P(x_{\tilde{b}}) ; P(x_1), P(x_2) \}, \] (3.14)

\[ = \{ P_{a3}, P_b ; \tilde{k}_1, \tilde{k}_2 \}. \]

The dipole term is written in Eq. (A13) as
\[ D(\tilde{a}\tilde{b}\tilde{p}1, (3) \rightarrow a_{3,b}) = -\frac{1}{s_{a3}} \frac{1}{s_{x_{\tilde{a}3}ab}} \frac{1}{C_F} V_{a3,b} \langle B1|T_{y_a} \cdot T_{y_b}|B1 \rangle. \] (3.15)

where the dipole splitting function \( V_{a3,b} \) and the Lorentz scalar \( x_{3,ab} \) are defined in Eqs. (A14) and (A48).

2. (b3,a) Set \( \{ \tilde{x} \} \) is defined as
\[ \{ \tilde{x} \} = \{ x_{\tilde{a}} , x_{\tilde{b}3} ; x_1, x_2 \}, \] (3.16)

\[ F(\{ \tilde{x} \}) = \{ u, \bar{u} ; \mu^-, \mu^+ \}, \] (3.17)

\[ P(\{ \tilde{x} \}) = \{ p_a, \tilde{p}_{b3} ; \tilde{k}_1, \tilde{k}_2 \}. \] (3.18)

Set \( \{ y \} \) is fixed in Eq. (3.6) and the field mapping is found as
\[ (y_a, y_b ; y_1, y_2) = (x_{\tilde{a}}, x_{\tilde{b}3}; x_1, x_2), \] (3.19)

which is abbreviated as \( (\tilde{a}, \tilde{b}3 ; 1, 2) \). The momenta are determined as
\[ P(\{ y \}) = \{ p_a, \tilde{p}_{b3} ; \tilde{k}_1, \tilde{k}_2 \}. \] (3.20)

The dipole term is written as
\[ D(\tilde{a}\tilde{b}\tilde{p}1, (3) \rightarrow b_{3,a}) = -\frac{1}{s_{b3}} \frac{1}{s_{x_{\tilde{a}3}ab}} \frac{1}{C_F} V_{b3,a} \langle B1|T_{y_b} \cdot T_{y_a}|B1 \rangle. \] (3.21)

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The output \( D(R_1) \) is written as
\[
D(R_1) = D(R_1, \text{dip1}) = D(R_1, \text{dip1}, (3)-2)_{a3,b} + D(R_1, \text{dip1}, (3)-2)_{b3,a}.
\] (3.22)

\( D(R_2) \) creation
The input process \( R_2 = ug \rightarrow \mu^- \mu^+ u \) determines set \( \{x\} \) as
\[
\{x\} = \{x_a, x_b; x_1, x_2, x_3\},
\] (3.23)
\[
F(\{x\}) = \{u, g; \mu^-, \mu^+, u\},
\] (3.24)
\[
\text{Moments} : \{p_a, p_b; p_1, p_2, p_3\}.
\] (3.25)

We create the dipole term as
\[
\text{Dipole4} (7)-2 : 1. (b3, a).
\] (3.26)

The dipole term has a reduced Born process of category Dipole4 as \( B4u(R_2) = u\bar{u} \rightarrow \mu^- \mu^+ \), which determines set \( \{y\} \) as
\[
\{y\} = \{y_a, y_b; y_1, y_2\},
\] (3.27)
\[
F(\{y\}) = \{u, \bar{u}; \mu^-, \mu^+\}.
\] (3.28)

Although the dipole term Dipole3 (6)-2 : (a3, b) is possible as a selection of the splitting, the reduced Born process \( B3u(R_2) = gg \rightarrow \mu^- \mu^+ \) does not exist at LO, and neither does the dipole term. Next we write down the concrete expression for the dipole term 1. (b3, a) in Eq. (3.26). Set \( \{\tilde{x}\} \) is defined as
\[
\{\tilde{x}\} = \{x_{\tilde{a}}, x_{\tilde{b}3}; x_1, x_2\},
\] (3.29)
\[
F(\{\tilde{x}\}) = \{u, \bar{u}; \mu^-, \mu^+\},
\] (3.30)
\[
P(\{\tilde{x}\}) = \{p_{\tilde{a}}, \tilde{p}_{b3}; \tilde{k}_1, \tilde{k}_2\}.
\] (3.31)

The field mapping to set \( \{y\} \) is found in Eq. (3.27) as
\[
(y_a, y_b; y_1, y_2) = (x_{\tilde{a}}, x_{\tilde{b}3}; x_1, x_2),
\] (3.32)
which is abbreviated as \((\tilde{a}, \tilde{b}3; 1, 2)\). The momenta are determined as
\[
P(\{y\}) = \{p_a, \tilde{p}_{b3}; \tilde{k}_1, \tilde{k}_2\}.
\] (3.33)

The dipole term is written in Eq. (A29) as
\[
D(\text{dip4}, (7)-2)_{b3,a} = -\frac{1}{s_{b3}} \frac{1}{x_3, ba} \frac{1}{C_F} V_{b3,a} \langle B4u | T_{y_b} \cdot T_{y_a} | B4u \rangle.
\] (3.34)

The output \( D(R_2) \) is written as
\[
D(R_2) = D(R_2, \text{dip4}) = D(R_2, \text{dip4}, (7)-2)_{b3,a}.
\] (3.35)

The dipole terms \( D(R_3) \) are created in a similar way to \( D(R_2) \).

\textit{Summary of creation}

The created dipole terms from the inputs \( \{R_1, R_2, R_3\} \) are summarized in Table 1.
Table 1. Summary table of dipole term creation.

<table>
<thead>
<tr>
<th>Dip j</th>
<th>B j</th>
<th>Splitting</th>
<th>(x_I x_J, x_K)</th>
<th>(y_a, y_b ; y_1, y_2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>B(R_1 = u u → μ^- μ^+) : S_{R_1} = 1</td>
<td>B1 = u u → μ^- μ^+</td>
<td>(3) - 2</td>
<td>1. (a_3, b)</td>
<td>(a_3̅, b̅ ; 1, 2)</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>2. (b_3, a)</td>
</tr>
<tr>
<td>D(R_2 = u g → μ^- μ^+ u) : S_{R_2} = 1</td>
<td>B4u = u u → μ^- μ^+</td>
<td>(7)u - 2</td>
<td>1. (b_3, a)</td>
<td>(a̅, b̅ ; 1, 2)</td>
</tr>
<tr>
<td>D(R_3 = u̅ g → μ^- μ^+ u̅) : S_{R_3} = 1</td>
<td>B4̅ = u̅ u̅ → μ^- μ^+</td>
<td>(7)u̅ - 2</td>
<td>1. (b_3, a)</td>
<td>(a̅, b̅ ; 1, 2)</td>
</tr>
</tbody>
</table>

Table 2. Summary table of I term creation.

<table>
<thead>
<tr>
<th>Leg - y_I</th>
<th>F(y_I)</th>
<th>(y_1, y_K)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(3) - 2</td>
<td>u</td>
<td>1. (a, b)</td>
</tr>
<tr>
<td></td>
<td>̅u</td>
<td>2. (b, a)</td>
</tr>
</tbody>
</table>

3.3. I term

In Step 3, we create the I term I(R_i) from the input B1(R_i). Among the real emission processes \{R_1, R_2, R_3\}, only the R_1 has the reduced Born process B1 as

\[ B1(R_1) = u u → μ^- μ^+. \] (3.36)

Set \{y\} is fixed in Eqs. (3.6) and (3.7) as

\[ \{y\} = \{y_a, y_b, y_1, y_2\}. \] (3.37)

\[ F(\{y\}) = \{u, ̅u; μ^-, μ^+\}. \] (3.38)

Following the order shown in Eq. (2.156), the I terms are created as

\[ (3) - 2: \ 1. (a, b), \ 2. (b, a). \] (3.39)

The concrete expressions are shown in Eq. (A52) as

1. \[ I(R_1)_{a,b} = -A_d \frac{V_f}{C_F} s_{aa}^{\epsilon} \langle B1 | T_{y_a} \cdot T_{y_b} | B1 \rangle, \] (3.40)

2. \[ I(R_1)_{b,a} = -A_d \frac{V_f}{C_F} s_{ba}^{\epsilon} \langle B1 | T_{y_b} \cdot T_{y_a} | B1 \rangle. \] (3.41)

The output I(R_1) is written as

\[ I(R_1) = I(R_1)_{a,b} + I(R_1)_{b,a} \]

\[ = -A_d \frac{V_f}{C_F} ([a,b] + [b,a]). \] (3.42)

with the notation in Eq. (2.166). The I terms created are summarized in Table 2.

3.4. P and K terms

In Step 4, we create the P and K terms P(R_i) and K(R_i) from the inputs R_i and B_j(R_i).
The only possible reduced Born process is \( B_1(R_1) \) as

\[ B_1(R_1) = u\bar{u} \rightarrow \mu^- \mu^+. \tag{3.46} \]

Set \( \{y\} \) is fixed in Eqs. (3.6) and (3.7) as

\[ \{y\} = \{y_a, y_b; y_1, y_2\}, \tag{3.47} \]

\[ F([y]) = \{u, \bar{u}; \mu^-, \mu^+\}. \tag{3.48} \]

The \( \mathbb{P} \) and \( \mathbb{K} \) terms are created in the order shown in Fig. A3 as

\[ \text{leg} - a : \text{Dipole 1} (3) : B_1 = \{u, \bar{u}; \mu^-, \mu^+\} \]

\[ (x_a, x_3) \rightarrow F(x_{\bar{a}3}) = u = F(y_a) \rightarrow 1. (y_a, y_0), \]

\[ \text{leg} - b : \text{Dipole 1} (3) : B_1 = \{u, \bar{u}; \mu^-, \mu^+\} \]

\[ (x_b, x_3) \rightarrow F(x_{\bar{b}3}) = \bar{u} = F(y_b) \rightarrow 3. (y_b, y_0), 4. (y_b, y_a). \tag{3.49} \]

The concrete expressions for the \( \mathbb{P} \) and \( \mathbb{K} \) terms with leg-\( a \) are shown in Eqs. (A60), (A66), and (A68) as

1. \( (y_a, y_0) \)

\[ K(R_1, x_a : B_1, y_a, y_0) = \frac{\alpha_s}{2\pi} \tilde{K}^{\bar{a}a}(x) \langle B_1 | B_1 \rangle, \tag{3.50} \]

2. \( (y_a, y_b) \)

\[ P(R_1, x_a : B_1, y_a, y_b) = \frac{\alpha_s}{2\pi} \frac{1}{C_P} \frac{p^{\bar{a}a}(x) \ln \frac{\mu_F^2}{x s_{x_a y_b}}}{x s_{x_a y_b}} \langle B_1 | T_{y_a} \cdot T_{y_b} | B_1 \rangle, \tag{3.51} \]

\[ K(R_1, x_a : B_1, y_a, y_b) = -\frac{\alpha_s}{2\pi} \frac{1}{C_F} \tilde{K}^{\bar{a}a}(x) \langle B_1 | T_{y_a} \cdot T_{y_b} | B_1 \rangle, \tag{3.52} \]

and with leg-\( b \) as

3. \( (y_b, y_0) \)

\[ K(R_1, x_b : B_1, y_b, y_0) = \frac{\alpha_s}{2\pi} \tilde{K}^{\bar{a}a}(x) \langle B_1 | B_1 \rangle, \tag{3.53} \]

4. \( (y_b, y_a) \)

\[ P(R_1, x_b : B_1, y_b, y_a) = \frac{\alpha_s}{2\pi} \frac{1}{C_F} \frac{p^{\bar{a}a}(x) \ln \frac{\mu_F^2}{x s_{x_b y_a}}}{x s_{x_b y_a}} \langle B_1 | T_{y_b} \cdot T_{y_a} | B_1 \rangle, \tag{3.54} \]

\[ K(R_1, x_b : B_1, y_b, y_a) = -\frac{\alpha_s}{2\pi} \frac{1}{C_F} \tilde{K}^{\bar{a}a}(x) \langle B_1 | T_{y_b} \cdot T_{y_a} | B_1 \rangle. \tag{3.55} \]

The output for the \( \mathbb{P} \) term is written in Eq. (2.191) as

\[ \mathbb{P}(R_1) = \{ P(R_1, x_a), P(R_1, x_b) \}, \tag{3.56} \]

where the elements \( P(R_1, x_{a/b}) \) are written as

\[ P(R_1, x_{a/b}) = P(R_1, x_{a/b} : B_1) = P(R_1, x_{a/b} : B_1, y_{a/b}, y_{b/a}). \tag{3.57} \]
Table 3. Summary tables of the P and K term creation.

<table>
<thead>
<tr>
<th>Leg→x_{a/b}</th>
<th>Dip j</th>
<th>B j</th>
<th>S_{Bj}</th>
<th>Splitting (y_{a/b}, y_k)</th>
</tr>
</thead>
<tbody>
<tr>
<td>P/K (R_1 = u\bar{u} \rightarrow \mu^-\mu^+ g)</td>
<td>Dip 1</td>
<td>B1 = u\bar{u} \rightarrow \mu^-\mu^+</td>
<td>S_{B1} = 1</td>
<td>(3) \rightarrow 0 \quad 1. (a, 0)</td>
</tr>
<tr>
<td>b</td>
<td>Dip 1</td>
<td>B1 = u\bar{u} \rightarrow \mu^-\mu^+</td>
<td>S_{B1} = 1</td>
<td>(3) \rightarrow 2 \quad 2. (a, b)</td>
</tr>
<tr>
<td>P/K (R_2 = u g \rightarrow \mu^-\mu^+ u)</td>
<td>Dip 4u</td>
<td>B4u = u\bar{u} \rightarrow \mu^-\mu^+</td>
<td>S_{B4u} = 1</td>
<td>(7) \rightarrow 0 \quad 1. (b, 0)</td>
</tr>
<tr>
<td>b</td>
<td>Dip 4u</td>
<td>B4\bar{u} = \bar{u}u \rightarrow \mu^-\mu^+</td>
<td>S_{B4u} = 1</td>
<td>(7) \rightarrow 2 \quad 2. (b, a)</td>
</tr>
</tbody>
</table>

The output for the K term is written in Eq. (2.201) as

\[ K(K_1) = \{K(K_1, x_a), K(K_1, x_b)\}, \quad (3.58) \]

where the elements K(R_1, x_{a/b}) are written as

\[ K(R_1, x_{a/b}) = K(R_1, x_{a/b} : B1) = K(R_1, x_{a/b} : B1, y_{a/b}, y_0) + K(R_1, x_{a/b} : B1, y_{a/b}, y_b/a). \quad (3.59) \]

P/K(R_2) creation

The process R_2 defines set {x} in Eq. (3.23). Only one reduced Born process, B4u(R_2) = u\bar{u} \rightarrow \mu^-\mu^+, exists, which fixes set {y} in Eq. (3.27). The P and K terms are created as

\[ \text{leg→b : Dipole 4u (7)} : \text{B4u} = \{u, \bar{u} ; \mu^-, \mu^+\} \]

\[ (x_b, x_3) \rightarrow F(x_{\bar{b}3}) = \bar{u} = F(y_b) \rightarrow 1. (y_b, y_0), \ 2. (y_b, y_a). \quad (3.60) \]

The concrete expressions are written down as

1. (y_b, y_0) \quad K(R_2, x_b : B4u, y_b, y_0) = \frac{\alpha_s}{2\pi} \tilde{K}^{\bar{u}u}(x) \langle B4u \mid B4u \rangle, \quad (3.61)

2. (y_b, y_a) \quad P(R_2, x_b : B4u, y_b, y_a) = \frac{\alpha_s}{2\pi} \frac{1}{C_F} \tilde{p}^{\bar{u}u}(x) \ln \frac{\mu_F}{x s x_{y_a}} \langle B4u \mid T_{y_b} \cdot T_{y_a} \mid B4u \rangle, \quad (3.62)

\quad K(R_2, x_b : B4u, y_b, y_a) = -\frac{\alpha_s}{2\pi} \frac{1}{C_F} \tilde{K}^{\bar{u}u}(x) \langle B4u \mid T_{y_b} \cdot T_{y_a} \mid B4u \rangle. \quad (3.63)

The outputs are written as

\[ P(R_2) = P(R_2, x_b) = P(R_2, x_b : B4u, y_b, y_a), \quad (3.64) \]

\[ K(K_2) = K(R_2, x_b) = K(R_2, x_b : B4u, y_b, y_0) + K(R_2, x_b : B4u, y_b, y_a). \quad (3.65) \]

The P and K terms created from the inputs of the real processes \{R_1, R_2, R_3\} are summarized in Table 3.

Summary of creation

The P and K terms created from the inputs of the real processes \{R_1, R_2, R_3\} are summarized in Table 3.
3.5. **NLO correction: σ\text{NLO}**

In Step 5, we obtain the NLO correction σ\text{NLO} in Eq. (2.3) as

\[
\sigma_{\text{NLO}} = \sum_{i=1}^{3} \sigma(R_i) \tag{3.66}
\]

The cross section \(\sigma(R_1)\) is concretely written as

\[
\sigma(R_1) = \int dx_1 \int dx_2 f_u(x_1) f_{\bar{u}}(x_2) \times \left[ (\hat{\sigma}_R(R_1) - \hat{\sigma}_D(R_1)) + (\hat{\sigma}_V(B1(R_1)) + \hat{\sigma}_I(R_1)) + \hat{\sigma}_P(R_1) + \hat{\sigma}_K(R_1) \right] \tag{3.67}
\]

where the finite combinations of the partonic cross sections are written separately as

\[
\hat{\sigma}_R(R_1) - \hat{\sigma}_D(R_1) = \frac{1}{S_{R_1}} \Phi(R_1) - \frac{1}{n_s(u)n_s(\bar{u})} D(R_1) \tag{3.68}
\]

\[
\hat{\sigma}_V(B1(R_1)) + \hat{\sigma}_I(R_1) = \frac{1}{S_{B1}} \Phi(B1) + \frac{1}{|M_{\text{virt}}(B1)|^2 + I(R_1)} \tag{3.69}
\]

\[
\hat{\sigma}_P(R_1) + \hat{\sigma}_K(R_1) = \int_0^1 dx \frac{1}{S_{B1}} \Phi_d(R_1 : B_1, x) \times (P(R_1, x_a : B_1) + K(R_1, x_a : B_1)) \tag{3.70}
\]

The subtraction terms \(D(R_1), I(R_1), P(R_1, x_a/b), \) and \(K(R_1, x_a/b)\) are written in Eqs. (3.22), (3.42), (3.57), and (3.59), respectively. \(\sigma(R_2)\) is written as

\[
\sigma(R_2) = \int dx_1 \int dx_2 f_u(x_1) f_{\bar{g}}(x_2) \left[ (\hat{\sigma}_R(R_2) - \hat{\sigma}_D(R_2)) + \hat{\sigma}_P(R_2) + \hat{\sigma}_K(R_2) \right] \tag{3.71}
\]

where the finite cross sections are written as

\[
\hat{\sigma}_R(R_2) - \hat{\sigma}_D(R_2) = \frac{1}{S_{R_2}} \Phi(R_2) - \frac{1}{n_s(u)n_s(g)} D(R_2) \tag{3.72}
\]

\[
\hat{\sigma}_P(R_2) + \hat{\sigma}_K(R_2) = \int_0^1 dx \frac{1}{S_{B_{4u}}} \Phi_b(R_2 : B_{4u}, x) \times [P(R_2, x_b : B_{4u}) + K(R_2, x_b : B_{4u})] \tag{3.73}
\]

The subtraction terms \(D(R_2), P(R_2, x_b), \) and \(K(R_2, x_b)\) are written in Eqs. (3.35), (3.64), and (3.65), respectively. \(\sigma(R_3)\) is written in similar expressions to \(\sigma(R_2)\). The contributions from the exchanged initial partons must be added. For example, in \(\sigma(R_1)\), the contribution from the process \(\bar{u}u \rightarrow \mu^+\mu^-g\) is added with the multiplication of the exchanged PDFs, \(f_{\bar{u}}(x_1) f_u(x_2)\). Furthermore, the contribution from the other quark flavors must be added. The full expression is clarified in Sect. 5.

4. **Dijet: \(pp \rightarrow 2\) jets +X**

In the present section, we apply the DSA to the dijet process. Like the Drell–Yan process in Sect. 3, the five steps are executed in Sects. 4.1, 4.2, 4.3, 4.4, and 4.5, respectively.
4.1. List of $R_i$

In Step 1 we make a list of the real processes $\{R_i\}$ as follows:

\begin{align*}
R_{1u} &= uu \to u\bar{u}g, \\
R_{2u} &= uu \to uug, \\
R_{3u} &= ug \to u\bar{u}u, \\
R_{4u} &= u\bar{u} \to d\bar{d}g, \\
R_{5ud} &= ud \to udg, \\
R_{6ud} &= u\bar{d} \to u\bar{d}g, \\
R_{7u} &= ug \to u\bar{d}d, \\
R_{8u} &= u\bar{u} \to ggg, \\
R_{9u} &= ug \to ugg, \\
R_{10u} &= gg \to u\bar{u}g, \\
R_{11} &= gg \to ggg. \\
\end{align*}

(4.1)

There are eleven independent processes so $n_{\text{real}} = 11$. The number of final states is $(n + 1) = 3$. We assume five massless quark flavors, $u, d, s, c,$ and $b$. The contributing real processes can be exhausted by the independent processes that are produced from the field contents of only two quark flavors and a gluon. We take the $u$ and $d$ quarks as the two flavors. The real processes in the curly brackets in Eq. (4.1) represent the processes that are obtained by the replacements of the quark flavors. For example, the process $R_{1d}$ is concretely written as $R_{1d} = d\bar{d} \to d\bar{d}g$, which is obtained by the replacements $u \to d$ and $\bar{u} \to \bar{d}$ in the process $R_{1u} = u\bar{u} \to u\bar{u}g$. It is sometimes useful to categorize the partonic processes by the crossing symmetry. The processes $R_{1u}, R_{2u},$ and $R_{3u}$ are categorized into the master process $0 \to uuu\bar{u}g$. The processes $R_{4u}, R_{5ud}, R_{6ud},$ and $R_{7u}$ are categorized into the process $0 \to u\bar{d}d$, and $R_{8u}, R_{9u},$ and $R_{10u}$ into $0 \to u\bar{u}gg$. The process $R_{11}$ is categorized into $0 \to gggg.$

4.2. $D$ term

In Step 2 we create the dipole terms $D(R_i)$ from the inputs $\{R_i\}$ in Eq. (4.1). Summary tables of all the dipole terms created are shown in Tables B1–B11 in Appendix B1. The details of the creation of $D(R_i)$ have already been presented with the concrete expressions of some dipole terms in Eqs. (2.90)–(2.145). In this section, we present only one dipole term in $D(R_{9u})$. Hereafter we drop the flavor index $u$, leaving $R_9$, for simplicity. The input process $R_9$ in Eq. (4.1) defines set $\{x\}$ as

\begin{align*}
\{x\} &= \{x_a, x_b; x_1, x_2, x_3\}, \\
F(\{x\}) &= \{u, g; u, g, g\}, \\
\text{Momenata :} &= \{p_a, p_b; p_1, p_2, p_3\}. \\
\end{align*}

(4.2)

(4.3)

(4.4)

We select the dipole term 16, $D(R_9, \xi \perp p_1, (4)−1)_{b2,1}$ from the 27 dipole terms in Table B9. For the process $R_9$, the reduced Born process $B1$ is fixed as

\begin{equation}
B1(R_9) = ug \to ug.
\end{equation}

(4.5)
which determines set \( \{ y \} \) as

\[
\{ y \} = \{ y_a, y_b; y_1, y_2 \},
\]

\[
F(\{ y \}) = \{ u, g; u, g \}.
\]

For the dipole term \( D_{b2,1} \), set \( \{ \tilde{x} \} \) is defined as

\[
\{ \tilde{x} \} = \{ x_{b2}, x_a; x_3, x_1 \},
\]

\[
F(\{ \tilde{x} \}) = \{ g, u; g, u \},
\]

\[
P(\{ \tilde{x} \}) = \{ \tilde{p}_{b2}, p_a; p_3, \tilde{p}_1 \},
\]

where the reduced momenta \( \tilde{p}_{b2} \) and \( \tilde{p}_1 \) are defined in Eqs. (A42) and (A43). The field mapping is specified as

\[
(y_a, y_b; y_1, y_2) = (x_a, x_{b2}; x_1, x_3),
\]

which is abbreviated as \( (a, \tilde{b}; \tilde{1}, 3) \) in Table B9. The field mapping determines the momenta of set \( \{ y \} \) as

\[
P(\{ y \}) = \{ p_a, \tilde{p}_{b2}; p_3, \tilde{p}_1 \}.
\]

The concrete expression of the dipole term is written in Eq. (A15) as

\[
D(\text{R}9, \text{dip}1, (4)−1)_{b2,1} = −\frac{1}{s_{b2}} \frac{1}{C_A} \frac{1}{n_s(u)n_s(g)} \{ \text{B}1 \mid T_{y_b} \cdot T_{y_1} \cdot V_{b2,1}^{y_b} \mid \text{B}1 \},
\]

where the dipole splitting function \( V_{b2,1} \) and the Lorentz scalar \( x_{21,b} \) are defined in Eqs. (A16) and (A44), respectively. The contribution to the cross section reads in Eq. (2.6) as

\[
\hat{\sigma}_D(\text{R}9) = \frac{1}{S_{\text{R}9}} \Phi(\text{R}9) \frac{1}{n_s(u)n_s(g)} \left[ D(\text{R}9, \text{dip}1, (4)−1)_{b2,1} + \cdots \right].
\]

It is noted that the symmetric factor of the reduced Born process \( \text{B}1(\text{R}9) \) is \( S_{\text{B}1} = 1 \), but the contribution of the dipole term is not divided by the factor. The contribution of all the dipole terms involved in \( D(\text{R}9) \) must be divided by the symmetric factor of the input real process \( \text{R}9, S_{\text{R}9} = 2 \).

### 4.3. \( I \) term

In **Step 3** we create the \( I \) terms \( I(\text{R}_i) \) from the inputs \( \text{B}1(\text{R}_i) \). Summary tables of all the \( I \) terms created are shown in Tables B12–B16 in Appendix B2. Since the real processes \( \text{R}3 \) and \( \text{R}7 \) do not have the reduced Born process \( \text{B}1 \), the \( I \) terms \( I(\text{R}_3) \) and \( I(\text{R}_7) \) do not exist. The details of the creation of \( I(\text{R}_1) \) have been explained in Sect. 2.3. Here we see \( I(\text{R}_9) \) concretely. The input to create \( I(\text{R}_9) \) is the process \( \text{B}1(\text{R}_9) \) in Eq. (4.5) and the associated set \( \{ y \} \) is defined in Eqs. (4.6) and (4.7). Twelve \( I \) terms are created and they are listed in Table B14. Referring to the formula in Eq. (A52), the concrete expression of \( I(\text{R}_9) \) is written as

\[
I(\text{R}_9) = −A_d \left[ \frac{Y_f}{C_F} \left[ [1, 2] + [1, a] + [1, b] + [a, 1] + [a, 2] + [a, b] \right] + \frac{Y_g}{C_A} \left[ [2, 1] + [2, a] + [2, b] + [b, 1] + [b, 2] + [b, a] \right] \right],
\]

where \( Y_f/g \) are defined in Eqs. (A54) and (A55). The contribution to the cross section reads in Eq. (2.8) as

\[
\hat{\sigma}_I(\text{R}_9) = \frac{1}{S_{\text{B}1}} \Phi(\text{B}1)_d \cdot I(\text{R}_9),
\]

where the cross section is divided by the symmetric factor of the \( \text{B}1(\text{R}_9), S_{\text{B}1} = 1 \).
4.4. P and K terms

In Step 4 we create the P and K terms P(R_\text{f}) and K(R_\text{f}). Summary tables of all the P and K terms created are shown in Tables B17–B27 in Appendix B3. The details of the creation of P/K (R_\text{f}) are presented with some examples in Sect. 2.4. In this section, we show only one P term in P (R_\text{f}) and one K term in K (R_\text{f}). The input R_\text{f} defines set \{x\} in Eqs. (4.2), (4.3), and (4.4). The possible reduced Born processes B_j (R_\text{f}) are fixed during the creation of the dipole term D (R_\text{f}), and are explicitly shown in Table B9 as

\[ B1: F(\{y\}) = \{u, g; u, g\}, \]  
\[ B3u: F(\{y\}) = \{g, g; g, g\}, \]  
\[ B4u: F(\{y\}) = \{u, \bar{u}; g, g\}. \]  

Here we show concrete expressions for the P and K terms 10. P/K(R_\text{f}, x_b : B1, y_b, y_1) in Table B25. The expressions are written in Eqs. (A60) and (A67) as

\[ P(R_\text{f}, x_b : B1, y_b, y_1) = \alpha_s \frac{1}{2\pi} \frac{1}{C_A} p^{gg}(x) \ln \frac{\mu_F^2}{x s_{xby_1}} \cdot \{B1 \mid T_{yb} \cdot T_{y_1} \mid B1\}. \]  
\[ K(R_\text{f}, x_b : B1, y_b, y_1) = \alpha_s \frac{\gamma_t}{2\pi} \frac{\gamma_u}{C_F} h(x) \cdot \{B1 \mid T_{yb} \cdot T_{y_1} \mid B1\}. \]

The functions of the argument \( x \), \( p^{gg}(x) \) and \( h(x) \), are defined in Eqs. (A63) and (A71). The contribution to the cross section is written in Eq. (2.217) as

\[ \hat{\sigma}_P(R_\text{f}) + \hat{\sigma}_K(R_\text{f}) = \int_0^1 dx \left[ \frac{1}{S_{B1}} \Phi_b(R_\text{f} : B1, x) \right. \\
\left. (P(R_\text{f}, x_b : B1, y_b, y_1) + K(R_\text{f}, x_b : B1, y_b, y_1) + \cdots) + \right. \\
\left. \frac{1}{S_{B4u}} \Phi_b(R_\text{f} : B4u, x) \cdot (P(R_\text{f}, x_b : B4u) + K(R_\text{f}, x_b : B4u) + \cdots) + \cdots \right]. \]  

The contributions of the P and K terms P/K(R_\text{f}, x_b : B1, y_b, y_1) are divided by the symmetric factor of the reduced Born process B1, \( S_{B1} = 1 \). The contributions from the terms P/K(R_\text{f}, x_b : B4u) are divided by the symmetric factor of the reduced Born process B4u, \( S_{B4u} = 2 \). In this way, the P and K terms are divided by the symmetric factor of the reduced Born processes, \( S_{B\text{f}} \), not by the symmetric factor of the input real process, \( S_{R\text{f}} \). In this sense, the symmetric factor for the P and K terms is used in an inverse manner to that for the dipole terms, which is explained at the end of Sect. 4.2.

4.5. NLO correction: \( \sigma_{NLO} \)

In Step 5 we obtain the NLO correction \( \sigma_{NLO} \) in Eq. (2.3) as

\[ \sigma_{NLO} = \sum_{i=1}^{11} \sigma(R_i), \]

where the summation over the different quark flavors is suppressed. The NLO cross sections \( \sigma(R_i) \) are written in the formula in Eq. (2.4) as

\[ \sigma(R_i) = \int dx_1 \int dx_2 f_{\bar{f}(x_1)}(x_1) f_{\bar{f}(x_2)}(x_2) \times \\
\left[ (\hat{\sigma}_R(R_i) - \hat{\sigma}_D(R_i)) + (\hat{\sigma}_V(B1(R_i)) + \hat{\sigma}_T(R_i)) + \hat{\sigma}_P(R_i) + \hat{\sigma}_K(R_i) \right]. \]
In the cases of the NLO cross sections $\sigma(R_3)$ and $\sigma(R_7)$, the formula is simplified as

$$\sigma(R_i) = \int dx_1 \int dx_2 f_F(x_1) f_{\bar{F}}(x_2) \left[ (\hat{\sigma}_R(R_i) - \hat{\sigma}_D(R_i)) + \hat{\sigma}_P(R_i) + \hat{\sigma}_K(R_i) \right]. \quad (4.25)$$

The formulae for the partonic cross sections are written in Eqs. (2.5)–(2.10). Like the Drell–Yan process in Sect. 3.5, the contribution from the exchanged PDFs must be added. Furthermore, the contributions from the remaining three quark flavors, $s$, $c$, and $b$, in addition to the $u$ and $d$ flavors, must be taken into account.

5. Analytical check in Drell–Yan

In the present section, we give an analytical check of the Drell–Yan process. In Sect. 5.1 we review the well known analytical results obtained by the traditional method. In Sect. 5.2 we obtain the analytical results by the DSA. We will show that both sets of results exactly coincide.

5.1. Traditional method

We review the well known results that were obtained for the first time in the pioneering works of Refs. [11–19]. The method used in these works became the traditional method to calculate QCD NLO corrections in hadron collider processes. In the method, both the real and virtual corrections are calculated in $d$ dimensions, i.e., the matrix elements, the PS integrations, and the spin-average factors are all defined in $d$ dimensions. In the method, not only the UV, soft, and collinear divergences in the virtual correction but also the soft and collinear divergences in the real correction are regularized as poles $1/\epsilon$ and $1/\epsilon^2$.

We start with a review of the LO contribution. The general formulae are given in Eqs. (2.18) and (2.19). In this method, we redefine the partonic cross section in Eq. (2.20) in $d$ dimensions. In the Drell–Yan process, we assume one quark flavor, $u$, and finally generalize to the five massless flavors. At LO, only one independent process exists as

$$L_1 = u\bar{u} \to \mu^- \mu^+. \quad (5.1)$$

A Feynman diagram is shown in Fig. 3^2. The contribution is written as

$$\sigma_{LO} = \sigma(L_1) = \int dx_1 \int dx_2 f_u(x_1) f_{\bar{u}}(x_2) \hat{\sigma}(L_1), \quad (5.2)$$

where the partonic cross section is defined in $d$ dimensions as

$$\hat{\sigma}(L_1) = \frac{1}{S_{L_1}} \Phi(L_1)_{d} \cdot |M(L_1)|_{d}^2. \quad (5.3)$$

We fix the kinematical values as follows. The squared energy of two protons in the initial state is denoted as

$$S = (P_a + P_b)^2 = 2P_a \cdot P_b \quad \text{with the momenta } P_{a/b}. \quad \text{The squared energy of two partons in the initial state is denoted as } \hat{s} = (p_a + p_b)^2 = 2p_a \cdot p_b = x_1 x_2 S, \quad \text{where the momenta are denoted as } p_{a/b} = x_{1/2} P_{a/b}. \quad \text{The square of the muon-pair invariant mass is denoted as}$$

\[ ^2 \text{All the Feynman diagrams in the present article are drawn using the JaxoDraw package [100]. } \]
\[ M_{\mu^+\mu^-}^2 = q^2 = (p_{\mu^+} + p_{\mu^-})^2 \]

With the momenta of the antimuon/muon \( p_{\mu^+}/p_{\mu^-} \). We write the total cross section as \( \hat{\sigma}_{\text{tot}}(L_1) = \hat{\sigma}_{\text{LO}}(q^2) \) and calculate it as

\[
\hat{\sigma}_{\text{LO}}(q^2) = \alpha^2 e^2 N_c \left( \frac{\Gamma(1/2)}{\Gamma(3/2 - \epsilon)} \right) (\frac{16\pi \mu^4}{q^2} - 1)^{-\epsilon} q^2 P(q^2), \tag{5.4}
\]

where the color degree of freedom of quarks is denoted as \( N_c = 3 \), and the symbol \( \mu \) represents a dimensionful free parameter introduced in the \( d \)-dimensional space-time. The last factor \( P(q^2) \) is defined as

\[
P(q^2) = \frac{1}{(q^2)^2} Q_u^2 Q_\mu^2 + \frac{2}{q^2(q^2 - M_4^2)} Q_u Q_\mu \frac{1}{(sc)^2} v_u v_l
\]

\[
+ \frac{1}{(q^2 - M_4^2)^2} \frac{1}{(sc)^4} (v_u^2 + a_u^2)(v_l^2 + a_l^2), \tag{5.5}
\]

with the electric charges \( (Q_u, Q_\mu) = (2/3, -1) \) and the constants from the Z-boson coupling \( (v_u, v_l, a_u, a_\mu) = (1/4 - 2s^2/3, -1/4 + s^2, -1/4, 1/4) \). The square of the weak-mixing angle and the Z-boson mass are denoted as \( s^2 = \sin^2 \theta_W \) and \( M_Z \), respectively. We omit the contribution from the decay of the on-shell Z boson for simplicity. For reference, the total cross section of the process \( L_{1d} = d \bar{d} \rightarrow \mu^- \mu^+ \) is obtained by the replacements \( (Q_u, v_u, a_u) \rightarrow (Q_d, v_d, a_d) = (-1/3, -1/4 + s^2/3, 1/4) \). We predict the distribution of the squared muon-pair invariant mass, \( M_{\mu^+\mu^-}^2 = q^2 \), because the observable may be the simplest and most typical one in the Drell–Yan event. The \( q^2 \)-distribution is calculated as

\[
\frac{d^2 \hat{\sigma}(L_1)}{dq^2} = \hat{\sigma}_{\text{LO}}(q^2) \delta(q^2 - \hat{s}) = \frac{1}{q^2} \delta(1 - z), \tag{5.6}
\]

where the variable \( z \) is defined as \( z = q^2/\hat{s} \).

The NLO corrections are written for general processes as

\[
\sigma_{\text{NLO}} = \sum_{i=1}^{n_{\text{real}}} \sigma_{\text{trad}}(R_i), \tag{5.7}
\]

where the processes \( \{R_i\} \) are the real emission processes, which are the same as those listed in Step 1 in the DSA. Each cross section \( \sigma_{\text{trad}}(R_i) \) is written as

\[
\sigma_{\text{trad}}(R_i) = \int dx_1 \int dx_2 \int \mathbb{F}(x_1)(x_1) \mathbb{F}(x_2)(x_2) \left[ \hat{\sigma}_R(R_i) + \hat{\sigma}_V(B1(R_i)) + \hat{\sigma}_C(R_i) \right], \tag{5.8}
\]

where the real correction \( \hat{\sigma}_R(R_i) \) is defined in \( d \) dimensions, in contrast with the quantity in Eq. (2.5). The virtual correction \( \hat{\sigma}_V(B1(R_i)) \) is the same as in Eq. (2.7). The symbol \( \hat{\sigma}_C(R_i) \) represents the collinear subtraction term, which is concretely shown later.
The Drell–Yan process has three independent real emission processes, listed in Eq. (3.1) as

\[ R_1 = u\bar{u} \rightarrow \mu^- \mu^+ g, \]
\[ R_2 = ug \rightarrow \mu^- \mu^+ u, \]
\[ R_3 = \bar{u}g \rightarrow \mu^- \mu^+ \bar{u}. \]

Each cross section is written as

\[ \sigma_{\text{trad}}(R_1) = \int dx_1 \int dx_2 f_u(x_1)f_{\bar{u}}(x_2) \left[ \hat{\sigma}_R(R_1) + \hat{\sigma}_V(B_1(R_1)) + \hat{\sigma}_C(R_1) \right], \]

\[ \sigma_{\text{trad}}(R_2) = \int dx_1 \int dx_2 f_u(x_1)f_g(x_2) \left[ \hat{\sigma}_R(R_2) + \hat{\sigma}_C(R_2) \right], \]

and the cross section \( \sigma_{\text{trad}}(R_3) \) is obtained by the replacement \( u \rightarrow \bar{u} \) in \( \sigma_{\text{trad}}(R_2) \). First we show the partonic cross sections included in the \( \sigma_{\text{trad}}(R_1) \) in Eq. (5.10).

\(\hat{\sigma}_R(R_1)\)

The real correction \( \hat{\sigma}_R(R_i) \) is defined in \( \delta \) dimensions as

\[ \hat{\sigma}_R(R_1) = \frac{1}{S_{R_1}} \Phi(R_1)_{\delta} \cdot |M(R_1)|^2_{\delta}. \]

A Feynman diagram of the real correction is shown in Fig. 4. The \( q^2 \)-distribution is calculated as

\[ \frac{d\hat{\sigma}_R(R_1)}{dq^2} = C(q^2, \epsilon) a_R, \]

where the factor \( C(q^2, \epsilon) \) is defined as

\[ C(q^2, \epsilon) = \frac{\hat{\sigma}_{\text{LO}}(q^2)}{q^2} \frac{\alpha_s}{\pi} C_F \left( \frac{4\pi\mu^2}{q^2} \right)^\epsilon \frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)}. \]

\( a_R \) is written as

\[ a_R = \delta(1-z) \frac{1}{\epsilon^2} + z(1+z^2) \left[ -\frac{1}{\epsilon} \frac{1}{1-z} + 2 \left( \frac{\ln(1-z)}{1-z} \right) + \frac{1}{1-z} \ln z \right] + O(\epsilon). \]
This expression includes the so-called (+)-distribution, which is defined for arbitrary functions \( g(z) \) and \( h(z) \) as
\[
\int_0^\infty dz \, g(z) \, (h(z))_+ = \int_0^\infty dz \, g(z) \, h(z) - \int_0^1 dz \, g(z = 1) \, h(z),
\]
with a value \( 0 \leq \tau < 1 \).

\( \hat{\sigma}_V(B1(R1)) \)

The virtual correction \( \hat{\sigma}_V(B1(R1)) \) is defined in Eq. (2.7) as
\[
\hat{\sigma}_V(B1) = \frac{1}{S_{B1}} \Phi(B1)_d \cdot |M_{\text{virt}}(B1)|_d^2.
\]

A Feynman diagram of the virtual one-loop correction is shown in Fig. 5. The \( q^2 \)-distribution is calculated as
\[
\frac{d\hat{\sigma}_V(B1)}{dq^2} = C(q^2, \epsilon) \, a_V,
\]
where the factor \( C(q^2, \epsilon) \) is the same as in Eq. (5.14) and, after the subtraction of the UV pole \( 1/\epsilon_{UV} \) by the renormalization program, \( a_V \) is obtained as
\[
a_V = \delta(1 - z) \left[ -\frac{1}{\epsilon^2} - \frac{3}{2} \epsilon + (\frac{\pi^2}{3} - 4) \right].
\]

\( \hat{\sigma}_C(R1) \)

The \( q^2 \)-distribution of the collinear subtraction term is written as
\[
\frac{d\hat{\sigma}_C(R1)}{dq^2} = -\frac{\alpha_s}{\pi} \frac{1}{\Gamma(1 - \epsilon)} \int_0^1 dx \left[ -\epsilon \left( \frac{4\pi\mu_F^2}{\mu_F^2} \right)^\epsilon \right] \frac{d\hat{\sigma}(L1, x\hat{s})}{dq^2},
\]
where the \( q^2 \)-distribution of the LO process \( L1 \) with the rescaled initial energy \( x\hat{s} \) is defined as
\[
\frac{d\hat{\sigma}(L1, x\hat{s})}{dq^2} = \hat{\sigma}_{LO}(q^2) \delta(q^2 - x\hat{s}) = \hat{\sigma}_{LO}(q^2) \frac{z}{q^2} \delta(z - x).
\]
Eq. (5.20) includes both contributions. The $q^2$-distribution is calculated as
\[ \frac{d \hat{\sigma}_C (R_1)}{dq^2} = C(q^2, \epsilon) a_C, \] (5.22)
where the factor $C(q^2, \epsilon)$ is the same as in Eq. (5.14) and $a_C$ is written as
\[ a_C = \frac{z P^{ff}(z)}{C_F} \left( \frac{1}{\epsilon} + \ln \frac{q^2}{\mu^2} \right) + O(\epsilon). \] (5.23)

The summation of the three contributions in Eqs. (5.13), (5.18), and (5.22), is free from divergences as follows:
\[ \frac{d \hat{\sigma}_{\text{trad}} (R_1)}{dq^2} = \frac{d \hat{\sigma}_R (R_1)}{dq^2} + \frac{d \hat{\sigma}_V (B1)}{dq^2} + \frac{d \hat{\sigma}_C (R_1)}{dq^2} = C(q^2, \epsilon) (a_R + a_V + a_C), \] (5.24)
with the summation
\[ a_V + a_R + a_C = \delta(1 - z) \left( \frac{\pi^2}{3} - 4 \right) + \frac{z P^{ff}(z)}{C_F} \ln \frac{q^2}{\mu^2} \]
\[ + z(1 + z^2) \left[ 2 \left( \frac{\ln(1 - z)}{1 - z} \right) + \frac{1}{1 - z} \ln z \right]. \] (5.25)

At this stage, with finite results in 4 dimensions, we can also return the common factor $C(q^2, \epsilon)$ to 4 dimensions as
\[ C(q^2, \epsilon = 0) = \frac{\hat{\sigma}^{(4)}_{\text{LO}} (q^2)}{q^2} \frac{\alpha_s}{\pi} C_F, \] (5.26)
with the total cross section at LO in 4 dimensions
\[ \hat{\sigma}^{(4)}_{\text{LO}} (q^2) = \alpha_s^2 \frac{4\pi}{3N_c} q^2 P(q^2). \] (5.27)

Next we show the partonic cross sections in $\sigma_{\text{trad}}(R_2)$ in Eq. (5.11).

- $\hat{\sigma}_R (R_2)$
  The real correction $\hat{\sigma}_R (R_2)$ is defined as
  \[ \hat{\sigma}_R (R_2) = \frac{1}{S_{R_2}} \Phi(R_2)_{d} \cdot |M(R_2)|^2_d. \] (5.28)
  Feynman diagrams of the real correction are shown in Fig. 6. The $q^2$-distribution is calculated as
  \[ \frac{d \hat{\sigma}_R (R_2)}{dq^2} = C(q^2, \epsilon) a_{R,ug}, \] (5.29)
  with
  \[ a_{R,ug} = \frac{z}{2C_F} \left[ -\frac{1}{\epsilon} P^{sf}(z) + \ln \frac{(1 - z)^2}{z} P^{sf}(z) + \frac{1}{4} (1 + 6z - 7z^2) \right] + O(\epsilon), \] (5.30)
  where the splitting function $P^{sf}(z)$ is written in Eq. (A65).
Fig. 6. Diagrams of the real emission process $R_2 = ug \rightarrow \mu^- \mu^+ u$.

○ $\hat{\sigma}_C(R_2)$

The $q^2$-distribution of the collinear subtraction term is written as

$$\frac{d \hat{\sigma}_C(R_2)}{dq^2} = -\frac{\alpha_s}{2\pi} \frac{1}{\Gamma(1-\epsilon)} \int_0^1 dx \left\{ \frac{1}{\epsilon} \left( \frac{4\pi \mu^2}{\mu_F^2} \right)^\epsilon \right\} P_{g\bar{u}}(x) \frac{d\hat{\sigma}(L_1, x\hat{s})}{dq^2} .$$  \hspace{1cm} (5.31)

where the $q^2$-distribution of the LO process is the same as in Eq. (5.21) and the splitting function $P_{g\bar{u}}(z) = P_{gf}(z)$ is shown in Eq. (A65). The distribution is written in the form

$$\frac{d \hat{\sigma}_C(R_2)}{dq^2} = C(q^2, \epsilon) a_{C, ug} ,$$ \hspace{1cm} (5.32)

$$a_{C, ug} = \frac{z}{C_F} P_{gf}(z) \left( \frac{1}{\epsilon} + \ln \frac{q^2}{\mu_F^2} \right) + O(\epsilon).$$ \hspace{1cm} (5.33)

The summation of the two contributions in Eqs. (5.29) and (5.32) is free from divergences as follows:

$$\frac{d \hat{\sigma}_{\text{trad}}(R_2)}{dq^2} = \frac{d \hat{\sigma}_R(R_2)}{dq^2} + \frac{d \hat{\sigma}_C(R_2)}{dq^2}$$

$$= C(q^2, \epsilon = 0) \left( a_{R, ug} + a_{C, ug} \right) ,$$ \hspace{1cm} (5.34)

with

$$a_{R, ug} + a_{C, ug} = \frac{z}{2C_F} \left[ P_{gf}(z) \cdot \ln \frac{(1-z)^2 q^2}{z \mu_F^2} + \frac{1}{4} (1 + 6z - 7z^2) \right] .$$ \hspace{1cm} (5.35)

Taking account of the exchanged initial states and all five massless flavors, we obtain the prediction of the $q^2$-distribution at NLO accuracy as

$$\frac{d \sigma_{\text{prediction}}}{dq^2} = \frac{d \sigma(L_1)}{dq^2} + \sum_{i=1}^3 \frac{d \sigma_{\text{trad}}(R_i)}{dq^2} .$$ \hspace{1cm} (5.36)

The contributions from the distributions $d \sigma(L_1)/dq^2$ and $d \sigma_{\text{trad}}(R_1)/dq^2$ are written as

$$\frac{d \sigma(L_1)}{dq^2} + \frac{d \sigma_{\text{trad}}(R_1)}{dq^2} = \sum_{q=u,c,d,b,s} \int_0^1 dx_1 \int_0^1 dz \frac{\tau}{x_1\tau^2} H_{q\bar{q}}(x_1, x_2)$$

$$\times \left[ \frac{d \hat{\sigma}(L_1)}{dq^2} + \frac{d \hat{\sigma}_{\text{trad}}(R_1)}{dq^2} \right] ,$$ \hspace{1cm} (5.37)

where the combination of PDFs is denoted as $H_{q\bar{q}}(x_1, x_2) = f_q(x_1) f_{\bar{q}}(x_2) + f_{\bar{q}}(x_1) f_q(x_2)$. The value $x_2$ is expressed as $x_2 = \tau/(x_1\tau)$ with $\tau = q^2/S$. The partonic distributions $d \hat{\sigma}(L_1)/dq^2$
and \(d\hat{\sigma}_{\text{trad}}(R_1)/dq^2\) are written in Eqs. (5.6) and (5.24), respectively. The contributions from the distributions \(d\sigma_{\text{trad}}(R_2)/dq^2\) and \(d\sigma_{\text{trad}}(R_3)/dq^2\) are written as

\[
\frac{d\sigma_{\text{trad}}(R_2)}{dq^2} + \frac{d\sigma_{\text{trad}}(R_3)}{dq^2} = \sum_{q=u,c,d,b,s} \int_0^1 dx_1 \int_0^{1/x_1} \frac{dz}{x_1 z^2} \left[ H_{qg}(x_1, x_2) \frac{d\hat{\sigma}_{\text{trad}}(R_2)}{dq^2} + H_{\bar{q}g}(x_1, x_2) \frac{d\hat{\sigma}_{\text{trad}}(R_3)}{dq^2} \right],
\]

with the PDFs \(H_{qg}(x_1, x_2) = f_q(x_1) f_{g}(x_2) + f_{\bar{q}}(x_1) f_q(x_2)\). The \(q^2\)-distribution of the partonic distribution \(d\hat{\sigma}_{\text{trad}}(R_2)/dq^2\) is written in Eq. (5.34). The expression of the partonic distribution \(d\hat{\sigma}_{\text{trad}}(R_3)/dq^2\) is identical to \(d\hat{\sigma}_{\text{trad}}(R_2)/dq^2\).

### 5.2. DSA

We use the dipole subtraction procedure through the DSA to obtain analytical results of the \(q^2\)-distribution in the Drell–Yan process. The DSA has already been applied to the Drell–Yan process in Sect. 3. The results are summarized in Sect. 3.5.

We start with the explicit calculation of \(\sigma(R_1)\) in Eq. (3.67). The subtracted real cross section \((\hat{\sigma}_R(R_1) - \hat{\sigma}_D(R_1))\) in Eq. (3.68) is defined in 4 dimensions. However, the analytical integration of the PS in 4 dimensions does not seem easy. Instead, we redefine the cross section in \(d\) dimensions and regularize the soft and collinear singularities as the poles of \(1/\epsilon\) and \(1/\epsilon^2\), which are produced by the \(d\)-dimensional PS integration. The poles from \(\hat{\sigma}_R(R_1)\) and \(\hat{\sigma}_D(R_1)\) cancel each other, and a finite analytical expression in 4 dimensions is obtained. The distribution \(d\hat{\sigma}_R(R_1)/dq^2\) is calculated in Eqs. (5.13) and (5.15). Then we proceed to the calculation of \(\hat{\sigma}_D(R_1)\), which is now defined in \(d\) dimensions as

\[
\hat{\sigma}_D(R_1) = \frac{1}{S_{R_1}} \Phi(R_1)_d \frac{1}{n_s(u) n_s(\bar{u})} D(R_1).
\]

There are two dipole terms for the process \(R_1\), as shown in Eq. (3.22). The first dipole term, \(D(\bar{a}||1, (3-2)a_3,b)\), is written in Eq. (3.15), which is now defined in \(d\) dimensions. The dipole splitting function \(V_{a_3,b}\) in \(d\) dimensions reads in Ref. [27] as

\[
V_{a_3,b} = 8\pi \mu^2 \epsilon \alpha_s C_F \left[ \frac{2}{1-x_{3,ab}} - 1 - x_{3,ab} - \epsilon (1 - x_{3,ab}) \right],
\]

and the square of the reduced Born process \(\langle B_1 | T \cdot T | B_1 \rangle\) is also defined in \(d\) dimensions. The contribution to the partonic cross section of the dipole terms can be analytically integrated over the PS region including the soft and collinear singularities. The analytical integration of the dipole terms for an arbitrary process is one core part of the construction of the dipole subtraction procedure [27]. The contribution of the first dipole term to the cross section in Eq. (5.39) is written as \(\hat{\sigma}_D(R_1 : D_{a_3,b})\), and the integration formula is applied to the cross section as

\[
\hat{\sigma}_D(R_1 : D_{a_3,b}) = - \int_0^1 dx \frac{1}{S_{R_1}} \Phi_a(R_1 : B_1, x)_d \frac{1}{n_s(u) n_s(\bar{u})} \frac{\alpha_s}{2\pi} \frac{1}{\Gamma(1 - \epsilon)} \left( \frac{4\pi \mu^2}{s_{ab}} \right)^\epsilon \tilde{V}^{\mu,u}(x; \epsilon) \frac{1}{C_F} \langle B_1 | T_{\gamma_a} \cdot T_{\gamma_b} | B_1 \rangle.
\]
where the Lorentz scalar is denoted as \( s_{ab} = 2 p_a \cdot p_b \) and the function \( \tilde{\nu}^{\mu,\nu}(x; \epsilon) \) reads in Ref. [27] as

\[
\tilde{\nu}^{\mu,\nu}(x; \epsilon) = -\frac{1}{\epsilon} P^{f \bar{f}}(x) + \delta(1 - x) \left[ \mathcal{V}_{f \bar{f}}(\epsilon) + C_F \left( \frac{\pi^2}{3} - 5 \right) \right] + C_F \left[ -\left( 4 \ln \frac{1}{1-x} + 1 - x - 2 (1 + x) \ln(1 - x) \right) \right].
\]

(5.42)

The correlation of the two color insertion operators in the square of the Born process \( B1 \) is fully factorized as \( \langle B1 | T_{y_a} \cdot T_{y_b} | B1 \rangle = -C_F \langle B1 | B1 \rangle \). The \( q^2 \)-distribution is written as

\[
d\hat{\sigma}_D(R_1 : D_{a3,b}) = \frac{\alpha_s}{2\pi} \frac{1}{\Gamma(1-\epsilon)} \left( \frac{4\pi\mu^2}{s_{ab}} \right)^\epsilon \int_0^1 dx \tilde{\nu}^{\mu,\nu}(x; \epsilon) \frac{d\hat{\sigma}(L_1, x\hat{s})}{dq^2} \]

(5.43)

where the distribution with the scaled initial energy, \( d\hat{\sigma}(L_1, x\hat{s})/dq^2 \), is defined in Eq. (5.21). The contribution from the second dipole in Eq. (3.21) is written identically. Then the contribution of all dipole terms in \( D(R_1) \) is written in the form

\[
d\hat{\sigma}_D(R_1) = C(q^2, \epsilon) a_D,
\]

(5.44)

where the common factor \( C(q^2, \epsilon) \) is the same as in Eq. (5.14), and \( a_D \) is written as

\[
a_D = \frac{1}{C_F} \frac{\Gamma(1-2\epsilon)}{\Gamma(1-\epsilon)^2} \left( \frac{q^2}{s_{ab}} \right)^\epsilon \tilde{\nu}^{f\bar{f}}(z; \epsilon) z,
\]

(5.45)

which is expanded by \( 1/\epsilon \) as

\[
a_D = \delta(1 - z) \frac{1}{\epsilon^2} + z \left[ -\frac{1}{\epsilon} + \frac{1 + z^2}{\epsilon (1 - z)_+} + \frac{4 \ln \frac{1}{1-z}}{1-z} \right] + 1 - z - z(1 + z) \ln(1 - z) - \frac{1 + z^2}{(1 - z)_+} \ln z + O(\epsilon).
\]

(5.46)

Recalling the expression of \( a_R \) in Eq. (5.15), the \( q^2 \)-distribution of the subtracted real correction is written as

\[
d\hat{\sigma}_R(R_1) = d\hat{\sigma}_D(R_1) = C(q^2, \epsilon = 0) (a_R - a_D),
\]

(5.47)

with the difference

\[
a_R - a_D = -z(1 - z).
\]

(5.48)

Next we calculate the subtracted virtual correction \( \hat{\sigma}_V(B1(R_1)) + \hat{\sigma}_I(R_1) \) in Eq. (3.69). The concrete expression of the virtual correction \( \hat{\sigma}_V(B1(R_1)) \) is written in Eqs. (5.18) and (5.19). The expression of the I term \( I(R_1) \) is given in Eq. (3.42) and the contribution to the partonic cross section
is written as

\[ \hat{\sigma}_1(R_1) = \frac{1}{s_{B1}} \Phi(B1)_d I(R_1) \]

\[ \frac{1}{s_{B1}} \Phi(B1)_d 2A_d \mathcal{V}_f s^\epsilon \langle B1 \mid B1 \rangle. \]

The \( q^2 \)-distribution is calculated in the form

\[ \frac{d\hat{\sigma}(R_1)}{dq^2} = C(q^2, \epsilon) a_1, \]

with

\[ a_1 = \delta(1 - z) \frac{1}{C_F} \frac{\Gamma(1 - 2\epsilon)}{\Gamma(1 - \epsilon)^2} \mathcal{V}_f \]

\[ = \delta(1 - z) \left[ \frac{1}{\epsilon^2} + \frac{3}{2} \frac{1}{\epsilon} - \frac{\pi^2}{3} + 5 \right] + O(\epsilon). \]

Then we obtain the \( q^2 \)-distribution of the subtracted virtual correction as

\[ \frac{d\hat{\sigma}_V(B1(R_1)) - d\hat{\sigma}_1(R_1)}{dq^2} = C(q^2, \epsilon = 0) (a_V - a_1), \]

with the difference

\[ a_V - a_1 = \delta(1 - z). \]

The cross section of the P and K terms is written in Eq. (3.70). The P and K terms are obtained in Eqs. (3.56) and (3.58), and the cross section is explicitly written down as

\[ \hat{\sigma}_P(R_1) + \hat{\sigma}_K(R_1) = \int_0^1 dx \frac{1}{s_{B1}} \Phi(\alpha_x) \frac{\alpha_x}{2\pi} \]

\[ \times \left[ -P^{\mu\nu}(x) \ln \frac{\mu_F^2}{x s_{ab}} + \tilde{K}^{\mu\nu}(x) + \tilde{K}^{\nu\mu}(x) \right] \langle B1 \mid B1 \rangle + (a \leftrightarrow b). \]

The \( q^2 \)-distribution is calculated as

\[ \frac{d\hat{\sigma}_P(R_1)}{dq^2} + \frac{d\hat{\sigma}_K(R_1)}{dq^2} = \int_0^1 dx \frac{\alpha_x}{\pi} \left[ -P^{\mu\nu}(x) \ln \frac{\mu_F^2}{x s_{ab}} + \tilde{K}^{\mu\nu}(x) + \tilde{K}^{\nu\mu}(x) \right] \frac{d\hat{\sigma}(L_1, x\hat{s})}{dq^2} \]

\[ = C(q^2, \epsilon = 0) a_{PK}, \]

with the expression

\[ a_{PK} = \frac{z}{C_F} \left[ -P^{\mu\nu}(z) \ln \frac{\mu_F^2}{q^2} + \tilde{K}^{\mu\nu}(z) + \tilde{K}^{\nu\mu}(z) \right] \]

\[ = \delta(1 - z) \left[ \frac{\pi^2}{3} - 5 \right] + \frac{z P^{\mu\nu}(z)}{C_F} \ln \frac{q^2}{\mu_F^2} + z(1 + z^2) \left[ \left( \frac{2}{1 - z} \ln(1 - z) \right) + \frac{1}{(1 - z) \ln z} \right]. \]

Then the \( q^2 \)-distribution \( d\hat{\sigma}(R_1)/dq^2 \) is written as

\[ \frac{d\hat{\sigma}(R_1)}{dq^2} = \frac{d}{dq^2} \left( \hat{\sigma}_R(R_1) - \hat{\sigma}_D(R_1) + \hat{\sigma}_V(B1(R_1)) + \hat{\sigma}_I(R_1) + \hat{\sigma}_P(R_1) + \hat{\sigma}_K(R_1) \right) \]

\[ = C(q^2, \epsilon = 0) \left[ (a_R - a_D) + (a_V + a_I) + a_{PK} \right]. \]
where the finite quantities \((a_R - a_D), (a_V + a_t),\) and \(a_{PK}\) are written in Eqs. (5.48), (5.54), and (5.58). The summation of the three contributions is calculated as

\[
(a_R - a_D) + (a_V + a_t) + a_{PK} = \delta(1 - z) \left( \frac{\pi^2}{3} - 4 \right) + z P^{ff}(z) \frac{q^2}{C_F} \ln \frac{q^2}{\mu^2_F} + z(1 + z^2) \left[ 2 \left( \frac{\ln(1 - z)}{1 - z} \right) + \frac{1}{1 - z} \ln z \right].
\]

The results exactly coincide with the results obtained by the traditional methods in Eq. (5.25).

Next we calculate the \(q^2\)-distribution of \(\sigma(R_2)\) in Eq. (3.71). The subtracted real correction is written in Eq. (3.72). Similarly to the case of \(\sigma(R_1)\), we redefine the cross section in \(d\) dimensions. The distribution \(d\hat{\sigma}(R_2)/dq^2\) is obtained in Eqs. (5.29) and (5.30). Then we proceed to the calculation of \(\hat{\sigma}_D(R_2)\), which is defined in \(d\) dimensions as

\[
\hat{\sigma}_D(R_2) = \frac{1}{\delta R_2} \Phi(R_2)_d \frac{1}{n_s(u)n_s(g)} D(R_2).
\]

For the process \(R_2\), only one dipole exists in Eq. (3.35) and the expression is given in Eq. (3.34). The cross section is analytically integrated over the soft and collinear regions, and the \(q^2\)-distribution is calculated as

\[
\frac{d\hat{\sigma}_D(R_2)}{dq^2} = \frac{\alpha_s}{2\pi} \frac{1}{r(1 - \epsilon)} \left( \frac{4\pi\mu^2}{s_{ab}} \right)^\epsilon \tilde{\gamma}^{gf}(z; \epsilon) \frac{\hat{\sigma}_{LO}(q^2)}{q^2} \frac{1}{z},
\]

where the function \(\tilde{\gamma}^{gf}(z; \epsilon)\) reads in Ref. [27] as

\[
\tilde{\gamma}^{gf}(z; \epsilon) = \frac{1}{\epsilon} P^{gf}(z) + 2 P^{gf}(z) \ln(1 - z) + T_R 2z(1 - z).
\]

The result is written in the form

\[
\frac{d\hat{\sigma}_D(R_2)}{dq^2} = C(q^2, \epsilon) a_{D,ug},
\]

with the factor

\[
a_{D,ug} = \frac{z}{2 C_F} \left[ -\frac{1}{\epsilon} P^{gf}(z) + 2 P^{gf}(z) \ln(1 - z) - P^{gf}(z) \ln z + T_R 2z(1 - z) \right].
\]

The difference between \(a_{R,ug}\) in Eq. (5.30) and \(a_{D,ug}\) in Eq. (5.65) is calculated as

\[
a_{R,ug} - a_{D,ug} = \frac{z}{8 C_F} (1 + 2z - 3z^2).
\]

The cross sections of the \(P\) and \(K\) terms are written in Eq. (3.73). The \(P\) and \(K\) terms \(P(R_2)\) and \(K(R_2)\) are created in Eqs. (3.64) and (3.65). The cross section is explicitly written as

\[
\hat{\sigma}_P(R_2) + \hat{\sigma}_K(R_2) = \int_0^1 dx \frac{1}{s_{B_{4u}}} \Phi_g(R_2 : B_{4u}, x) \frac{\alpha_s}{2\pi} \times \left[ -P^{gf}(x) \ln \frac{\mu^2_F}{x} + \tilde{K}^{gf}(x) + \tilde{K}^{gf}(x) \right] \langle B_{4u} | B_{4u} \rangle.
\]

Then the \(q^2\)-distribution is calculated in the form

\[
\frac{d\hat{\sigma}_P(R_2)}{dq^2} + \frac{d\hat{\sigma}_K(R_2)}{dq^2} = C(q^2, \epsilon = 0) a_{PK,ug},
\]
with the factor

\[ a_{PK,ug} = \frac{z}{2C_F} \left[ P_{gf}(z) \ln \frac{(1-z)^2q^2}{z\mu_F^2} + T_R z(1-z) \right]. \tag{5.69} \]

Finally, we obtain the \( q^2 \)-distribution \( d\hat{\sigma}(R_2)/dq^2 \) as

\[
\frac{d\hat{\sigma}(R_2)}{dq^2} = \frac{d}{dq^2} \left( \hat{\sigma}_R(R_2) - \hat{\sigma}_D(R_2) + \hat{\sigma}_P(R_2) + \hat{\sigma}_K(R_2) \right)
= C(q^2, \epsilon = 0) \left[ (a_{R,ug} - a_{D,ug}) + a_{PK,ug} \right]. \tag{5.70}
\]

with the difference

\[ a_{R,ug} - a_{D,ug} + a_{PK,ug} = \frac{z}{2C_F} \left[ P_{gf}(z) \ln \frac{(1-z)^2q^2}{z\mu_F^2} + \frac{1}{4} \left( 1 + 6z - 7z^2 \right) \right]. \tag{5.71} \]

The results exactly coincide with the results in Eq. (5.35). The calculation of the distribution \( d\hat{\sigma}(R_3)/dq^2 \) is completely analogous to \( d\hat{\sigma}(R_2)/dq^2 \), and the results are identical. In this way, it is shown that the analytical results by the DSA exactly agree with the well known results by the traditional methods as

\[
\frac{d\hat{\sigma}(R_i)}{dq^2} = \frac{d\hat{\sigma}_{\text{trad}}(R_i)}{dq^2} \quad \text{for} \quad i = 1, 2, \text{and } 3. \tag{5.72}
\]

6. Summary

We study the QCD NLO corrections in hadron collider processes. In simple processes, the analytical results are obtained by the traditional method, which may originate in the pioneering works on the Drell–Yan process [11–19]. The traditional method is reviewed in Sect. 5.1. In complex processes, namely, multiparton leg processes like \( pp \rightarrow n \) jets, it is almost impossible to obtain analytical results for the NLO corrections. The dipole subtraction procedure overcomes some difficulties of the calculations and makes it possible to obtain NLO corrections in the multiparton leg processes. The price of employing the dipole subtraction mainly involves two things. One is that many subtraction terms are created and the expressions are not so simple. The other one is that a large amount of the calculation is executed as numerical evaluation of the Monte Carlo integration. As a consequence, the person who has obtained the results of the NLO corrections has difficulty confirming whether the obtained results are true or false. For the other person who does not do the calculations him- or herself, the confirmation is more difficult. In order to solve some of the difficulties, we need a practical algorithm to use the dipole subtraction that allows clear presentation of the following items:

- Input, output, creation order, and all formulae in the document,
- Necessary information to specify each subtraction term,
- Summary table of all created subtraction terms,
- Associated proof algorithm.

A clear definition of the practical algorithm in the document produces the merit that we can precisely communicate about the subtraction terms through a common language, and quickly compare those by two or more people, and also that, when the algorithm is implemented as a computer package, the users can understand the outputs properly. In this article, we have presented such an algorithm that satisfies all the above requirements. We call it the dipole splitting algorithm (DSA) and define
it in Sect. 2. The master formulae and all the steps are defined in Sect. 2.1 and the formulae for the subtraction terms are collected in Appendix A. The creation algorithm and concrete expressions for the D, I, and P/K terms are explained in Sects. 2.2, 2.3, and 2.4, respectively. The advantage of the DSA is clarified in Sect. 2.5. We demonstrate the DSA in the Drell–Yan process in Sect. 3 and in the dijet process in Sect. 4. Summary tables for the Drell–Yan process are shown in Sect. 3. Summary tables for the dijet process are shown in Appendix B. These tables can be a template for summary tables specifying all the subtraction terms created by the DSA. Regarding the use of the summary tables, we intend that the other person who does not execute the DSA himself or herself can specify the subtraction terms created and write down concrete expressions for them simply by reading the tables in a document. As one particularly reliable confirmation of the DSA, we have made an analytical check against the results by the traditional method in the Drell–Yan process in Sect. 5. The associated algorithm proving the consistency relation is presented in Ref. [98].

We plan to study the following subjects in the future. We will apply the DSA to some processes at the LHC and make predictions at NLO accuracy. The DSA can be applied to processes at $e^-e^+$ and $e^-p$ colliders as well. The application is easy, because the DSA becomes simpler than the hadron collider case. The DSA is presented for an arbitrary process including only massless quarks in this article. We will present the DSA for processes including massive quarks. The extension should be straightforward, because the construction algorithm of the subtraction terms for the massive quarks presented in Ref. [28] is the same as in the case of the massless quarks in Ref. [27]. Regarding automation of the DSA as a computer package, AutoDipole [93] is a good candidate for its implementation, because the creation algorithm of the dipole and I terms implemented in AutoDipole is essentially the same as the DSA and only the creation algorithm of the P and K terms is different from the DSA. Thus, it is sufficient that only some of the code to create the P and K terms is modified. We hope that the DSA will help users to obtain reliable predictions at QCD NLO accuracy.

Acknowledgements

We are grateful to V. Ravindran for helpful discussion. The present article was completed during a stay at the Harish-Chandra Research Institute. We are grateful for the warm support of the institute.

Funding

Open Access funding: SCOAP³.

Appendix A. Formulae for DSA

A.1. Step 2: D term

$$\hat{\sigma}_D(R_i) = \frac{1}{S_{R_i}} \Phi(R_i) \frac{1}{n_s(a)n_s(b)} D(R_i),$$

\begin{equation}
D(R_i, \text{dip}j)_{IJ,K} = -\frac{1}{s_{IJ} x_{IK}} \frac{1}{T_{\gamma_{\text{emi}}}^F} \left\langle B_j \mid T_{\gamma_{\text{emi}}} \cdot T_{\gamma_{\text{spe}}} V_{IJ,K}^{\gamma_{\text{emi}}} \mid B_j \right\rangle.
\end{equation}

\[D_{ij,k} = -\frac{1}{s_{ij}} C_F V_{ij,k} \langle B1 \mid T \cdot T \mid B1 \rangle,\]

\[V_{ij,k} = 8\pi \alpha_s C_F \left[ \frac{2}{1 - z_i(1 - y_{ij,k})} - 1 - z_i \right].\]
Fig. A1. The creation order of the dipole terms is shown. There are seven possible splittings (1)–(7). Splittings (1)–(4) in the upper half are diagonal splittings, which are grouped into category Dipole 1. Splittings (5)–(7) in the lower half are nondiagonal splittings. The indices $a$ and $i/j$ represent the legs in the initial and final states, respectively.

Dipole 1 (1)-2: $(ij, a) = (fg, a)$

$$D_{ij,a} = -\frac{1}{s_{ij}} \frac{1}{x_{ij,a}} \frac{1}{C_{F}} V_{ij,a} \langle B1 \mid T \cdot T \mid B1 \rangle,$$  \hspace{0.5cm} (A5)

$$V_{ij,a} = 8\pi \alpha_s C_F \left[ \frac{2}{1 - z_i \left( 1 - x_{ij,a} \right)} - 1 - z_i \right].$$  \hspace{0.5cm} (A6)

Dipole 1 (2)-1: $(ij, k) = (gg, k)$

$$D_{ij,k} = -\frac{1}{s_{ij}} \frac{1}{C_{A}} \langle B1 \mid T \cdot T \mid V_{ij,k} \mid B1 \rangle,$$  \hspace{0.5cm} (A7)

$$V_{ij,k}^{\mu \nu} = 16\pi \alpha_s C_A \left[ -g^{\mu \nu} \left( \frac{1}{1 - z_i \left( 1 - y_{ij,k} \right)} + \frac{1}{1 - z_j \left( 1 - y_{ij,k} \right)} - 2 \right) ight.$$ \hspace{0.5cm} \hspace{0.5cm} \hspace{0.5cm} \hspace{0.5cm} + \frac{1}{p_i p_j} \left( z_i p_i^\mu - z_j p_j^\mu \right) \left( z_i p_i^\nu - z_j p_j^\nu \right) \bigg].$$  \hspace{0.5cm} (A8)

Dipole 1 (2)-2: $(ij, a) = (gg, a)$

$$D_{ij,a} = -\frac{1}{s_{ij}} \frac{1}{x_{ij,a}} \frac{1}{C_{A}} \langle B1 \mid T \cdot T \mid V_{ij,a} \mid B1 \rangle,$$  \hspace{0.5cm} (A9)

$$V_{ij,a}^{\mu \nu} = 16\pi \alpha_s C_F \left[ -g^{\mu \nu} \left( \frac{1}{1 - z_i \left( 1 - x_{ij,a} \right)} + \frac{1}{1 - z_j \left( 1 - x_{ij,a} \right)} - 2 \right) ight.$$ \hspace{0.5cm} \hspace{0.5cm} \hspace{0.5cm} \hspace{0.5cm} + \frac{1}{p_i p_j} \left( z_i p_i^\mu - z_j p_j^\mu \right) \left( z_i p_i^\nu - z_j p_j^\nu \right) \bigg].$$  \hspace{0.5cm} (A10)
Dipole1 (3)-1: \((ai, k) = (fg, k)\)

\[
D_{ai,k} = -\frac{1}{s_{ai}} \frac{1}{x_{ik,a}} \frac{1}{C_F} V_{ai,k} \langle B1 | T \cdot T | B1 \rangle,
\]
\[
V_{ai,k} = 8\pi \alpha_s C_F \left[ \frac{2}{1 - x_{ik,a} + u_i} - 1 - x_{ik,a} \right].
\] (A11)

Dipole1 (3)-2: \((ai, b) = (fg, b)\)

\[
D_{ai,b} = -\frac{1}{s_{ai}} \frac{1}{x_{i,ab}} \frac{1}{C_F} V_{ai,b} \langle B1 | T \cdot T | B1 \rangle,
\]
\[
V_{ai,b} = 8\pi \alpha_s C_F \left[ \frac{2}{1 - x_{i,ab}} - 1 - x_{i,ab} \right].
\] (A13)

Dipole1 (4)-1: \((ai, k) = (gg, k)\)

\[
D_{ai,k} = -\frac{1}{s_{ai}} \frac{1}{x_{ik,a}} C_A \langle B1 | T \cdot T V_{ai,k} | B1 \rangle,
\]
\[
V_{ai,k}^{\mu\nu} = 16\pi \alpha_s C_A \left[ -g^{\mu\nu} \left( \frac{1}{1 - x_{ik,a} + u_i} - 1 + x_{ik,a} (1 - x_{ik,a}) \right) + \frac{1 - x_{ik,a} u_i (1 - u_i)}{x_{ik,a}} \left( \frac{p_i^\mu}{u_i} - \frac{p_k^\mu}{1 - u_i} \right) \left( \frac{p_i^\nu}{u_i} - \frac{p_k^\nu}{1 - u_i} \right) \right].
\] (A15)

Dipole1 (4)-2: \((ai, b) = (gg, b)\)

\[
D_{ai,b} = -\frac{1}{s_{ai}} \frac{1}{x_{i,ab}} C_A \langle B1 | T \cdot T V_{ai,b} | B1 \rangle,
\]
\[
V_{ai,b}^{\mu\nu} = 16\pi \alpha_s C_A \left[ -g^{\mu\nu} \left( \frac{x_{i,ab}}{1 - x_{i,ab}} + x_{i,ab} (1 - x_{i,ab}) \right) + \frac{1 - x_{i,ab}}{x_{i,ab}} \left( \frac{p_a}{p_i} \cdot \frac{p_b}{p_i} \cdot \frac{p_i}{p_a} \cdot \frac{p_i}{p_b} \right) \left( \frac{p_i^\mu}{p_i} - \frac{p_i^\mu}{p_b p_a} \right) \left( \frac{p_i^\nu}{p_i} - \frac{p_i^\nu}{p_b p_a} \right) \right].
\] (A17)

Dipole2 (5)-1: \((ij, k) = (f \bar{f}, k)\)

\[
D_{ij,k} = -\frac{1}{s_{ij}} C_A \langle B2 | T \cdot T V_{ij,k} | B2 \rangle,
\]
\[
V_{ij,k}^{\mu\nu} = 8\pi \alpha_s T_R \left[ -g^{\mu\nu} - \frac{2}{p_i p_j} (z_i p_i^\mu - z_j p_i^\mu) (z_i p_i^\nu - z_j p_i^\nu) \right].
\] (A19)

Dipole2 (5)-2: \((ij, b) = (f \bar{f}, b)\)

\[
D_{ij,a} = -\frac{1}{s_{ij}} \frac{1}{x_{ij,a}} C_A \langle B2 | T \cdot T V_{ij,a} | B2 \rangle,
\]
\[
V_{ij,a}^{\mu\nu} = 8\pi \alpha_s T_R \left[ -g^{\mu\nu} - \frac{2}{p_i p_j} (z_i p_i^\mu - z_j p_i^\mu) (z_i p_i^\nu - z_j p_i^\nu) \right].
\] (A21)
Dipole3 (6)-1: \((a_i, k) = (f f, k)\)

\[
D_{a_i,k} = -\frac{1}{s_{a_i}} \frac{1}{x_{i,k,a}} \frac{1}{C_A} (B3 | T \cdot | V_{a_i,k} | B3),
\]

\[
V_{a_i,k}^{\mu\nu} = 8\pi \alpha_s C_F \left[-g^{\mu\nu} x_{i,k,a} + \frac{1 - x_{i,k,a}}{x_{i,k,a}} \frac{2 u_i (1 - u_i)}{p_i p_k} \left(\frac{p_i^\mu}{u_i} - \frac{p_k^\mu}{1 - u_i}\right)\right].
\]

Dipole3 (6)-2: \((a_i, b) = (f f, b)\)

\[
D_{a_i,b} = -\frac{1}{s_{a_i}} \frac{1}{x_{i,ab}} \frac{1}{C_A} (B3 | T \cdot | V_{a_i,b} | B3),
\]

\[
V_{a_i,b}^{\mu\nu} = 8\pi \alpha_s C_F \left[-g^{\mu\nu} x_{i,ab} + \frac{1 - x_{i,ab}}{x_{i,ab}} \frac{2 p_a \cdot p_b}{p_i \cdot p_a p_i \cdot p_b} \left(\frac{p_i^\mu}{p_i} - \frac{p_i p_a}{p_b p_a p_b^\nu}\right)\right].
\]

Dipole4 (7)-1: \((a_i, k) = (g f, k)\)

\[
D_{a_i,k} = -\frac{1}{s_{a_i}} \frac{1}{x_{i,k,a}} \frac{1}{C_F} V_{a_i,k} (B4 | T \cdot | B4),
\]

\[
V_{a_i,k} = 8\pi \alpha_s T_R \left[1 - 2x_{i,k,a}(1 - x_{i,k,a})\right].
\]

Dipole4 (7)-2: \((a_i, b) = (g f, b)\)

\[
D_{a_i,b} = -\frac{1}{s_{a_i}} \frac{1}{x_{i,ab}} \frac{1}{C_F} V_{a_i,b} (B4 | T \cdot | B4),
\]

\[
V_{a_i,b} = 8\pi \alpha_s T_R \left[1 - 2x_{i,ab}(1 - x_{i,ab})\right].
\]

Reduced momenta and some functions:

\(D_{i,j,k}:\) Final-final dipole

Dipole 1 (1)-1, (2)-1,

Dipole 2 (5)-1:

\[
\tilde{p}_{ij}^\mu = p_i^\mu + p_j^\mu - \frac{y_{i,j,k}}{1 - y_{i,j,k}} p_k^\mu,
\]

\[
\tilde{p}_k^\mu = \frac{1}{1 - y_{i,j,k}} p_k^\mu,
\]

\[x_{ijk} = 1,\]

\[y_{i,j,k} = \frac{p_i \cdot p_j}{p_i \cdot p_j + p_j \cdot p_k + p_k \cdot p_i},\]

\[z_i = \frac{p_i \cdot p_k}{(p_i + p_j) \cdot p_k},\]

\[z_j = 1 - z_i.\]
**D_{ij,a}: Final-initial dipole**

Dipole 1 (1)-2, (2)-2,
Dipole 2 (5)-2:

\[
\tilde{p}_{ij}^\mu = p_i^\mu + p_j^\mu - (1 - x_{ij,b}) p_b^\mu, \\
\tilde{p}_{a}^\mu = x_{ij,a} p_a^\mu, \\
x_{ija} = x_{ij,a} = \frac{p_i \cdot p_a + p_j \cdot p_a - p_i \cdot p_j}{(p_i + p_j) \cdot p_a}, \\
z_i = \frac{p_i \cdot p_a}{(p_i + p_j) \cdot p_a}, \\
z_j = 1 - z_i.
\]

**D_{ai,k}: Initial-final dipole**

Dipole 1 (3)-1, (4)-1,
Dipole 3 (6)-1,
Dipole 4 (7)-1:

\[
\tilde{p}_{ai}^\mu = x_{ik,a} p_a^\mu, \\
\tilde{p}_{k}^\mu = p_i^\mu + p_k^\mu - (1 - x_{ik,a}) p_a^\mu, \\
x_{aik} = x_{ik,a} = \frac{p_i \cdot p_a + p_k \cdot p_a - p_i \cdot p_k}{(p_i + p_k) \cdot p_a}, \\
u_i = \frac{p_i \cdot p_a}{(p_i + p_k) \cdot p_a}.
\]

**D_{ai,b}: Initial-initial dipole**

Dipole 1 (3)-2, (4)-2,
Dipole 3 (6)-2,
Dipole 4 (7)-2:

\[
\tilde{p}_{ai}^\mu = x_{i,ab} p_a^\mu, \\
\tilde{k}_j^\mu = k_j^\mu - \frac{2k_j \cdot (K + \tilde{K})}{(K + \tilde{K})^2} (K + \tilde{K})^\mu + \frac{2k_j \cdot K}{K^2} \tilde{K}^\mu, \\
x_{ai,ab} = x_{i,ab} = \frac{p_a \cdot p_b - p_i \cdot p_a - p_i \cdot p_b}{p_a \cdot p_b}, \\
K^\mu = p_a + p_b - p_i, \\
\tilde{K}^\mu = \tilde{p}_{ai}^\mu + p_b^\mu.
\]

**A.2. Step 3: I term**

\[
\hat{\sigma}_1(R_i) = \frac{1}{S_{B1}} \Phi(B1)_{d} \cdot I(R_i), \\
I_{I,K} = -A_d \cdot \frac{1}{T^2_{F(t)}} \gamma_{F(t)} \cdot s^{\gamma}_{T_R} (T_I \cdot T_K), \\
A_d = \frac{\alpha_s (4\pi \mu^2)^{\epsilon}}{2\pi \Gamma(1 - \epsilon)}.
\]
Fig. A2. The creation order of the I terms is shown.

Universal singular functions:

\[ V_f = V_{fg}(\epsilon), \]  
\[ V_g = \frac{1}{2} V_{gg}(\epsilon) + N_f V_{f\bar{f}}(\epsilon), \]  
\[ \text{with} \]
\[ V_{fg}(\epsilon) = C_F \left[ \frac{1}{\epsilon^2} + \frac{3}{2\epsilon} + 5 - \frac{\pi^2}{2} \right], \]  
\[ V_{gg}(\epsilon) = 2C_A \left[ \frac{1}{\epsilon^2} + \frac{11}{6\epsilon} + \frac{50}{9} - \frac{\pi^2}{2} \right], \]  
\[ V_{f\bar{f}}(\epsilon) = T_R \left[ -\frac{2}{3\epsilon} - \frac{16}{9} \right]. \]

A.3. Step 4: P and K terms

\[ \hat{\sigma}_{P/K}(R_i) = \int_0^1 dx \sum_{B_j} \frac{1}{\Delta_{B_j}} \Phi_a(R_i : B_j, x) \cdot P/K(R_i, x_a : B_j, x_{pa}) + (a \leftrightarrow b). \]

P term

\[ P(R_i, x_a : B_j, y_a, y_K) = A_4 \cdot \frac{1}{T_{F(y_a)}} P^{F(x_a)F(y_a)}(x) \ln \frac{\mu^2_F}{x s_{x_a y_K}} (B_j \mid T_{y_a} \cdot T_{y_K} \mid B_j). \]

\[ A_4 = \frac{\alpha_s}{2\pi}. \]

Altarelli–Parisi splitting functions:

\[ \text{Dipole1 (3): } (a, i) = (f, g) \]

\[ P^{ff}(x) = C_F \left[ \frac{1 + x^2}{1 - x} \right]_+ = C_F \left[ \frac{1 + x^2}{(1 - x)_+} + \frac{3}{2} \delta(1 - x) \right]. \]
Fig. A3. The creation order of the P and K terms is shown.

Dipole 1 (4): \((a, i) = (g, g)\)

\[
P^{gg}(x) = 2CA \left[ \frac{1}{1-x} + \frac{1-x}{x} - 1 + x(1-x) \right] + \delta(1-x) \left( \frac{11}{6}CA - \frac{2}{3}NfTR \right).
\] (A63)

Dipole 3 (6): \((a, i) = (f, f)\)

\[
P^{fg}(x) = CF \frac{1 + (1-x)^2}{x}.
\] (A64)

Dipole 4 (7): \((a, i) = (g, f)\)

\[
P^{gf}(x) = TR \left[ x^2 + (1-x)^2 \right].
\] (A65)

K term

\[
K(R_i, x_a : B_j, y_a, y_0) = A_4 \cdot \tilde{K}^{F(x_a)F(y_a)}(x) \cdot \langle B_j | B_j \rangle,
\] (A66)

\[
K(R_i, x_a : B_1, y_a, y_k) = A_4 \cdot \frac{\gamma F(y_k)}{T^2_{F(y_k)}} h(x) \cdot \langle B_1 | T_{y_a} \cdot T_{y_k} | B_1 \rangle,
\] (A67)

\[
K(R_i, x_a : B_j, y_a, y_b) = -A_4 \cdot \frac{1}{T^2_{F(y_a)}} \tilde{K}^{F(x_a)F(y_a)}(x) \cdot \langle B_j | T_{y_a} \cdot T_{y_b} | B_j \rangle.
\] (A68)

\[
\gamma_f = \frac{3}{2}CF,
\] (A69)

\[
\gamma_g = \frac{11}{6}CA - \frac{2}{3}TRNf.
\] (A70)

\[
h(x) = \left( \frac{1}{1-x} \right)_+ + \delta(1-x).
\] (A71)
\[ \bar{K}^{\text{ff}}(x) = C_F \left[ \left( \frac{2}{1-x} \ln \frac{1-x}{x} \right)_+ - (1+x) \ln \frac{1-x}{x} + (1-x) - \delta(1-x)(5 - \pi^2) \right], \]  
\[ \bar{K}^{\text{ff}}(x) = P^{\text{ff}}_{\text{reg}}(1-x) + C_F \left[ \left( \frac{2}{1-x} \ln(1-x) \right)_+ - \frac{\pi^2}{3} \delta(1-x) \right]. \]  

Dipole1 (4): \((a, i) = (g, g)\)

\[ \bar{K}^{gg}(x) = 2C_A \left[ \left( \frac{1}{1-x} \ln \frac{1-x}{x} \right)_+ + \left( \frac{1-x}{x} - 1 + x(1-x) \right) \ln \frac{1-x}{x} \right] 
- \delta(1-x) \left[ \left( \frac{50}{9} - \pi^2 \right) C_A - \frac{16}{9} T_R N_f \right], \]  
\[ \tilde{K}^{gg}(x) = P^{gg}_{\text{reg}}(1-x) + C_A \left[ \left( \frac{2}{1-x} \ln(1-x) \right)_+ - \frac{\pi^2}{3} \delta(1-x) \right]. \]  

Dipole3 (6): \((a, i) = (f, f)\)

\[ \bar{K}^{fg}(x) = P^{fg}(x) \ln \frac{1-x}{x} + C_F x, \]  
\[ \bar{K}^{fg}(x) = P^{fg}(x) \ln(1-x). \]  

Dipole4 (7): \((a, i) = (g, f)\)

\[ \bar{K}^{gf}(x) = P^{gf}(x) \ln \frac{1-x}{x} + T_R 2x(1-x), \]  
\[ \tilde{K}^{gf}(x) = P^{gf}(x) \ln(1-x). \]  

Regular part of the Altarelli–Parisi splitting function:

\[ P^{\text{ff}}_{\text{reg}}(x) = -C_F(1+x), \]  
\[ P^{gg}_{\text{reg}}(x) = 2C_A \left[ \frac{1-x}{x} - 1 + x(1-x) \right]. \]

The above expressions for the K terms are in the $\overline{\text{MS}}$ subtraction scheme. The other factorization scheme is specified by the term $K^{ab}_{\text{FS}}(x)$, which is exactly defined in Eq. (6.6) of Ref. [27]. The scheme-dependent term is introduced in the dipole subtraction procedure at the K term. The inclusion is realized by the replacement of $\bar{K}^{F(x_b)F(y_a)}_{\text{FS}}(x)$ in Eq. (A66) as

\[ \bar{K}^{F(x_b)F(y_a)}_{\text{FS}}(x) \rightarrow \bar{K}^{F(x_b)F(y_a)}_{\text{FS}}(x) - K^{F(x_b)F(y_a)}_{\text{FS}}(x). \]

Appendix B. Summary tables for the dijet process

B.1. D term

\[ \hat{\sigma}_D(R_i) = \frac{1}{S_{R_i}} \Phi(R_i) A \cdot \frac{1}{n_s(a)n_s(b)} D(R_i), \]

60/81
Table B1. Summary table of $D(R_{1u})$: $S_{R_{1u}} = 1$

<table>
<thead>
<tr>
<th>Dip $j$</th>
<th>B $j$</th>
<th>Splitting</th>
<th>$(x_I, x_J, x_K)$</th>
<th>$(y_u, y_{\bar{u}} : y_1, y_2)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dip 1</td>
<td>B1</td>
<td>(1) - 1</td>
<td>1. (13, 2)</td>
<td>$(a, b; \bar{13}, \bar{2})$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>2. (23, 1)</td>
<td>$(a, b; \bar{1}, \bar{2})$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>(1) - 2</td>
<td>$(\bar{a}, b; \bar{13}, 2)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>3. (13, a)</td>
<td>$(\bar{a}, b; \bar{13}, 2)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>4. (13, b)</td>
<td>$(\bar{a}, \bar{b}; \bar{1}, \bar{2})$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>5. (23, a)</td>
<td>$(\bar{a}, \bar{b}; \bar{1}, \bar{2})$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>6. (23, b)</td>
<td>$(\bar{a}, \bar{b}; \bar{1}, \bar{2})$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(3) - 1</td>
<td>7. (a3, 1)</td>
<td>$(\bar{a}3, b; \bar{1}, \bar{2})$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>8. (a3, 2)</td>
<td>$(\bar{a}3, b; \bar{1}, \bar{2})$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>9. (b3, 1)</td>
<td>$(\bar{a}, \bar{b}3; \bar{1}, \bar{2})$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>10. (b3, 2)</td>
<td>$(\bar{a}, \bar{b}3; \bar{1}, \bar{2})$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(3) - 2</td>
<td>11. (a3, b)</td>
<td>$(\bar{a}3, \bar{b}; 1, 2)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>12. (b3, a)</td>
<td>$(\bar{a}3, \bar{b}; 1, 2)$</td>
</tr>
<tr>
<td>Dip 2$u$</td>
<td>B2$u$</td>
<td>(5) - 1</td>
<td>13. (12, 3)</td>
<td>$(a, b; \bar{12}, \bar{3})$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>14. (12, a)</td>
<td>$(\bar{a}, \bar{b}; \bar{12}, \bar{3})$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>15. (12, b)</td>
<td>$(\bar{a}, \bar{b}; \bar{12}, \bar{3})$</td>
</tr>
<tr>
<td>Dip 3$u$</td>
<td>B3$u$</td>
<td>(6) - 1</td>
<td>16. (a1, 2)</td>
<td>$(\tilde{a}1, b; \tilde{2}, 3)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>17. (a1, 3)</td>
<td>$(\tilde{a}1, b; 2, \tilde{3})$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>18. (a1, b)</td>
<td>$(\tilde{a}1, \bar{b}; 2, \bar{3})$</td>
</tr>
<tr>
<td>Dip 3$\bar{u}$</td>
<td>B3$\bar{u}$</td>
<td>(6) - 1</td>
<td>19. (b2, 1)</td>
<td>$(a, \bar{b}2; \tilde{1}, \tilde{3})$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>20. (b2, 3)</td>
<td>$(a, \bar{b}2; 1, \tilde{3})$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>21. (b2, a)</td>
<td>$(\bar{a}, \bar{b}2; 1, 3)$</td>
</tr>
</tbody>
</table>

$$D(R_i, dip j)_{IJ,K} = \frac{1}{s_{IJ}} \frac{1}{s_{IJK}} \frac{1}{s_{\bar{I}K}} T_{\gamma_{\bar{I}K}}^2 \left( B_j | T_{\gamma_{\bar{I}K}Y_{\gamma_{\bar{I}K}}} \cdot V_{\gamma_{\bar{I}K}} | B_j \right).$$
Table B2. Summary table of D (R2u).

\[ D(\text{R}_2u = uu \rightarrow uug) : \quad S_{R_2} = 2 \]

<table>
<thead>
<tr>
<th>Dip ( j )</th>
<th>B( j )</th>
<th>Splitting</th>
<th>((x_I, x_J, x_K))</th>
<th>((y_a, y_b : y_1, y_2))</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dip 1</td>
<td>B1</td>
<td>(1) - 1</td>
<td>1. (13, 2)</td>
<td>(a, b : 1\overline{3}, \bar{2})</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>2. (23, 1)</td>
<td>(a, b : \bar{1}, \bar{2})</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>(1) - 2</td>
<td>\bar{a}, b : \bar{1}\overline{3}, 2</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>3. (13, a)</td>
<td>(a, \bar{b} : \bar{1}\overline{3}, \bar{2})</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>4. (13, b)</td>
<td>(a, \bar{b} : \bar{1}\overline{3}, \bar{2})</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>5. (23, a)</td>
<td>(\bar{a}, b : 1, \bar{2})</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>6. (23, b)</td>
<td>(a, b : 1, \bar{2})</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(3) - 1</td>
<td>7. (a3, 1)</td>
<td>(\bar{a}\overline{3}, b : \bar{1}, \bar{2})</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>8. (a3, 2)</td>
<td>(\bar{a}\overline{3}, b : 1, \bar{2})</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>9. (b3, 1)</td>
<td>(a, \bar{b}\overline{3} : \bar{1}, \bar{2})</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>10. (b3, 2)</td>
<td>(a, \bar{b}\overline{3} : 1, \bar{2})</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(3) - 2</td>
<td>11. (a3, b)</td>
<td>(\bar{a}\overline{3}, \bar{b} : 1, \bar{2})</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>12. (b3, a)</td>
<td>(\bar{a}, \bar{b}\overline{3} : 1, \bar{2})</td>
</tr>
<tr>
<td>Dip 3u</td>
<td>B3u</td>
<td>(6) - 1</td>
<td>13. (a1, 2)</td>
<td>(\bar{a}\overline{1}, b : 2, \bar{2}, \bar{3})</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>14. (a1, 3)</td>
<td>(\bar{a}\overline{1}, b : 2, \bar{3})</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>15. (a2, 1)</td>
<td>(\bar{a}\overline{2}, b : \bar{1}, \bar{3})</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>16. (a2, 3)</td>
<td>(\bar{a}\overline{2}, b : 1, \bar{3})</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>17. (b1, 2)</td>
<td>(\bar{b}\overline{1}, a : \bar{2}, \bar{3})</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>18. (b1, 3)</td>
<td>(\bar{b}\overline{1}, a : 2, \bar{3})</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>19. (b2, 1)</td>
<td>(\bar{b}\overline{2}, a : \bar{1}, \bar{3})</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>20. (b2, 3)</td>
<td>(\bar{b}\overline{2}, a : 1, \bar{3})</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(6) - 2</td>
<td>21. (a1, b)</td>
<td>(\bar{a}\overline{1}, \bar{b} : 2, \bar{3})</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>22. (a2, b)</td>
<td>(\bar{a}\overline{2}, \bar{b} : 1, \bar{3})</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>23. (b1, a)</td>
<td>(\bar{b}\overline{1}, a : \bar{2}, \bar{3})</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>24. (b2, a)</td>
<td>(\bar{b}\overline{2}, a : 1, \bar{3})</td>
</tr>
</tbody>
</table>
### Table B3. Summary table of $D(R_{3u})$.

<table>
<thead>
<tr>
<th>Dip $j$</th>
<th>B $j$</th>
<th>Splitting</th>
<th>$(x_l, x_j, x_K)$</th>
<th>$(y_u, y_b : y_1, y_2)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dip2u</td>
<td>B2u</td>
<td>(5) − 1</td>
<td>1. (13, 2)</td>
<td>$(a, b ; 13, 2)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>2. (23, 1)</td>
<td>$(a, b ; 23, 1)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>(5) − 2</td>
<td>$(a, b ; 13, 2)$</td>
</tr>
<tr>
<td>Dip3u</td>
<td>B3u</td>
<td>(6) − 1</td>
<td>7. (a1, 2)</td>
<td>$(\tilde{a}, b ; 2, 3)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>8. (a1, 3)</td>
<td>$(\tilde{a}, b ; 2, 3)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>9. (a2, 1)</td>
<td>$(\tilde{a}, b ; 1, 3)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>10. (a2, 3)</td>
<td>$(\tilde{a}, b ; 1, 3)$</td>
</tr>
<tr>
<td>Dip4u</td>
<td>B4u</td>
<td>(7) − 1</td>
<td>13. (b1, 2)</td>
<td>$(a, \tilde{b} ; 2, 3)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>14. (b1, 3)</td>
<td>$(a, \tilde{b} ; 2, 3)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>15. (b2, 1)</td>
<td>$(a, \tilde{b} ; 1, 3)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>16. (b2, 3)</td>
<td>$(a, \tilde{b} ; 1, 3)$</td>
</tr>
<tr>
<td>Dip4u</td>
<td>B4u</td>
<td>(7) − 2</td>
<td>17. (b1, a)</td>
<td>$(\tilde{a}, \tilde{b} ; 2, 3)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>18. (b2, a)</td>
<td>$(\tilde{a}, \tilde{b} ; 2, 3)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>(7) − 2</td>
<td>$(\tilde{a}, \tilde{b} ; 2, 3)$</td>
</tr>
<tr>
<td>Dip2u</td>
<td>B2u</td>
<td>(5) − 1</td>
<td>13. (b3, 1)</td>
<td>$(a, \tilde{b} ; 3, 1)$</td>
</tr>
<tr>
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<td>14. (b3, 2)</td>
<td>$(a, \tilde{b} ; 3, 1)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>(5) − 2</td>
<td>$(a, \tilde{b} ; 3, 1)$</td>
</tr>
</tbody>
</table>

### Table B4. Summary table of $D(R_{4u})$.

<table>
<thead>
<tr>
<th>Dip $j$</th>
<th>B $j$</th>
<th>Splitting</th>
<th>$(x_l, x_j, x_K)$</th>
<th>$(y_u, y_b : y_1, y_2)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dip1</td>
<td>B1</td>
<td>(1) − 1</td>
<td>1. (13, 2)</td>
<td>$(a, b ; 13, 2)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>2. (23, 1)</td>
<td>$(a, b ; 13, 2)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>(1) − 2</td>
<td>$(a, b ; 13, 2)$</td>
</tr>
<tr>
<td>Dip2d</td>
<td>B2d</td>
<td>(5) − 1</td>
<td>13. (12, 3)</td>
<td>$(a, b ; 12, 3)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>14. (12, a)</td>
<td>$(a, b ; 12, 3)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>(5) − 2</td>
<td>$(a, b ; 12, 3)$</td>
</tr>
</tbody>
</table>
Table B5. Summary table of D (R_{\text{dud}}).

\[ D (R_{\text{dud}} = ud \rightarrow udg) : \quad S_{R_{\text{dud}}} = 1 \]

<table>
<thead>
<tr>
<th>Dip  j</th>
<th>Bj</th>
<th>Splitting</th>
<th>((x_1, x_j, x_K))</th>
<th>((y_a, y_b : y_1, y_2))</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dip 1</td>
<td>B1 = ud → ud</td>
<td>(1) - 1</td>
<td>(13, 2)</td>
<td>((a, b : 13, 2))</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(2) - 1</td>
<td>(23, 1)</td>
<td>((a, b : 23, 2))</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(1) - 2</td>
<td>(13, a)</td>
<td>((\tilde{a}, b : 13, 2))</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(3) - 2</td>
<td>(a3, 1)</td>
<td>((a\tilde{a}, b : 12, 2))</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(3) - 2</td>
<td>(a3, b)</td>
<td>((a\tilde{a}, \tilde{b} : 12, 2))</td>
</tr>
<tr>
<td>Dip 3u</td>
<td>B3u = gd → dg</td>
<td>(6) - 1</td>
<td>(a1, 2)</td>
<td>((\tilde{a}, b : 2, 3))</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(6) - 2</td>
<td>(a1, 3)</td>
<td>((a\tilde{a}, b : 2, \tilde{3}))</td>
</tr>
<tr>
<td>Dip 3d</td>
<td>B3d = ug → ug</td>
<td>(6) - 1</td>
<td>(b2, 1)</td>
<td>((a\tilde{b}, \tilde{b} : 1, 3))</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(6) - 2</td>
<td>(b2, a)</td>
<td>((a\tilde{b}, \tilde{b} : 1, 3))</td>
</tr>
</tbody>
</table>

Table B6. Summary table of D (R_{\text{dud}}).

\[ D (R_{\text{dud}} = ud \rightarrow udg) : \quad S_{R_{\text{dud}}} = 1 \]

<table>
<thead>
<tr>
<th>Dip  j</th>
<th>Bj</th>
<th>Splitting</th>
<th>((x_1, x_j, x_K))</th>
<th>((y_a, y_b : y_1, y_2))</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dip 1</td>
<td>B1 = ud → ud</td>
<td>(1) - 1</td>
<td>(13, 2)</td>
<td>((a, b : 13, 2))</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(2) - 1</td>
<td>(23, 1)</td>
<td>((a, b : 23, 2))</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(1) - 2</td>
<td>(13, a)</td>
<td>((\tilde{a}, b : 13, 2))</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(3) - 2</td>
<td>(a3, 1)</td>
<td>((a\tilde{a}, b : 12, 2))</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(3) - 2</td>
<td>(a3, b)</td>
<td>((a\tilde{a}, \tilde{b} : 12, 2))</td>
</tr>
<tr>
<td>Dip 3u</td>
<td>B3u = gd → dg</td>
<td>(6) - 1</td>
<td>(a1, 2)</td>
<td>((\tilde{a}, b : 2, 3))</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(6) - 2</td>
<td>(a1, 3)</td>
<td>((a\tilde{a}, b : 2, \tilde{3}))</td>
</tr>
<tr>
<td>Dip 3d</td>
<td>B3d = ug → ug</td>
<td>(6) - 1</td>
<td>(b2, 1)</td>
<td>((a\tilde{b}, \tilde{b} : 1, 3))</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(6) - 2</td>
<td>(b2, a)</td>
<td>((a\tilde{b}, \tilde{b} : 1, 3))</td>
</tr>
</tbody>
</table>
Table B7. Summary table of: D (R\(_7\)u).

\(D (R\(_7\)u = ug \rightarrow udd): \ S_{R\(_7\)u} = 1\)

<table>
<thead>
<tr>
<th>Dip j</th>
<th>Bj</th>
<th>Splitting</th>
<th>((x_I x_J, x_K))</th>
<th>((y_a, y_b : y_1, y_2))</th>
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<tbody>
<tr>
<td>Dip 2u</td>
<td>B2u = ug \rightarrow ug</td>
<td>(5) (-) 1</td>
<td>1. (23, 1)</td>
<td>((a, b; 1, 23))</td>
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<td></td>
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<td></td>
<td>2. (23, a)</td>
<td>((\tilde{a}, b; 1, 23))</td>
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<td></td>
<td></td>
<td></td>
<td>3. (23, b)</td>
<td>((a, \tilde{b}; 1, 23))</td>
</tr>
<tr>
<td>Dip 3u</td>
<td>B3u = gg \rightarrow d\bar{d}</td>
<td>(6) (-) 1</td>
<td>4. (a1, 2)</td>
<td>((\tilde{a}1, b; 2, 3))</td>
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<td></td>
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<td>5. (a1, 3)</td>
<td>((\tilde{a}1, b; 2, \tilde{3}))</td>
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<td>6. (a1b)</td>
<td>((\tilde{a}1, \tilde{b}; 2, 3))</td>
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<tr>
<td>Dip 4u</td>
<td>B4u = u\bar{u} \rightarrow d\bar{d}</td>
<td>(7) (-) 1</td>
<td>7. (b1, 2)</td>
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<td>8. (b1, 3)</td>
<td>((a, \tilde{b}1; 2, \tilde{3}))</td>
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<td>9. (b1, a)</td>
<td>((\tilde{a}, \tilde{b}1; 2, 3))</td>
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<tr>
<td>Dip 4d</td>
<td>B4d = ud \rightarrow u\bar{d}</td>
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<td>12. (b2, a)</td>
<td>((\tilde{a}, \tilde{b}2; 1, 3))</td>
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<td>Dip 4\bar{d}</td>
<td>B4\bar{d} = ud \rightarrow ud</td>
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<td>((a, \tilde{b}3; 1, 2))</td>
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<td>14. (b3, 2)</td>
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<td>15. (b3, a)</td>
<td>((\tilde{a}, \tilde{b}3; 1, 2))</td>
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<td>$(a, b; 12, 3)$</td>
</tr>
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<td>2. $(13, 2)$</td>
<td>$(a, b; 13, 2)$</td>
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<td>3. $(23, 1)$</td>
<td>$(a, b; 23, 1)$</td>
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<td>4. $(12, a)$</td>
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<td>5. $(12, b)$</td>
<td>$(\bar{a}, \bar{b}; 12, 3)$</td>
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<td>6. $(13, a)$</td>
<td>$(\bar{a}, b; 13, 2)$</td>
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<td>7. $(13, b)$</td>
<td>$(\bar{a}, \bar{b}; 13, 2)$</td>
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<td>9. $(23, b)$</td>
<td>$(\bar{a}, \bar{b}; 23, 1)$</td>
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<td>10. $(a1, 2)$</td>
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<td>13. $(a2, 3)$</td>
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<td>15. $(a3, 2)$</td>
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<td>16. $(b1, 2)$</td>
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<td>18. $(b2, 1)$</td>
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<td>19. $(b2, 3)$</td>
<td>$(a, \bar{b}2; 1, 3)$</td>
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<td>20. $(b3, 1)$</td>
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<td>21. $(b3, 2)$</td>
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<td>22. $(a1, b)$</td>
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<td>23. $(a2, b)$</td>
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<td>24. $(a3, b)$</td>
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<td>25. $(b1, a)$</td>
<td>$(\bar{a}, b1; 2, 3)$</td>
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<td>$(\bar{a}, \bar{b}2; 1, 3)$</td>
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<td>27. $(b3, a)$</td>
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Table B9. Summary table of D (R_{9u}).

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<th>Bj</th>
<th>Splitting</th>
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<th>(y_1, y_2 : y_1, y_2)</th>
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<tbody>
<tr>
<td>Dip 1</td>
<td>B1 = ug → ug</td>
<td>(1) − 1</td>
<td>1. (12, 3)</td>
<td>(a, b; 12, 3)</td>
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<td>2. (13, 2)</td>
<td>(a, b; 13, 2)</td>
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<td></td>
<td></td>
<td></td>
<td>(1) − 2</td>
<td>(a, b; 12, 3)</td>
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<td></td>
<td></td>
<td></td>
<td>3. (12, a)</td>
<td>(a, b; 12, 3)</td>
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<td></td>
<td>4. (12, b)</td>
<td>(a, b; 12, 3)</td>
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<td>5. (13, a)</td>
<td>(a, b; 13, 2)</td>
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<td>6. (13, b)</td>
<td>(a, b; 13, 2)</td>
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<td></td>
<td>(2) − 1</td>
<td>7. (23, 1)</td>
<td>(a, b; 23)</td>
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<td>(2) − 2</td>
<td>8. (23, a)</td>
<td>(a, b; 23)</td>
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<td>9. (23, b)</td>
<td>(a, b; 23)</td>
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<td></td>
<td>(3) − 1</td>
<td>10. (a2, 1)</td>
<td>(a2, b; 1, 2)</td>
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<td></td>
<td>11. (a2, 3)</td>
<td>(a2, b; 1, 2, 3)</td>
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<td></td>
<td>12. (a3, 1)</td>
<td>(a3, b; 1, 2)</td>
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<td>13. (a3, 2)</td>
<td>(a3, b; 1, 2)</td>
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<td>(3) − 2</td>
<td>14. (a2, b)</td>
<td>(a2, b; 1, 3)</td>
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<td>15. (a3, b)</td>
<td>(a3, b; 1, 2)</td>
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<td>(4) − 1</td>
<td>16. (b2, 1)</td>
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<td>17. (b2, 3)</td>
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<td>18. (b3, 1)</td>
<td>(a, b3; 1, 2)</td>
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<td>19. (b3, 2)</td>
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<td>(4) − 2</td>
<td>20. (b2, a)</td>
<td>(a, b2; 1, 3)</td>
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<td>21. (b3, a)</td>
<td>(a, b3; 1, 2)</td>
</tr>
<tr>
<td>Dip 3u</td>
<td>B3u = gg → gg</td>
<td>(6) − 1</td>
<td>22. (a1, 2)</td>
<td>(a1, b; 2, 3)</td>
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<td>23. (a1, 3)</td>
<td>(a1, b; 2, 3)</td>
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<td>(6) − 2</td>
<td>(a1, b; 2, 3)</td>
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<tr>
<td>Dip 4u</td>
<td>B4u = uū → gg</td>
<td>(7) − 1</td>
<td>25. (b1, 2)</td>
<td>(a, b1; 2, 3)</td>
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<td>26. (b1, 3)</td>
<td>(a, b1; 2, 3)</td>
</tr>
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<td>(7) − 2</td>
<td>27. (b1, a)</td>
<td>(a, b1; 2, 3)</td>
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</tbody>
</table>
Table B10. Summary table of D (R_{10u}).

\[ \text{D (R}_{10u} = gg \rightarrow u\bar{u}g) : \ S_{R_{10u}} = 1 \]

<table>
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<tr>
<th>Dip</th>
<th>Bj</th>
<th>Splitting</th>
<th>((x_j, x_j, x_R))</th>
<th>((y_a, y_b : y_1, y_2))</th>
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</thead>
<tbody>
<tr>
<td>Dip 1</td>
<td>B1 = gg \rightarrow u\bar{u}</td>
<td>(1) - 1</td>
<td>1. (13, 2)</td>
<td>(a, b; \tilde{1}, \tilde{2})</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>2. (23, 1)</td>
<td>(a, b; \tilde{1}, \tilde{2})</td>
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<td></td>
<td>(1) - 2</td>
<td>(a, b; \tilde{1}, \tilde{2})</td>
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<td></td>
<td>3. (13, a)</td>
<td>(a, b; \tilde{1}, \tilde{2})</td>
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<td></td>
<td>4. (13, b)</td>
<td>(a, b; \tilde{1}, \tilde{2})</td>
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<td>5. (23, a)</td>
<td>(a, b; \tilde{1}, \tilde{2})</td>
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<td>6. (23, b)</td>
<td>(a, b; \tilde{1}, \tilde{2})</td>
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<td>(4) - 1</td>
<td>(a, b; \tilde{1}, \tilde{2})</td>
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<td>8. (a3, 2)</td>
<td>(a, b; \tilde{1}, \tilde{2})</td>
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<td>9. (b3, 1)</td>
<td>(a, b; \tilde{1}, \tilde{2})</td>
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<td>(a, b; \tilde{1}, \tilde{2})</td>
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<td>B2u = gg \rightarrow gg</td>
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<td>13. (12, 3)</td>
<td>(a, b; \tilde{1}, \tilde{2})</td>
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<td>(5) - 2</td>
<td>(a, b; \tilde{1}, \tilde{2})</td>
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<td>(a, b; \tilde{1}, \tilde{2})</td>
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<td>18. (b1, 2)</td>
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<td>19. (b1, 3)</td>
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<tr>
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<td>(a, b; \tilde{1}, \tilde{1})</td>
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<td>23. (a2, 3)</td>
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<td>24. (b2, 1)</td>
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<td>25. (b2, 3)</td>
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<td>(a, b; \tilde{1}, \tilde{1})</td>
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<td>26. (a2, b)</td>
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<td>27. (b2, a)</td>
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Table B11. Summary table of D (R₁₁).

\[\text{D (R₁₁ = gg → ggg): } S_{R₁₁} = 6\]

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<tr>
<th>Dip (j)</th>
<th>Bj</th>
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<th>((y_a, y_b : y_1, y_2))</th>
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<td>Dip 1</td>
<td>B1 = gg → gg</td>
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<td>1. (12, 3)</td>
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<td>2. (13, 2)</td>
<td>((a, \tilde{b}; \tilde{13}, \tilde{2}))</td>
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<td>3. (23, 1)</td>
<td>((a, b; \tilde{23}, \tilde{1}))</td>
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<td>6. (13, a)</td>
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<td>7. (13, b)</td>
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<td>8. (23, a)</td>
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<td>11. (a1, 3)</td>
<td>((\alpha1, b; \tilde{2}, 3))</td>
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<td>12. (a2, 1)</td>
<td>((\alpha2, b; \tilde{1}, 3))</td>
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<td>13. (a2, 3)</td>
<td>((\alpha2, b; \tilde{1}, \tilde{3}))</td>
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<td>14. (a3, 1)</td>
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<td></td>
<td>15. (a3, 2)</td>
<td>((\alpha3, b; \tilde{1}, 2))</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>16. (b1, 2)</td>
<td>((a, \tilde{b1}; \tilde{2}, 3))</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>17. (b1, 3)</td>
<td>((a, \tilde{b1}; \tilde{2}, \tilde{3}))</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>18. (b2, 1)</td>
<td>((a, \tilde{b2}; \tilde{1}, 3))</td>
</tr>
<tr>
<td></td>
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<td></td>
<td>19. (b2, 3)</td>
<td>((a, \tilde{b2}; \tilde{1}, \tilde{3}))</td>
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<td></td>
<td></td>
<td></td>
<td>20. (b3, 1)</td>
<td>((a, \tilde{b3}; \tilde{1}, 2))</td>
</tr>
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<td></td>
<td></td>
<td></td>
<td>21. (b3, 2)</td>
<td>((a, \tilde{b3}; \tilde{1}, \tilde{2}))</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(4) - 2</td>
<td>22. (a1, b)</td>
<td>((\alpha1, b; \tilde{2}, 3))</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>23. (a2, b)</td>
<td>((\alpha2, b; \tilde{1}, 3))</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>24. (a3, b)</td>
<td>((\alpha3, \tilde{b}; \tilde{1}, 2))</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>25. (b1, a)</td>
<td>((\alpha, \tilde{b1}; \tilde{2}, 3))</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>26. (b2, a)</td>
<td>((\alpha, \tilde{b2}; \tilde{1}, 3))</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>27. (b3, a)</td>
<td>((\alpha, \tilde{b3}; \tilde{1}, 2))</td>
</tr>
</tbody>
</table>
B.2. I term

\[ \hat{\sigma}_I(R_i) = \frac{1}{S_{B1}} \Phi(B1)_d \cdot I(R_i), \]  
(B1)

\[ I_{I, K} = -A_d \cdot \frac{1}{T^2_{F(I)}} \mathcal{V}_{F(I)} \cdot [I, K], \]  
(B2)

with the definition

\[ A_d = \frac{\alpha_s}{2\pi} \frac{(4\pi \mu^2)^\epsilon}{\Gamma(1-\epsilon)}, \quad \text{and} \quad [I, K] = s^{-\epsilon}_{IK} \{T_I \cdot T_K\}. \]  
(B3)

Table B12. Summary table of I(R_{1u}). The I terms I(R_{2u}) are created from the inputs B1(R_{2u}) = uu \rightarrow uu, B1(R_{4u}) = uu \rightarrow d\bar{d}, B1(R_{6u}) = ud \rightarrow ud, and B1(R_{8u}) = ud \rightarrow ud, respectively. The creations of the I terms are completely analogous to the creation of I(R_{1u}). The summary tables are identical except for differences in the field species \( F(y_I) = u, d, \bar{u}, \) or \( \bar{d} \) and the symmetric factor \( S_{B2u} = 2. \)

<table>
<thead>
<tr>
<th>I(R_{1u}) : B1 = uu \rightarrow u\bar{u}, \quad S_{B1} = 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>Leg (-y_I)</td>
</tr>
<tr>
<td>----------------</td>
</tr>
<tr>
<td>(1)-1</td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td>(1)-2</td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td></td>
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<tr>
<td></td>
</tr>
<tr>
<td>(3)-1</td>
</tr>
<tr>
<td></td>
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<tr>
<td></td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td>(3)-2</td>
</tr>
<tr>
<td></td>
</tr>
</tbody>
</table>
Table B13. Summary table of $I(R_{8u})$.

$I(R_{8u}) : B1 = u\bar{u} \rightarrow gg, \quad S_{B1} = 2$

<table>
<thead>
<tr>
<th>Leg $- y_I$</th>
<th>$\mathcal{V}<em>{F(I)}/T</em>{F(I)}^2$ (y_I, y_K)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(2)-1</td>
<td>$\mathcal{V}_{f}/C_A$ 1. (1, 2)</td>
</tr>
<tr>
<td></td>
<td>2. (2, 1)</td>
</tr>
<tr>
<td>(2)-2</td>
<td>3. (1, a)</td>
</tr>
<tr>
<td></td>
<td>4. (1, b)</td>
</tr>
<tr>
<td></td>
<td>5. (2, a)</td>
</tr>
<tr>
<td></td>
<td>6. (2, b)</td>
</tr>
<tr>
<td>(3)-1</td>
<td>$\mathcal{V}_{f}/C_F$ 7. (a, 1)</td>
</tr>
<tr>
<td></td>
<td>8. (a, 2)</td>
</tr>
<tr>
<td></td>
<td>9. (b, 1)</td>
</tr>
<tr>
<td></td>
<td>10. (b, 2)</td>
</tr>
<tr>
<td>(3)-2</td>
<td>11. (a, b)</td>
</tr>
<tr>
<td></td>
<td>12. (b, a)</td>
</tr>
</tbody>
</table>

Table B14. Summary table of $I(R_{9u})$.

$I(R_{9u}) : B1 = u\bar{g} \rightarrow u\bar{g}, \quad S_{B1} = 1$

<table>
<thead>
<tr>
<th>Leg $- y_I$</th>
<th>$\mathcal{V}<em>{F(I)}/T</em>{F(I)}^2$ (y_I, y_K)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1)-1</td>
<td>$\mathcal{V}_{f}/C_F$ 1. (1, 2)</td>
</tr>
<tr>
<td></td>
<td>2. (1, a)</td>
</tr>
<tr>
<td></td>
<td>3. (1, b)</td>
</tr>
<tr>
<td>(2)-1</td>
<td>$\mathcal{V}_{f}/C_A$ 4. (2, 1)</td>
</tr>
<tr>
<td></td>
<td>5. (2, a)</td>
</tr>
<tr>
<td></td>
<td>6. (2, b)</td>
</tr>
<tr>
<td>(2)-2</td>
<td>7. (a, 1)</td>
</tr>
<tr>
<td></td>
<td>8. (a, 2)</td>
</tr>
<tr>
<td>(3)-1</td>
<td>$\mathcal{V}_{f}/C_F$ 9. (a, b)</td>
</tr>
<tr>
<td></td>
<td>10. (b, 1)</td>
</tr>
<tr>
<td></td>
<td>11. (b, 2)</td>
</tr>
<tr>
<td>(3)-2</td>
<td>12. (b, a)</td>
</tr>
<tr>
<td>(4)-1</td>
<td>13. (a, b)</td>
</tr>
<tr>
<td></td>
<td>14. (b, a)</td>
</tr>
<tr>
<td>(4)-2</td>
<td>15. (a, a)</td>
</tr>
</tbody>
</table>
Table B15. Summary table of $I (R_{10})$.

$B_1 = gg \to u \bar{u}$, $S_{B_1} = 1$

<table>
<thead>
<tr>
<th>Leg - $y_I$</th>
<th>$V_{F(t)}/T_{F(t)}^2$</th>
<th>$(y_I, y_K)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1)-1</td>
<td>$V_f/C_F$</td>
<td>1. (1, 2)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>2. (2, 1)</td>
</tr>
<tr>
<td>(1)-2</td>
<td></td>
<td>3. (1, a)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>4. (1, b)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>5. (2, a)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>6. (2, b)</td>
</tr>
<tr>
<td>(4)-1</td>
<td>$V_g/C_A$</td>
<td>7. (a, 1)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>8. (a, 2)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>9. (b, 1)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>10. (b, 2)</td>
</tr>
<tr>
<td>(4)-2</td>
<td></td>
<td>11. (a, b)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>12. (b, a)</td>
</tr>
</tbody>
</table>

Table B16. Summary table of $I (R_{11})$.

$B_1 = gg \to gg$, $S_{B_1} = 2$

<table>
<thead>
<tr>
<th>Leg - $y_I$</th>
<th>$V_{F(t)}/T_{F(t)}^2$</th>
<th>$(y_I, y_K)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(2)-1</td>
<td>$V_g/C_A$</td>
<td>1. (1, 2)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>2. (2, 1)</td>
</tr>
<tr>
<td>(2)-2</td>
<td></td>
<td>3. (1, a)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>4. (1, b)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>5. (2, a)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>6. (2, b)</td>
</tr>
<tr>
<td>(4)-1</td>
<td></td>
<td>7. (a, 1)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>8. (a, 2)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>9. (b, 1)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>10. (b, 2)</td>
</tr>
<tr>
<td>(4)-2</td>
<td></td>
<td>11. (a, b)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>12. (b, a)</td>
</tr>
</tbody>
</table>
B.3. $P$ and $K$ terms

$$
\hat{\sigma}_{P/K}(R_i) = \int_0^1 dx \sum_{B_j} \frac{1}{S_{B_j}} \Phi_{\alpha}(R_i : B_j, x) \cdot P/K(R_i, \text{dip}_j, x_a) + (a \leftrightarrow b).
$$

$$
P(R_i, \text{dip}_j, x_{a/b}, \gamma_{\text{emi}}, \gamma_{\text{spe}}) = A_4 \cdot \frac{1}{T_{F(y_{\text{emi}})}^2} p^{F(y_{a/b})F(y_{\text{emi}})}(x) \ln \frac{\mu_2}{x s_{x_{a/b}} y_{\text{spe}}} \times \{B_j|T_{y_{\text{emi}}}, T_{y_{\text{spe}}}| B_j\}. \tag{B4}
$$

$$
K(R_i, \text{dip}1/3/4, (3)/(4)/(6)/(7)-0, x_{a/b}, \gamma_{\text{emi}}, \gamma_0) = A_4 \cdot \tilde{K}^{F(y_{a/b})F(y_{\text{emi}})}(x) \cdot \langle B_j \rangle, \tag{B5}
$$

$$
K(R_i, \text{dip}1, (3)/(4)-1, x_{a/b}, \gamma_{\text{emi}}, \gamma_{\text{spe}}) = A_4 \cdot \frac{\gamma_{F(y_{\text{spe}})}}{T_{F(y_{\text{spe}})}^2} h(x) : \{B1|T_{y_{\text{emi}}}, T_{y_{\text{spe}}}| B1\}, \tag{B6}
$$

$$
K(R_i, \text{dip}1/3/4, (3)/(4)/(6)/(7)-2, x_{a/b}, \gamma_{\text{emi}}, \gamma_{\text{spe}}) = -A_4 \cdot \frac{1}{T_{F(y_{\text{emi}})}^2} \tilde{K}^{F(y_{a/b})F(y_{\text{emi}})}(x) \times \{B_j|T_{y_{\text{emi}}}, T_{y_{\text{spe}}}| B_j\}. \tag{B7}
$$

### Table B17. Summary table of $P/K$ ($R_{1u}$).

<table>
<thead>
<tr>
<th>Leg</th>
<th>Dip</th>
<th>Bj</th>
<th>$S_{B_j}$</th>
<th>Splitting</th>
<th>$p^{F(y_{a/b})F(y_{\text{emi}})}/T_{F(y_{\text{emi}})}^2$</th>
<th>$(\gamma_{\text{emi}}, \gamma_{\text{spe}})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a$</td>
<td>Dip 1</td>
<td>B1 = $u\bar{u} \rightarrow u\bar{u}$</td>
<td>1</td>
<td>(3) - 0</td>
<td>$P^{ff}/C_F$</td>
<td>1. $(a, 0)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>2. $(a, 1)$</td>
</tr>
<tr>
<td></td>
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<td></td>
<td></td>
<td></td>
<td>3. $(a, 2)$</td>
</tr>
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<td></td>
<td></td>
<td></td>
<td></td>
<td>4. $(a, b)$</td>
</tr>
<tr>
<td></td>
<td>Dip 3u</td>
<td>B3u = $g\bar{u} \rightarrow \bar{u}g$</td>
<td>1</td>
<td>(6) - 0</td>
<td>$P^{ff}/C_A$</td>
<td>5. $(a, 0)$</td>
</tr>
<tr>
<td></td>
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<td></td>
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<td></td>
<td>6. $(a, 1)$</td>
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<td></td>
<td></td>
<td></td>
<td></td>
<td>7. $(a, 2)$</td>
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<td></td>
<td></td>
<td></td>
<td></td>
<td>8. $(a, b)$</td>
</tr>
<tr>
<td>$b$</td>
<td>Dip 1</td>
<td>B1 = $u\bar{u} \rightarrow u\bar{u}$</td>
<td>1</td>
<td>(3) - 0</td>
<td>$P^{ff}/C_F$</td>
<td>9. $(b, 0)$</td>
</tr>
<tr>
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<td></td>
<td></td>
<td>10. $(b, 1)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>11. $(b, 2)$</td>
</tr>
<tr>
<td></td>
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<td></td>
<td></td>
<td></td>
<td>12. $(b, a)$</td>
</tr>
<tr>
<td></td>
<td>Dip 3u</td>
<td>B3u = $ug \rightarrow ug$</td>
<td>1</td>
<td>(6) - 0</td>
<td>$P^{ff}/C_A$</td>
<td>13. $(b, 0)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
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<td></td>
<td>14. $(b, 1)$</td>
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<td></td>
<td>15. $(b, 2)$</td>
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<td></td>
<td></td>
<td>16. $(b, a)$</td>
</tr>
</tbody>
</table>
Table B18. Summary table of $P/K (R_{2u})$. 

\[
P/K (R_{2u} = uu → uug)
\]

<table>
<thead>
<tr>
<th>Leg</th>
<th>Dip j</th>
<th>B j</th>
<th>$S_{B_j}$</th>
<th>Splitting</th>
<th>$P^{(x_a/b)F(Y_{em})}/T^2_{F(Y_{em})}$</th>
<th>($y_{em}, y_{spe}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>Dip 1</td>
<td>B1</td>
<td>uu</td>
<td>(3) - 0</td>
<td>1. (a, 0)</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(3) - 1</td>
<td>2. (a, 1)</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(3) - 2</td>
<td>3. (a, 2)</td>
<td></td>
</tr>
<tr>
<td>Dip 3u</td>
<td>B3u = gu → ug</td>
<td>1</td>
<td>(6) - 0</td>
<td>1. (a, 0)</td>
<td>5. (a, 0)</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(6) - 1</td>
<td>6. (a, 1)</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(6) - 2</td>
<td>7. (a, 2)</td>
<td></td>
</tr>
<tr>
<td>b</td>
<td>Dip 1</td>
<td>B1</td>
<td>uū</td>
<td>(3) - 0</td>
<td>9. (b, 0)</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(3) - 1</td>
<td>10. (b, 1)</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(3) - 2</td>
<td>11. (b, 2)</td>
<td></td>
</tr>
<tr>
<td>Dip 3u</td>
<td>B3u = gu → ug</td>
<td>1</td>
<td>(6) - 0</td>
<td>13. (a, 0)</td>
<td>14. (a, 1)</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(6) - 1</td>
<td>15. (a, 2)</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(6) - 2</td>
<td>16. (a, b)</td>
<td></td>
</tr>
</tbody>
</table>

Table B19. Summary table of $P/K (R_{3u})$. 

\[
P/K (R_{3u} = ug → uuuu)
\]

<table>
<thead>
<tr>
<th>Leg</th>
<th>Dip j</th>
<th>B j</th>
<th>$S_{B_j}$</th>
<th>Splitting</th>
<th>$P^{(x_a/b)F(Y_{em})}/T^2_{F(Y_{em})}$</th>
<th>($y_{em}, y_{spe}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>Dip 3u</td>
<td>B3u = gg → uū</td>
<td>1</td>
<td>(6) - 0</td>
<td>1. (a, 0)</td>
<td>1. (a, 0)</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(6) - 1</td>
<td>2. (a, 1)</td>
<td>2. (a, 1)</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(6) - 2</td>
<td>3. (a, 2)</td>
<td>3. (a, 2)</td>
</tr>
<tr>
<td>Dip 4u</td>
<td>B4u = uū → uū</td>
<td>1</td>
<td>(7) - 0</td>
<td>5. (b, 0)</td>
<td>5. (b, 0)</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(7) - 1</td>
<td>6. (b, 1)</td>
<td>6. (b, 1)</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(7) - 2</td>
<td>8. (b, a)</td>
<td>8. (b, a)</td>
</tr>
<tr>
<td>Dip 4ū</td>
<td>B4ū = uu → uu</td>
<td>2</td>
<td>(7) - 0</td>
<td>9. (b, 0)</td>
<td>9. (b, 0)</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(7) - 1</td>
<td>10. (b, 1)</td>
<td>10. (b, 1)</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(7) - 2</td>
<td>11. (b, 2)</td>
<td>11. (b, 2)</td>
</tr>
<tr>
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<td></td>
<td></td>
<td>(7) - 3</td>
<td>12. (b, a)</td>
<td>12. (b, a)</td>
</tr>
</tbody>
</table>
Table B20. Summary table of $P/K (R_{4u})$.

$P/K (R_{4u} = u\bar{u} \to d\bar{d}g)$

<table>
<thead>
<tr>
<th>Leg</th>
<th>Dip j</th>
<th>Bj = $u\bar{u} \to d\bar{d}$</th>
<th>$S_{B_j}$</th>
<th>Splitting</th>
<th>$p^{F_{(x,y)}} F_{(y_{em})}/T^2_{F(y_{em})}$</th>
<th>$(y_{em}, y_{spe})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a$</td>
<td>Dip1</td>
<td></td>
<td>1</td>
<td>(3) - 0</td>
<td>$P^{ff}/C_F$</td>
<td>1. $(a, 0)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(3) - 1</td>
<td></td>
<td>2. $(a, 1)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(3) - 2</td>
<td></td>
<td>3. $(a, 2)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>4. $(a, b)$</td>
</tr>
<tr>
<td>$b$</td>
<td>Dip1</td>
<td></td>
<td>1</td>
<td>(3) - 0</td>
<td>$P^{ff}/C_F$</td>
<td>5. $(b, 0)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(3) - 1</td>
<td></td>
<td>6. $(b, 1)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(3) - 2</td>
<td></td>
<td>7. $(b, 2)$</td>
</tr>
</tbody>
</table>

Table B21. Summary table of $P/K (R_{5ud})$.

$P/K (R_{5ud} = ud \to udg)$

<table>
<thead>
<tr>
<th>Leg</th>
<th>Dip j</th>
<th>Bj = $ud \to ud$</th>
<th>$S_{B_j}$</th>
<th>Splitting</th>
<th>$p^{F_{(x,y)}} F_{(y_{em})}/T^2_{F(y_{em})}$</th>
<th>$(y_{em}, y_{spe})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a$</td>
<td>Dip1</td>
<td></td>
<td>1</td>
<td>(3) - 0</td>
<td>$P^{ff}/C_F$</td>
<td>1. $(a, 0)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(3) - 1</td>
<td></td>
<td>2. $(a, 1)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(3) - 2</td>
<td></td>
<td>3. $(a, 2)$</td>
</tr>
<tr>
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<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>4. $(a, b)$</td>
</tr>
<tr>
<td>Dip3u</td>
<td>B3u = $gd \to dg$</td>
<td>1</td>
<td></td>
<td>(6) - 0</td>
<td>$P^{fs}/C_A$</td>
<td>5. $(a, 0)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(6) - 1</td>
<td></td>
<td>6. $(a, 1)$</td>
</tr>
<tr>
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<td></td>
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<td>7. $(a, 2)$</td>
</tr>
<tr>
<td>$b$</td>
<td>Dip1</td>
<td></td>
<td>1</td>
<td>(6) - 2</td>
<td>$P^{ff}/C_F$</td>
<td>8. $(a, b)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(3) - 0</td>
<td></td>
<td>9. $(b, 0)$</td>
</tr>
<tr>
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<td></td>
<td>(3) - 1</td>
<td></td>
<td>10. $(b, 1)$</td>
</tr>
<tr>
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<td></td>
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<td>(3) - 2</td>
<td></td>
<td>11. $(b, 2)$</td>
</tr>
<tr>
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<td></td>
<td></td>
<td></td>
<td>12. $(b, a)$</td>
</tr>
<tr>
<td>Dip3d</td>
<td>B3d = $ug \to ug$</td>
<td>1</td>
<td></td>
<td>(6) - 0</td>
<td>$P^{fs}/C_A$</td>
<td>13. $(b, 0)$</td>
</tr>
<tr>
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<td></td>
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<td>(6) - 1</td>
<td></td>
<td>14. $(b, 1)$</td>
</tr>
<tr>
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<td>15. $(b, 2)$</td>
</tr>
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<td>(6) - 2</td>
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<td>16. $(b, a)$</td>
</tr>
</tbody>
</table>
**Table B22.** Summary table of P/K ($R_{ud\bar{d}}$).

<table>
<thead>
<tr>
<th>Leg</th>
<th>Dip j</th>
<th>Bj</th>
<th>$S_{B_i}$</th>
<th>Splitting</th>
<th>$P^{F(\nu_{\tau\nu})F_{\nu_{\tau\nu}}}/\Gamma_{\nu_{\tau\nu}}^2$</th>
<th>($y_{emi}$, $y_{spe}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>Dip 1</td>
<td>B1 $= ud \rightarrow ud$</td>
<td>1</td>
<td>(3) − 0</td>
<td>$P^{ff}/C_F$</td>
<td>1. ($a$, 0)</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(3) − 1</td>
<td></td>
<td>2. ($a$, 1)</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(3) − 2</td>
<td></td>
<td>3. ($a$, 2)</td>
</tr>
<tr>
<td>Dip3u</td>
<td>B3u $= g\bar{d} \rightarrow \bar{d}g$</td>
<td>1</td>
<td>(6) − 0</td>
<td>$P^{fs}/C_A$</td>
<td>5. ($a$, 0)</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(6) − 1</td>
<td></td>
<td>6. ($a$, 1)</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(6) − 2</td>
<td></td>
<td>7. ($a$, 2)</td>
</tr>
<tr>
<td>b</td>
<td>Dip 1</td>
<td>B1 $= ud \rightarrow ud$</td>
<td>1</td>
<td>(3) − 0</td>
<td>$P^{ff}/C_F$</td>
<td>9. ($b$, 0)</td>
</tr>
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<td>(3) − 1</td>
<td></td>
<td>10. ($b$, 1)</td>
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<td></td>
<td>(3) − 2</td>
<td></td>
<td>11. ($b$, 2)</td>
</tr>
<tr>
<td>Dip3d</td>
<td>B3d $= ug \rightarrow ug$</td>
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<td>(6) − 0</td>
<td>$P^{fs}/C_A$</td>
<td>12. ($b$, a)</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(6) − 1</td>
<td></td>
<td>13. ($b$, 0)</td>
</tr>
<tr>
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<td></td>
<td></td>
<td></td>
<td>(6) − 2</td>
<td></td>
<td>14. ($b$, 1)</td>
</tr>
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<td></td>
<td>15. ($b$, 2)</td>
</tr>
</tbody>
</table>

<table>
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<tr>
<th>Leg</th>
<th>Dip j</th>
<th>Bj</th>
<th>$S_{B_i}$</th>
<th>Splitting</th>
<th>$P^{F(\nu_{\tau\nu})F_{\nu_{\tau\nu}}}/\Gamma_{\nu_{\tau\nu}}^2$</th>
<th>($y_{emi}$, $y_{spe}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>Dip 3u</td>
<td>B3u $= gg \rightarrow d\bar{d}$</td>
<td>1</td>
<td>(6) − 0</td>
<td>$P^{fs}/C_A$</td>
<td>1. ($a$, 0)</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(6) − 1</td>
<td></td>
<td>2. ($a$, 1)</td>
</tr>
<tr>
<td></td>
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<td></td>
<td></td>
<td>(6) − 2</td>
<td></td>
<td>3. ($a$, 2)</td>
</tr>
<tr>
<td>b</td>
<td>Dip 4u</td>
<td>B4u $= u\bar{u} \rightarrow d\bar{d}$</td>
<td>1</td>
<td>(7) − 0</td>
<td>$P^{sf}/C_F$</td>
<td>5. ($b$, 0)</td>
</tr>
<tr>
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<td></td>
<td></td>
<td></td>
<td>(7) − 1</td>
<td></td>
<td>6. ($b$, 1)</td>
</tr>
<tr>
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<td></td>
<td></td>
<td>(7) − 2</td>
<td></td>
<td>7. ($b$, 2)</td>
</tr>
<tr>
<td>Dip4d</td>
<td>B4d $= ud \rightarrow ud$</td>
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<td>(7) − 0</td>
<td>$P^{sf}/C_F$</td>
<td>9. ($b$, 0)</td>
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<td>(7) − 1</td>
<td></td>
<td>10. ($b$, 1)</td>
</tr>
<tr>
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<td>(7) − 2</td>
<td></td>
<td>11. ($b$, 2)</td>
</tr>
<tr>
<td>Dip4d</td>
<td>B4d $= ud \rightarrow ud$</td>
<td>1</td>
<td>(7) − 0</td>
<td>$P^{sf}/C_F$</td>
<td>12. ($b$, a)</td>
<td></td>
</tr>
<tr>
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<td></td>
<td></td>
<td>(7) − 1</td>
<td></td>
<td>13. ($b$, 0)</td>
</tr>
<tr>
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<td></td>
<td></td>
<td></td>
<td>(7) − 2</td>
<td></td>
<td>14. ($b$, 1)</td>
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<td>15. ($b$, 2)</td>
</tr>
</tbody>
</table>
Table B24. Summary table of P/K (R_{8u}).

<table>
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<th>Leg</th>
<th>Dip j</th>
<th>B_j = uū → gg</th>
<th>S_{B_j}</th>
<th>Splitting</th>
<th>( \frac{P^{F(x_{a,b})F(y_{emi})}}{T_{F(y_{emi})}^2} )</th>
<th>( (y_{emi}, y_{spe}) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>Dip 1</td>
<td>B1 = uū → gg</td>
<td>2</td>
<td>(3) – 0</td>
<td>( P^{ff}/C_F )</td>
<td>1. ((a, 0))</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(3) – 1</td>
<td>( P^{ff}/C_F )</td>
<td>2. ((a, 1))</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>( P^{ff}/C_F )</td>
<td>3. ((a, 2))</td>
<td></td>
</tr>
<tr>
<td>b</td>
<td>Dip 1</td>
<td>B1 = uū → gg</td>
<td>2</td>
<td>(3) – 2</td>
<td>( P^{ff}/C_F )</td>
<td>4. ((a, b))</td>
</tr>
<tr>
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<td></td>
<td>(3) – 0</td>
<td>( P^{ff}/C_F )</td>
<td>5. ((b, 0))</td>
</tr>
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<td></td>
<td></td>
<td>(3) – 1</td>
<td>( P^{ff}/C_F )</td>
<td>6. ((b, 1))</td>
</tr>
<tr>
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<td></td>
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<td></td>
<td>( P^{ff}/C_F )</td>
<td>7. ((b, 2))</td>
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</tr>
</tbody>
</table>

Table B25. Summary table of P/K (R_{9u}).

<table>
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<th>Leg</th>
<th>Dip j</th>
<th>B_j = ug → ug</th>
<th>S_{B_j}</th>
<th>Splitting</th>
<th>( \frac{P^{F(x_{a,b})F(y_{emi})}}{T_{F(y_{emi})}^2} )</th>
<th>( (y_{emi}, y_{spe}) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>Dip 1</td>
<td>B1 = ug → ug</td>
<td>1</td>
<td>(3) – 0</td>
<td>( P^{ff}/C_F )</td>
<td>1. ((a, 0))</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(3) – 1</td>
<td>( P^{ff}/C_F )</td>
<td>2. ((a, 1))</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>( P^{ff}/C_F )</td>
<td>3. ((a, 2))</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(3) – 2</td>
<td>( P^{ff}/C_F )</td>
<td>4. ((a, b))</td>
</tr>
<tr>
<td></td>
<td>Dip 3u</td>
<td>B3u = gg → gg</td>
<td>2</td>
<td>(6) – 0</td>
<td>( P^{ff}/C_F )</td>
<td>5. ((a, 0))</td>
</tr>
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<td>(6) – 1</td>
<td>( P^{ff}/C_F )</td>
<td>6. ((a, 1))</td>
</tr>
<tr>
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<td></td>
<td></td>
<td></td>
<td>( P^{ff}/C_F )</td>
<td>7. ((a, 2))</td>
<td></td>
</tr>
<tr>
<td>b</td>
<td>Dip 1</td>
<td>B1 = ug → ug</td>
<td>1</td>
<td>(6) – 2</td>
<td>( P^{ff}/C_F )</td>
<td>8. ((a, b))</td>
</tr>
<tr>
<td></td>
<td></td>
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<td></td>
<td>(6) – 0</td>
<td>( P^{ff}/C_F )</td>
<td>9. ((b, 0))</td>
</tr>
<tr>
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<td>(4) – 1</td>
<td>( P^{ff}/C_F )</td>
<td>10. ((b, 1))</td>
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<td>( P^{ff}/C_F )</td>
<td>11. ((b, 2))</td>
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<tr>
<td></td>
<td>Dip 4u</td>
<td>B4u = uū → gg</td>
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<td>(7) – 0</td>
<td>( P^{ff}/C_F )</td>
<td>12. ((b, a))</td>
</tr>
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<td>(7) – 1</td>
<td>( P^{ff}/C_F )</td>
<td>13. ((b, 0))</td>
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<td>( P^{ff}/C_F )</td>
<td>14. ((b, 1))</td>
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<td>( P^{ff}/C_F )</td>
<td>15. ((b, 2))</td>
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<td>( P^{ff}/C_F )</td>
<td>16. ((b, a))</td>
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</tbody>
</table>
Table B26. Summary table of \( P/K(R_{10u}) \).

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<th>Leg</th>
<th>Dip</th>
<th>Bj</th>
<th>( S_{Bj} )</th>
<th>Splitting</th>
<th>( \frac{F^{(x_{y_{i}})F(y_{spec})}}{T^{2}<em>{F(y</em>{spec})}} )</th>
<th>( (y_{emi}, y_{spe}) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>Dip1</td>
<td>B1 = ( gg \rightarrow u \bar{u} )</td>
<td>1</td>
<td>(4) - 0</td>
<td>( \frac{P_{gg}}{C_{A}} )</td>
<td>1. (a, 0)</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(4) - 1</td>
<td>( \frac{P_{gg}}{C_{A}} )</td>
<td>2. (a, 1)</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(4) - 2</td>
<td>( \frac{P_{gg}}{C_{A}} )</td>
<td>3. (a, 2)</td>
</tr>
<tr>
<td></td>
<td>Dip4u</td>
<td>B4u = ( \bar{u}g \rightarrow \bar{u}g )</td>
<td>1</td>
<td>(7) - 0</td>
<td>( \frac{P_{gf}}{C_{F}} )</td>
<td>4. (a, b)</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(7) - 1</td>
<td>( \frac{P_{gf}}{C_{F}} )</td>
<td>5. (a, 0)</td>
</tr>
<tr>
<td></td>
<td>Dip4( u )</td>
<td>B4( u ) = ( ug \rightarrow ug )</td>
<td>1</td>
<td>(7) - 2</td>
<td>( \frac{P_{gf}}{C_{F}} )</td>
<td>6. (a, 1)</td>
</tr>
<tr>
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<td></td>
<td></td>
<td>(7) - 1</td>
<td>( \frac{P_{gf}}{C_{F}} )</td>
<td>7. (a, 2)</td>
</tr>
<tr>
<td>b</td>
<td>Dip1</td>
<td>B1 = ( gg \rightarrow u \bar{u} )</td>
<td>1</td>
<td>(4) - 0</td>
<td>( \frac{P_{gg}}{C_{A}} )</td>
<td>8. (a, b)</td>
</tr>
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<td>(4) - 1</td>
<td>( \frac{P_{gg}}{C_{A}} )</td>
<td>9. (a, 0)</td>
</tr>
<tr>
<td></td>
<td>Dip4u</td>
<td>B4u = ( \bar{u}g \rightarrow \bar{u}g )</td>
<td>1</td>
<td>(4) - 2</td>
<td>( \frac{P_{gg}}{C_{A}} )</td>
<td>10. (a, 1)</td>
</tr>
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<td></td>
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<td>(7) - 0</td>
<td>( \frac{P_{gf}}{C_{F}} )</td>
<td>11. (a, 2)</td>
</tr>
<tr>
<td></td>
<td>Dip4( u )</td>
<td>B4( u ) = ( ug \rightarrow ug )</td>
<td>1</td>
<td>(7) - 2</td>
<td>( \frac{P_{gf}}{C_{F}} )</td>
<td>12. (a, b)</td>
</tr>
<tr>
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<td></td>
<td></td>
<td></td>
<td>(7) - 1</td>
<td>( \frac{P_{gf}}{C_{F}} )</td>
<td>13. (a, 0)</td>
</tr>
</tbody>
</table>

Table B27. Summary table of \( P/K(R_{11}) \).

<table>
<thead>
<tr>
<th>Leg</th>
<th>Dip</th>
<th>Bj</th>
<th>( S_{Bj} )</th>
<th>Splitting</th>
<th>( \frac{F^{(x_{y_{i}})F(y_{spec})}}{T^{2}<em>{F(y</em>{spec})}} )</th>
<th>( (y_{emi}, y_{spe}) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>Dip1</td>
<td>B1 = ( gg \rightarrow gg )</td>
<td>2</td>
<td>(4) - 0</td>
<td>( \frac{P_{gg}}{C_{A}} )</td>
<td>1. (a, 0)</td>
</tr>
<tr>
<td></td>
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<td></td>
<td></td>
<td>(4) - 1</td>
<td>( \frac{P_{gg}}{C_{A}} )</td>
<td>2. (a, 1)</td>
</tr>
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<td>(4) - 2</td>
<td>( \frac{P_{gg}}{C_{A}} )</td>
<td>3. (a, 2)</td>
</tr>
<tr>
<td>b</td>
<td>Dip1</td>
<td>B1 = ( gg \rightarrow gg )</td>
<td>2</td>
<td>(4) - 2</td>
<td>( \frac{P_{gg}}{C_{A}} )</td>
<td>4. (a, b)</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
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<td>( \frac{P_{gg}}{C_{A}} )</td>
<td>5. (b, 0)</td>
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<td>6. (b, 1)</td>
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<td>( \frac{P_{gg}}{C_{A}} )</td>
<td>7. (b, 2)</td>
</tr>
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<td>(4) - 2</td>
<td>( \frac{P_{gg}}{C_{A}} )</td>
<td>8. (b, a)</td>
</tr>
</tbody>
</table>

References


