Structure of Supersymmetric Gauge Theories

Index calculation by means of harmonic expansion

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Received October 9, 2015; Accepted October 23, 2015; Published November 16, 2015

We review derivation of superconformal indices by means of supersymmetric localization and spherical harmonic expansion for 3d $\mathcal{N}=2$, 4d $\mathcal{N}=1$, and 6d $\mathcal{N}=(1,0)$ supersymmetric gauge theories. We demonstrate calculation of indices for vector multiplets in each dimension by analyzing energy eigenmodes in $S^p \times \mathbb{R}$. For the 6d index we consider the perturbative contribution only. We focus on the technical details of harmonic expansion rather than physical applications.

Subject Index  B00, B16

1. Introduction

Indices are a special type of partition function in supersymmetric theories defined in such a way that the contributions of bosonic and fermionic states partially cancel each other and only modes satisfying some BPS conditions make non-trivial contributions. Thanks to the supersymmetry, indices are protected from uncontrollable quantum corrections, and it is possible to calculate them exactly even in strongly coupled theories. They are powerful tools for analyzing dynamics of supersymmetric theories and have been used to test various dualities.

There are a variety of indices. The simplest one is the Witten index $[1]$, which is defined by

$$I_W = \text{tr} \left[ (-1)^F e^{-\beta H} \right],$$

where $H$ and $F$ are the Hamiltonian and the fermion number, respectively. The trace is taken over all gauge-invariant states. This gives an integer, which is the difference between the numbers of bosonic and fermionic vacua. To calculate $I_W$ the system is put in a torus to avoid IR divergence.

We can generalize $I_W$ by replacing the background space with another compact manifold $M$ (or a non-compact manifold with appropriate boundary conditions). If we take $M$ to be $S^p$, the index gives the BPS spectrum of the theory in $S^p \times \mathbb{R}$. We can map the background to $\mathbb{R}^{p+1}$ by a Weyl transformation, and for superconformal theories the index can be associated with local BPS operators via the state–operator correspondence. This kind of index, which we will focus on in this paper, is called a superconformal index $[2–4]$. The purpose of this paper is to demonstrate derivation of the superconformal indices for 3d, 4d, and 6d supersymmetric theories using harmonic expansion.

For a given $p$-dimensional manifold $M$, the index is defined and calculated as follows. We consider a $p+1$-dimensional supersymmetric field theory defined on $M \times \mathbb{R}$. We suppose the theory has $k$ commuting bosonic conserved charges including the Hamiltonian $H$ and another $k-1$ charges $F_i$ ($i=1, \ldots, k-1$). Let $Q$ be a supercharge, and $\overline{Q}$ be another supercharge such that
\( \{ Q, \overline{Q} \} = H + c_i F_i \). The index is defined by

\[
I \left( e^{-\beta}, e^{-\gamma_i} \right) = \text{tr} \left[ (-1)^F e^{-\beta H - \gamma_i F_i} \right].
\]

The operator \( \beta H + \gamma_i F_i \) in the exponent must commute with \( Q \). Therefore, only \( k - 1 \) variables in \( (\beta, \gamma_i) \) are independent. It is easy to show that the deformation of the Hamiltonian by a \( Q \)-exact term

\[
H \rightarrow H' = H + t \{ Q, V \}
\]

does not change (2). If we take \( V = \overline{Q} \), the deformation is equivalent to the shift of parameters \( (\beta, \gamma_i) \rightarrow (\beta, \gamma_i) + t (1, c_i) \). This means that the index depends on the parameters through \( \gamma_i - \beta c_i \), and the index is a function of \( k - 2 \) variables.

If we choose \( V \) so that the deformation term contains quadratic terms of fields, the theory becomes weakly coupled when we take the \( t \rightarrow \infty \) limit. This enables us to calculate the index exactly from the information of eigenmodes of the deformed Hamiltonian in the weak coupling limit.

Let \( \Phi \) be fields in the theory. We expand the solution of the free field equations by eigenmodes of charges \( (H, F_i, t_I) \) labeled by \( n \), where \( t_I \) are Cartan generators of the gauge group. Let \( (\omega_n, f_{i,n}, t_{I,n}) \) be the eigenvalues of a mode \( n \). The solution is given by

\[
\Phi(\tau, x) = \sum_n c_n e^{-\omega_n \tau} Y_n(x),
\]

where \( \tau \) is a Euclidean time and \( x = (x^1, \ldots, x^P) \) are coordinates in \( M \). The frequencies \( \omega_n \) are always non-zero in the theories we consider in the following sections except for gauge degrees of freedom, which have to be dealt with separately.

The spectrum of multi-particle states in a free theory is uniquely determined by the spectrum of single-particle states. It is convenient to define the single-particle index in a similar way to (2) by

\[
I_{sp} \left( e^{-\beta}, e^{-\gamma_i}, e^{iaI} \right) = \text{tr} \left[ (-1)^F e^{-\beta H - \gamma_i F_i + ia_I t_I} \right]
\]

\[
= \sum_n (-1)^F (e^{-\beta \omega_n - \gamma_i f_{i,n} + ia_I t_{I,n}}) \text{sign} \omega_n,
\]

where the trace is taken over all single-particle states including charged states. The exponent, \( \text{sign} \omega_n \), represents the fact that negative frequency modes correspond to anti-particles, which carry opposite quantum numbers to those of particles corresponding to positive frequency modes. If the field is complex we have to take account of both positive and negative frequency modes, while for a real field we need to include only positive (or negative) frequency modes. The index for multi-particle states is given by

\[
I_{mp} \left( e^{-\beta}, e^{-\gamma_i}, e^{iaI} \right) = \text{P exp} I_{sp} \left( e^{-\beta}, e^{-\gamma_i}, e^{iaI} \right),
\]

where \( \text{P exp} \) is the plethystic exponential defined by

\[
\text{P exp} f(x, y, \ldots) = \exp \left( \sum_{n=1}^{\infty} \frac{1}{n} f(x^n, y^n, \ldots) \right).
\]

This multi-particle index includes the contribution of charged states as well as gauge-invariant states. To obtain the index for physical states, we need to pick up the contribution of gauge-invariant states by integrating over \( a_I \). To determine the measure of this integral let us rewrite (2) in the path integral form

\[
I \left( e^{-\beta}, e^{-\gamma} \right) = \int [d\Phi] e^{-S - S'},
\]
where $S$ is the supersymmetric action of the theory defined on the background $M \times S^1$, and $S' = t \delta_Q V$ is a $Q$-exact deformation term corresponding to the deformation in (3). The parameters $\beta$ and $\gamma_i$ in (3) are taken in the path integral formulation as the period of $S^1$ and the Wilson lines associated with the global symmetries, respectively. In the weak coupling limit the saddle point approximation gives the exact answer. For a gauge theory we have to fix the gauge symmetry and carefully take the associated ghost factor into account. Let $\alpha$ be weights in the adjoint representation of the gauge group $G$, including both roots and Cartan generators, and $t_{\alpha}$ be the corresponding generators. Let $A$ be a fluctuation of the gauge potential around a saddle point. It belongs to the adjoint representation and is expanded as

$$A = \sum_{\alpha} A^\alpha t_{\alpha} = \sum_{\alpha \neq 0} A^\alpha t_{\alpha} + \sum_{I} A^I t_{I},$$

(9)

where $\alpha \neq 0$ means the sum is taken over only roots, and $I$ labels the Cartan generators. We divide the gauge field in $M \times S^1$ into two parts, the $x$-independent part $A^0$ and the $x$-dependent part $A' \equiv A - A^0$. The gauge fixing for $A'$ will be done in the following sections. We focus here only on $A^0 = A^0(\tau) d\tau$. The path integral of $A^0$ essentially gives the integral over the Wilson line $u = P \exp \int_0^1 A^0 d\tau \in G$. We take the static gauge

$$A_{\tau}(\tau) = \frac{1}{\beta} \sum_I a_I t_{I},$$

(10)

where $a_I$ are $\tau$-independent constants, which are identified with the chemical potentials for $t_{I}$ in (5). In this gauge the integral over $u$ can be rewritten as

$$\frac{1}{\text{Vol } G} \int du \text{Imp} = \frac{1}{|W|} \int \prod_{I=1}^r \frac{da_I}{2\pi} \prod_{\alpha \neq 0} \left(1 - e^{i\alpha(a)}\right) \text{Imp},$$

(11)

where $r = \text{rank } G$ and $|W|$ is the number of elements of the Weyl group. $\alpha(a)$ is defined by $[\alpha, t_{\alpha}] = \alpha(a) t_{\alpha}$ for a Cartan element $a$ of the gauge algebra. As well as the non-zero mode contribution (6), the measure factor in (11) is also written in the form of the plethystic exponential of the single-particle index,

$$I_{\text{sp}}^{(gh)}(e^{ia_I}) = - \sum_{\alpha \neq 0} e^{i\alpha(a)} = \sum_{\alpha} e^{i\alpha(a)} (\delta_{\alpha,0} - 1).$$

(12)

We can interpret this as the contribution of ghost constant modes.

After all this, we obtain the following final expression for the index:

$$I = \frac{1}{|W|} \int \prod_{I=1}^r \frac{da_I}{2\pi} P \exp I_{\text{sp}}^{(\text{tot})}, \quad I_{\text{sp}}^{(\text{tot})} = I_{\text{sp}} + I_{\text{sp}}^{(gh)}. $$

(13)

In the rest of the paper we calculate the single-particle index $I_{\text{sp}}$ for vector multiplets in 3-, 4-, and 6-dimensional theories. For 6d theories we only take account of the perturbative contribution.

2. **Spherical harmonics**

In this section, we define spherical harmonics on $S^p$ in preparation for the analysis in the following sections.
2.1. Local frame on \( S^p \)

Let \( f_a (a = -1, 0, 1, \ldots, p - 1) \) be \( p + 1 \) unit vectors that form an orthonormal basis in \( \mathbb{R}^{p+1} \). We denote the position vector by \( y \), and the orthonormal coordinates \( y^a \) are defined by \( y = f_a y^a \), or, equivalently, \( y^a = f_a \cdot y \). The unit sphere \( S^p \) is defined by \( y_a y_a = y \cdot y = 1 \).

Let \( \hat{T}_{ab} \) be the generators of the rotation group \( G \equiv SO(p + 1) \) that act on \( f_a \) as

\[
\hat{T}_{ab} f_c = f_a \delta_{bc} - f_b \delta_{ac} = f_d \rho^V_{dc}(T_{ab}),
\]

where \( \rho^V_{cd}(T_{ab}) := \delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc} \) are representation matrices for the vector representation. For an anti-symmetric tensor \( \lambda_{ab} \) we denote the corresponding \( G \) generator \( \frac{1}{2}\lambda_{ab} \hat{T}_{ab} \) by \( \hat{\lambda} \). It acts on a vector \( v = f_a v^a \) as

\[
\hat{\lambda} v = \frac{1}{2} \lambda_{ab} \hat{T}_{ab} (f_c v^c) = f_a \lambda_{ab} v^b.
\]

\( S^p \) can be given as the coset \( G/H \), where \( H = SO(p) \) is the subgroup of \( G \) that does not move a specific vector \( n \in S^p \). We choose \( n \) to be \( n = f_{-1} \). \( S^p = G/H \) means that there exists a projection map \( \pi : G \to S^p \) defined by

\[
\pi : g \in G \to y = gn \in S^p.
\]

In other words, \( G \) is an \( H \) fibration over \( S^p \). This fiber bundle is called the frame bundle of \( S^p \).

Let \( g \) be a section of the frame bundle. Namely, \( g \) is a map from \( S^p \) to \( G \) satisfying \( y = g(y)n \). With this section we can define a local basis \( \xi^{(y)}_i \) at every point \( y \in S^p \) by

\[
\xi^{(y)}_i = g(y)f_i \quad (i = 0, 1, \ldots, p - 1).
\]

We call \( g(y) \) a frame section.

The vielbein 1-form \( e^i \) and the spin connection 1-form \( \omega_{ij} \) are defined by the relations

\[
dy = \xi_i^{(y)} e^i, \quad d\xi_i^{(y)} = \xi_j^{(y)} \omega_{ji} + \kappa_i y,
\]

where \( \kappa_i \) is the extrinsic curvature 1-form, which we are not interested in. The two equations in (18) are equivalent to

\[
g^{-1} \, dg = \hat{\epsilon} + \hat{\omega}, \quad \hat{\epsilon} = e^i \hat{T}_{i(-1)}, \quad \hat{\omega} = \frac{1}{2} \omega_{ij} \hat{T}_{ij},
\]

and \( \kappa_i = -e^i \). Equation (19) is used in the following to rewrite covariant derivatives of harmonics in an algebraic form.

A change of the frame section by \( g'(y) = g(y)h(y), h(y) \in H \) reproduces the local frame rotation

\[
\hat{\epsilon}' = h^{-1} \hat{\epsilon} h, \quad \hat{\omega}' = h^{-1} (\hat{\omega} + d) h.
\]

2.2. Spherical harmonics

A scalar harmonic \( Y^0 : S^p \to \mathbb{R} \) with angular momentum \( \ell \) is given by a homogeneous polynomial of the orthonormal coordinates \( y^a = y \cdot f_a \) of order \( \ell \).

\[
Y^0(y) = c_{a_1 \cdots a_\ell} y^{a_1} \cdots y^{a_\ell}.
\]

The coefficients \( c_{a_1 \cdots a_\ell} \) are components of a totally symmetric traceless tensor satisfying \( c_{bba_3 \cdots a_\ell} = 0 \). We can rewrite (21) as

\[
Y^0(y) = c_{a_1 \cdots a_\ell} (y \otimes \cdots \otimes y) \cdot (f_{a_1} \otimes \cdots \otimes f_{a_\ell})
\]

\[
= g(y) (n \otimes \cdots \otimes n) \cdot c_{a_1 \cdots a_\ell} (f_{a_1} \otimes \cdots \otimes f_{a_\ell})
\]

\[
= g(y) N \cdot F,
\]

\[
= \rho^V_{cd}(T_{ab}) = f_d \rho^V_{dc}(T_{ab}),
\]
where

\[ F = c_{a_1 \cdots a_\ell} (f_{a_1} \otimes \cdots \otimes f_{a_\ell}) \]  

is a traceless symmetric \( \ell \)-tensor, and

\[ N = (n \otimes \cdots \otimes n) \]  

is an \( H \)-invariant \( \ell \)-tensor. Equation (22) is a scalar function on \( S^p \) because it is invariant under the change of the frame section \( g(y) \rightarrow g'(y) = g(y)h(y) \).

Next, let us consider vector harmonics. An arbitrary \( \mathbb{R}^{p+1} \)-vector function \( S^p \rightarrow \mathbb{R}^{p+1} \) that in general has both normal and tangential components to \( S^p \) can be expanded by a set of vector functions of the form

\[ Y^1(\mathbf{y}) = \mathbf{f}_b c_{b,a_1 a_2 \cdots a_{\ell}} y_{a_1} \cdots y_{a_{\ell}}. \]  

A vector harmonic \( Y^1 : S^p \rightarrow T S^p \) can be obtained by projecting away the normal component from (25). Its components are given by

\[ Y^1_i(y) = \xi^{(y)}_i b c_{b,a_1 a_2 \cdots a_{\ell}} y_{a_1} \cdots y_{a_{\ell}} = g(y) N_i \cdot F, \]  

where \( F \) and \( N_i \) are defined by

\[ F = c_{b,a_1 a_2 \cdots a_{\ell}} (f_{b} \otimes f_{a_1} \otimes \cdots \otimes f_{a_{\ell}}), \quad N_i = (f_i \otimes n \otimes \cdots \otimes n). \]  

It is easy to generalize the above construction of the scalar and the vector harmonics to general spins. Let \( S \) and \( R \) be a spin and an angular momentum, which are representations of \( H \) and \( G \), respectively. We suppose these are irreducible. Let \( V_R \) and \( \tilde{V}_R \) be the representation space of \( R \) and its dual space, respectively. Although \( R \) is an irreducible representation of \( G \), it may be reducible as an \( H \) representation. For the existence of the \( S \) harmonics with angular momentum \( R \), \( S \) must appear in the \( H \)-irreducible decomposition of \( R \). In other words, there must be an \( H \)-invariant subspace \( V_S \subset V_R \) and its dual space \( \tilde{V}_S \subset \tilde{V}_R \) associated with the spin representation \( S \).

Let \( E_\mu \in V_R \) (\( \mu = 1, \ldots, \text{dim} R \)) be basis vectors of \( V_R \), and \( \tilde{E}_\mu \in \tilde{V}_R \) be the dual vectors satisfying \( \tilde{E}_\mu, E_\nu = \delta_{\mu \nu} \). \( G \) acts on these vectors as

\[ g E_\mu = E_\nu \rho^R_{\mu \nu}(g), \quad g \tilde{E}_\mu = \rho^{R^\dagger}_{\mu \nu}(g^{-1}) \tilde{E}_\nu, \]  

where \( \rho^R_{\mu \nu}(g) = (\tilde{E}_\mu, g E_\nu) \) is the representation matrix for \( R \). We also introduce basis vectors \( E_\alpha \in V_S \) and \( \tilde{E}_\alpha \in \tilde{V}_S \) that are transformed by \( H \) in a similar way to (28).

Spin \( S \) harmonics with angular momentum \( R \) are given by

\[ Y^S_{a\mu}(\mathbf{y}) = (g(\mathbf{y}) \tilde{E}_\alpha, E_\mu), \]  

where \( \alpha = 1, \ldots, \text{dim} S \) is a spin index, and \( \mu = 1, \ldots, \text{dim} R \) labels harmonics belonging to \( R \). The harmonics are transformed under the local frame rotation \( g'(\mathbf{y}) = g(\mathbf{y}) h(\mathbf{y}) \) by

\[ Y^S_{a\mu}(\mathbf{y}) \rho^S_{\alpha \beta} (h^{-1}(\mathbf{y})) Y^S_{\beta \mu}(\mathbf{y}). \]  

This means \( Y^S_{a\mu} \) have spin \( S \). Under an isometry transformation \( \mathbf{y}' = \mathbf{g}^{-1} \mathbf{y} \) by \( \mathbf{g} \in G \), \( Y^S_{a\mu} \) are transformed as

\[ Y^S_{a\mu}(\mathbf{y}') = \rho^S_{\alpha \beta} (h(\mathbf{y}')) Y^S_{\beta \mu}(\mathbf{y}) \rho^R_{\nu \mu}(\mathbf{g}), \]  

where \( h(\mathbf{y}) = g^{-1}(\mathbf{y}) \mathbf{g}(\mathbf{g}^{-1} \mathbf{y}) \in H \) is the local frame rotation compensating the change of the local frame due to the isometry rotation. The relation (31) shows that the harmonics have angular momentum \( R \).
For a fixed spin $S$ there are an infinite number of representations $R$ that contain $S$ in their $H$-irreducible decomposition. We can show by using Peter–Weyl theorem that the collection of all $Y^{SR}_{a\mu}$ for such representations form a complete basis of spin $S$ fields.

2.3. Covariant derivatives

The harmonics we defined above are eigenfunctions of the Laplacian on the sphere. This is easily shown by expressing the covariant derivatives in an algebraic form.

The covariant exterior derivative $D \equiv e^I D_i \equiv d + \hat{\omega}$ of a spin-$S$ harmonic $Y^{SR}_{a\mu} = (g(y)\tilde{E}_a, E_\mu)$ is given by

$$
DY^{SR}_{a\mu} = dy^{SR}_{a\mu} + \rho^S_{\alpha\beta}(\hat{\omega})Y^{SR}_{\beta\mu}
= (dg(y)\tilde{E}_a, E_\mu) + \rho^S_{\alpha\beta}(\hat{\omega}) (g(y)\tilde{E}_\beta, E_\mu)
= (g(y)(g^{-1}(y)dg(y) - \hat{\omega})\tilde{E}_a, E_\mu)
= (g(y)\hat{\epsilon}\tilde{E}_a, E_\mu).
$$

(32)

At the last step we used (19). This is equivalent to

$$
D_i Y^{SR}_{a\mu} = (g(y)\hat{T}_{i(-1)}\tilde{E}_a, E_\mu).
$$

(33)

Note that (33) is again a harmonic, and has the general form (29) of harmonics with $\tilde{E}_a$ replaced by $\hat{T}_{i(-1)}\tilde{E}_a$. The second derivative of $Y^{SR}_{a\mu}$ is obtained in the same way as

$$
D_i D_j Y^{SR}_{a\mu} = (g\hat{T}_{i(-1)}\hat{T}_{j(-1)}\tilde{E}_a, E_\mu).
$$

(34)

By contracting indices $i$ and $j$, we obtain

$$
\Delta Y^{SR}_{a\mu} = -\lambda Y^{SR}_{a\mu}, \quad \lambda = -\hat{T}_{i(-1)}\hat{T}_{i(-1)} = C^G_2(R) - C^H_2(S),
$$

(35)

where $C^SO(n)(R)$ is the quadratic Casimir of an $SO(n)$ representation $R$ defined by

$$
\frac{1}{2} \sum_{a,b=1}^n \rho^R(\hat{T}_{ab}) \rho^R(\hat{T}_{ab}) = -C^SO(n)(R).
$$

(36)

It is also easy to obtain the curvature tensor $R_{ij}^{\ kl} = \delta_{ik}\delta_{jl} - \delta_{ij}\delta_{kj}$ from the anti-symmetric part of (34).

3. $\mathcal{N} = 1$ superconformal index in 4d

In this section we demonstrate the calculation of the superconformal index of vector multiplets in a 4d $\mathcal{N} = 1$ supersymmetric gauge theory. The index is defined in [2,3], and the relation to short and long multiplets of the superconformal algebra is investigated. The superconformal indices for extended supersymmetric theories are also defined in [3]. In particular, the index for the $\mathcal{N} = 4$ supersymmetric Yang–Mills theory is calculated, and the agreement in the large $N$ limit with the corresponding quantity in the gravity dual is confirmed. The $\mathcal{N} = 1$ superconformal index for a general $\mathcal{N} = 1$ supersymmetric gauge theory is derived in [5] using the Lagrangian of theories in $S^3 \times \mathbb{R}$ constructed in [2]. The index is used in [5] to test Seiberg duality [6].

There are two kinds of multiplets of 4d $\mathcal{N} = 1$ supersymmetry: vector multiplets and chiral multiplets. Due to limitations of space we only consider vector multiplets. A vector multiplet consists
of a vector field $A_{\mu}$, a gaugino field $\lambda$, and an auxiliary field $D$. All component fields belong to the adjoint representation of the gauge group.

### 3.1. $S^3$ harmonics

We consider a gauge theory in $S^3 \times \mathbb{R}$. In this section we label coordinates $y^a \in \mathbb{R}^4$ in a different way from the previous section. We use $(y^4, y^1, y^2, y^3)$ instead of $(y^{-1}, y^0, y^1, y^2)$. For $S^3 \times \mathbb{R}$ coordinates we use $x^a (a = 1, 2, 3, 4), x^i (i = 1, 2, 3)$ for $S^3$ and $x^4$ for $\mathbb{R}$. The theory has four commuting conserved charges: the Hamiltonian $H = -\partial / \partial x^4$, a $U(1)_R$ charge $R$, and the angular momenta $J_1 = \frac{1}{7}T_{12}$ and $J_2 = \frac{1}{7}T_{34}$. The Dirac matrices used in this section are

$$
\gamma^i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} \quad (i = 1, 2, 3), \quad \gamma^4 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \gamma^5 = -\gamma_{1234} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
$$

(37)

The generators $\hat{T}_{ab} (a, b = 1, 2, 3, 4)$ of $G = SO(4) = SU(2)_L \times SU(2)_R$ are related to $SU(2)_L$ and $SU(2)_R$ generators $\hat{T}_k^L$ and $\hat{T}_k^R$ by

$$
\hat{T}_{ij} = \epsilon_{ijk} \hat{T}_k^D = \epsilon_{ijk} (\hat{T}_k^L + \hat{T}_k^R), \quad \hat{T}_{i4} = \hat{T}_i^L - \hat{T}_i^R,
$$

(38)

where $\hat{T}_k^D$ are generators of the diagonal subgroup $H = SU(2)_D$.

A spin representation $S$ is specified by $s \in (1/2)\mathbb{Z}_{\geq 0}$, while a $G$ representation $R$ is specified by the left and the right angular momenta $j_L, j_R \in (1/2)\mathbb{Z}_{\geq 0}$. The $G$ representation $R = (j_L, j_R)$ must contain the $H$ representation $s$ in its $H$-irreducible decomposition. This requires the triangular inequality

$$
|j_L - j_R| \leq s \leq j_L + j_R.
$$

(39)

$S^3$ harmonics are labeled by six half integers: $y^s \sigma^{(j_L, j_R)}_{\alpha (m_L, m_R)}$. When we do not need to explicitly show $(j_L, j_R)$ and $(m_L, m_R)$, we often omit them to simplify the expression. The eigenvalues of the Laplacian are given by

$$
\Delta Y^s_{\alpha} = -[2j_L(j_L + 1) + 2j_R(j_R + 1) - s(s + 1)] Y^s_{\alpha}.
$$

(40)

We define

$$
\text{rot } Y^s_{\alpha} \equiv -\rho^s_{\alpha \beta} (\hat{T}_k^D) D_k Y^s_{\beta}.
$$

(41)

We denote the operator by “rot” because this becomes the rotation for vector fields; rot $Y^1_{\alpha} = \epsilon_{ijk} D_j Y^1_{\beta}$. For spinor fields, this becomes the Dirac operator up to a numerical factor; rot $Y^3_{\alpha} = -\frac{i}{2} (\sigma_k)_{\alpha \beta} D_k Y^3_{\beta}$. We can calculate eigenvalues of this operator as

$$
\text{rot } Y^s_{\alpha} = -\rho^s_{\alpha \beta} \left( \hat{T}_k^D \right) \left( g(y) \hat{T}_{k(-1)} \hat{E}_\beta, E_{(m_L, m_R)} \right)
$$

$$
= \left( g(y) \hat{T}_{k(-1)} \hat{T}_k^D \hat{E}_\alpha, E_{(m_L, m_R)} \right).
$$

$$
= \sigma Y^s_{\alpha},
$$

(42)

where $\sigma$ is given by

$$
\sigma = \hat{T}_{k(-1)} \hat{T}_k^D = \left( \hat{T}_k^L \right)^2 - \left( \hat{T}_k^R \right)^2 = -j_L(j_L + 1) + j_R(j_R + 1).
$$

(43)
3.2. Killing spinors

Supersymmetry transformations in a 4d conformally flat background are parameterized by a left-handed spinor $\epsilon$ and a right-handed spinor $\bar{\epsilon}$ satisfying the Killing spinor equations

$$D_a \epsilon = \gamma_a \bar{\kappa}, \quad D_a \bar{\epsilon} = \gamma_a \kappa, \quad (a = 1, 2, 3, 4).$$

(44)

To calculate the superconformal index, we need to determine Killing spinors in $S^3 \times \mathbb{R}$. From (44) we can show $\Delta_{s^3} \epsilon = -\frac{3}{2} \epsilon$ (and the same equation for $\bar{\epsilon}$). Comparing this with (40), we find $\epsilon$ and $\bar{\epsilon}$ must have $(J_L, J_R) = \left( \frac{1}{2}, 0 \right)$ or $(J_L, J_R) = \left( 0, \frac{1}{2} \right)$. There are four linearly independent Killing spinors for $\epsilon$:

$$\epsilon_\alpha (y) \propto Y_2^2 \left( \frac{1}{2}, 0 \right)_a, \quad \epsilon_\alpha (y) \propto Y_2^\frac{1}{2} \left( \frac{1}{2}, 0 \right)_a \propto Y_2 \left( 0, \frac{1}{2} \right)_a.$$

(45)

We also have four Killing spinors for $\bar{\epsilon}$. By using the general formula (33) we obtain

$$D_l \epsilon = -\frac{i}{2} \gamma_{\text{iso}} \sigma_l \epsilon = -\frac{1}{2} \gamma_{\text{iso}} \gamma_l \gamma_4 \epsilon, \quad D_l \bar{\epsilon} = -\frac{i}{2} \gamma_{\text{iso}} \sigma_l \bar{\epsilon} = \frac{1}{2} \gamma_{\text{iso}} \gamma_l \gamma_4 \bar{\epsilon},$$

(46)

where $\gamma_{\text{iso}}$ is the chirality operator for the isometry group $G = SO(4)$, and $\gamma_{\text{iso}} = +1 (-1)$ for Killing spinors belonging to $R = \left( \frac{1}{2}, 0 \right)$ [$R = \left( 0, \frac{1}{2} \right)$]. From these equations, we find

$$\bar{\kappa} = -\frac{1}{2} \gamma_{\text{iso}} \gamma_4 \epsilon, \quad \kappa = \frac{1}{2} \gamma_{\text{iso}} \gamma_4 \bar{\epsilon}.$$

(47)

The equations in (44) with $a = 4$ determine the $x^4$ dependence of the Killing spinors;

$$\epsilon \propto e^{-\frac{\gamma_{\text{iso}} x^4}{2}}, \quad \bar{\epsilon} \propto e^{+\frac{\gamma_{\text{iso}} x^4}{2}}.$$

(48)

We have obtained eight linearly independent Killing spinors (four for $\epsilon$ and four for $\bar{\epsilon}$). To perform localization, we choose the following specific ones:

$$\epsilon_\alpha = e^{\frac{1}{2} x^4} Y_2 \left( \frac{1}{2}, 0 \right)_a \propto Y_2 \left( \frac{1}{2}, 0 \right)_a, \quad \bar{\epsilon}_\alpha = e^{\frac{1}{2} x^4} Y_2 \left( \frac{1}{2}, 0 \right)_a \propto Y_2 \left( \frac{1}{2}, 0 \right)_a.$$

(49)

with quantum numbers

$$\epsilon : (H, J_1, J_2, R) = \left( \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, 1 \right), \quad \bar{\epsilon} : (H, J_1, J_2, R) = \left( -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -1 \right).$$

(50)

We denote the supercharges corresponding to $\epsilon$ and $\bar{\epsilon}$ by $Q$ and $\overline{Q}$, respectively.

Among linear combinations of four conserved charges $H, J_1, J_2,$ and $R$, the following three commute with $Q$ and $\overline{Q}$:

$$\{ Q, \overline{Q} \} = H - J_1 - J_2 - \frac{3}{2} R, \quad J_1 + \frac{R}{2}, \quad J_2 + \frac{R}{2}.$$

(51)

We define the superconformal index by

$$I(z_1, z_2) = \text{tr} \left[ (-1)^F q^{H - J_1 - J_2 - \frac{3}{2} R - \frac{J_1 + \frac{R}{2}}{2} - \frac{J_2 + \frac{R}{2}}{2}} \right].$$

(52)

Because the spinors in (50) are $SU(2)_R$ singlet, the cancellation occurs between modes with the same $SU(2)_R$ quantum numbers. In the following analysis it is convenient to separate the $SU(2)_R$ quantum number from others. We define Cartan generators of $SU(2)_L$ and $SU(2)_R$ by

$$J_L = \frac{1}{2} (J_1 + J_2) = \frac{1}{i} \hat{T}_L, \quad J_R = \frac{1}{2} (J_1 - J_2) = \frac{1}{i} \hat{T}_R,$$

(53)

and rewrite the index (52) by introducing $z_1 = tx$ and $z_2 = t/x$ as

$$I(z_1, z_2) = \text{tr} \left[ (-1)^F q^{H - 2 J_L - \frac{3}{2} R - 2 J_L + R} x^2 J_R \right].$$

(54)
3.3. Mode analysis

The transformation rules of 4d $\mathcal{N} = 1$ vector multiplets are conformally invariant, and we can use the rules for the flat background except that the parameters $\epsilon$ and $\bar{\epsilon}$ are Killing spinors satisfying (44);

$$
\delta A_\mu = i(\epsilon \gamma_\mu \bar{\epsilon}) + i(\bar{\epsilon} \gamma_\mu \lambda),
$$

$$
\delta \lambda = i \bar{\epsilon} \epsilon + D \epsilon,
$$

$$
\delta \bar{\epsilon} = -i \bar{\epsilon} \epsilon + D \bar{\epsilon},
$$

$$
\delta D = (\epsilon R \bar{\epsilon}) + (\bar{\epsilon} R \lambda),
$$

(55)

where $R \equiv (1/2)\gamma^{ab} F_{ab}$ and $Q \equiv \gamma^a D_a$. To perform index calculation, we deform the action by adding $Q$-exact terms. A standard form of the action used in the literature is $S \sim \delta Q ((\delta Q \psi)^\dagger \psi)$, where $\psi$ denotes fermions in the theory. An advantage of this action is that the bosonic part is manifestly positive definite, and the path integral is well defined automatically. However, this also has the disadvantage that it contains spinor bilinear $\epsilon^\dagger \gamma^i \epsilon$, which breaks the rotational symmetry $G$. To avoid this problem, we adopt another action,

$$
\mathcal{L} = \frac{1}{(\epsilon \bar{\epsilon})} \delta (\bar{\epsilon}) \delta (\epsilon) \left( -\frac{1}{4} (\bar{\lambda} \lambda) \right),
$$

(56)

where $\bar{\epsilon}'$ is a Killing spinor such that $\bar{\epsilon} \bar{\epsilon}'$ is a $G$-invariant scalar. For example, we can use the Killing spinor obtained from $\bar{\epsilon}$ by replacing $Y_{a(1/2,0)}$ with $Y_{a(-1/2,0)}$ as $\bar{\epsilon}'$. The absence of $\bar{\epsilon} \gamma^i \bar{\epsilon}'$ in the action is guaranteed by the algebra $\{\delta (\bar{\epsilon}), \delta (\epsilon)\} = 0$, and the spinors $\bar{\epsilon}$ and $\bar{\epsilon}'$ appear in the action only through $\bar{\epsilon} \bar{\epsilon}'$, and the Lagrangian is $G$-invariant;

$$
\mathcal{L} = \frac{1}{4} F_{ab}^2 - \frac{1}{8} \epsilon^{abcd} F_{ab} F_{cd} - (\bar{\lambda} \gamma^a D_a \lambda) - \frac{1}{2} D^2.
$$

(57)

We fix the gauge symmetry by the gauge-fixing function $V_{gf} = D_i A_i$. The corresponding ghost Lagrangian is $\mathcal{L}_{gh} = \bar{\epsilon} \partial_i D^i c'$, where primes indicate the ghost and the anti-ghost fields do not include the constant modes on $S^3$. They are expanded by the scalar harmonics $Y_{0(j,j)}^{(m_L,m_R)}$ ($j \geq 1/2$), and the path integral gives the factor $-4j(j + 1)$ for each $j$ and $(m_L, m_R)$. Note that $V_{gh}$ is not the full 4d divergence $D_a D_a = D_i A_i + D_4 A_4$ but the 3d divergence in $S^3$. Therefore, the gauge fixing is partial, and the gauge transformation with parameter that depends only on $x^4$ is not fixed. The fixing of this remaining gauge symmetry has already been discussed in Sect. 1, and gives the Jacobian factor in (11).

The bosonic part of the Lagrangian including the gauge-fixing term is

$$
\mathcal{L} = \frac{1}{2} V_i^2 + \frac{1}{2} V_{gf}^2, \quad V_i = \frac{1}{2} \epsilon_{ijk} F_{jk} - F_{i4}, \quad i, j, k = 1, 2, 3,
$$

(58)

where the $D^2$ term is neglected because the path integral of $D$ gives just a constant. $(V_i, V_{gf})$ and $A_a$ are related by

$$
V = D_A A
$$

(59)

with a differential operator $D_A$. The energy eigenmodes are obtained by solving $D_A A = 0$. This can be easily solved by harmonic expansion. We expand $A$ and $V$ as follows:

$$
A_i = a_1 Y_{i(m_L,m_R)}^{1,(j+1,j)} + a_2 Y_{i(m_L,m_R)}^{1,(j-1,j)} + a_3 Y_{i(m_L,m_R)}^{1,(j,j)} \quad A_4 = a_4 Y_{(m_L,m_R)}^{0,(j,j)},
$$

$$
V_i = f_1 Y_{i(m_L,m_R)}^{1,(j+1,j)} + f_2 Y_{i(m_L,m_R)}^{1,(j-1,j)} + f_3 Y_{i(m_L,m_R)}^{1,(j,j)} \quad V_{gf} = f_4 Y_{(m_L,m_R)}^{0,(j,j)}.
$$

(60)
Table 1. Bosonic physical modes in $S^3 \times \mathbb{R}$ are shown. We denote $J_L$ eigenvalues by $m$. $J_R$ eigenvalues always take values between $-j$ and $j$, and we do not explicitly show them in the table.

<table>
<thead>
<tr>
<th>ID</th>
<th>$H = -\partial_4$</th>
<th>$D - 2 J_L - \frac{3}{2} R$</th>
<th>$2 J_L + R$</th>
<th>Range of $m$</th>
</tr>
</thead>
<tbody>
<tr>
<td>[A1]</td>
<td>$-2(j + 1)$</td>
<td>$-2 j - 2 - 2 m$</td>
<td>$2 m$</td>
<td>$-j - 1 \leq m \leq j + 1$</td>
</tr>
<tr>
<td>[A2]</td>
<td>$2 j$</td>
<td>$2 j - 2 m$</td>
<td>$2 m$</td>
<td>$-j + 1 \leq m \leq j - 1$</td>
</tr>
</tbody>
</table>

Table 2. Fermionic physical spectrum in $S^3 \times \mathbb{R}$. We denote $J_L$ eigenvalues by $m'$.

<table>
<thead>
<tr>
<th>ID</th>
<th>$H = -\partial_4$</th>
<th>$D - 2 J_L - \frac{3}{2} R$</th>
<th>$2 J_L + R$</th>
<th>Range of $m'$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda 1$</td>
<td>$-(2 j + \frac{3}{2})$</td>
<td>$-2 j - 3 - 2 m'$</td>
<td>$2 m' + 1$</td>
<td>$-j - \frac{1}{2} \leq m' \leq j + \frac{1}{2}$</td>
</tr>
<tr>
<td>$\lambda 2$</td>
<td>$2 j + \frac{1}{2}$</td>
<td>$2 j - 1 - 2 m'$</td>
<td>$2 m' + 1$</td>
<td>$-j + \frac{1}{2} \leq m' \leq j - \frac{1}{2}$</td>
</tr>
</tbody>
</table>

Because $D_A$ is $G$-invariant, modes with different $(j_L, j_R)$ and $(m_L, m_R)$ do not mix. For $(j_L, j_R) = (j, j)$ ($j \geq \frac{1}{2}$), there are two components for each $A$ and $V$, and the coefficients are related by

$$
\begin{pmatrix}
  f_3 \\
  f_4
\end{pmatrix} =
\begin{pmatrix}
  \partial_4 & -1 \\
  -4 j (j + 1) & 0
\end{pmatrix}
\begin{pmatrix}
  a_3 \\
  a_4
\end{pmatrix} \quad \left( |m_L| \leq j \geq \frac{1}{2} \right).
$$

(61)

The determinant of the matrix in (61) is

$$
\det = -4 j (j + 1).
$$

(62)

This is canceled by the ghost factor, and does not correspond to any physical modes. When $(j_L, j_R) = (0, 0)$, $a_3$ mode does not exist, and we have only $a_4$ mode. This corresponds to the Wilson line integrated in (11).

The $(j + 1, j)$ mode $f_1$ and $(j - 1, j)$ mode $f_2$ depend only on $a_1$ and $a_2$, respectively. By using formula (42) we obtain

$$
f_1 = [\partial_4 - 2(j + 1)] a_1 \quad (|m_L| \leq j + 1),
$$

$$
f_2 = [\partial_4 + 2 j] a_2 \quad (|m_L| \leq j - 1).
$$

These give the physical modes in Table 1.

For the gaugino $\lambda$, we need to solve the Dirac equation $\mathcal{D} \lambda = 0$ in $S^3 \times \mathbb{R}$. We expand $\lambda$ and $\mathcal{D} \lambda$ by the harmonics as follows:

$$
\lambda_\alpha = c Y_\alpha^{\frac{1}{2},(j+\frac{1}{2},j)} + d Y_\alpha^{\frac{1}{2},(-j-\frac{1}{2},j)}, \quad \mathcal{D} \lambda_\alpha = s Y_\alpha^{\frac{1}{2},(j+\frac{1}{2},j)} + t Y_\alpha^{\frac{1}{2},(-j-\frac{1}{2},j)}.
$$

(63)

By using (42) we obtain

$$
s = -i (\partial_4 - 2 j - \frac{3}{2}) c, \quad t = i (\partial_4 + 2 j + \frac{1}{2}) d.
$$

(64)

The corresponding eigenmodes are summarized in Table 2.

Let us compare the bosonic modes in Table 1 and the fermionic modes in Table 2. If we replace $m'$ in Table 2 by $m - 1/2$, we see that the quantum numbers $D - 2 J_L - \frac{3}{2} R$ and $2 J_L + R$ in the two tables completely match. However, the ranges of $m$ are different. $m = m' + 1/2$ runs over the range $-j \leq m \leq j + 1$ for $[\lambda 1]$ and $-j + 1 \leq m \leq j$ for $[\lambda 2]$. This means the [A1] mode with $m = -j - 1$ and the [\lambda 2] mode with $m = m' + 1/2 = j$ do not have partners and make contributions
to the index. The $[A1]$ modes with $m = -j - 1$ contribute to the index by

$$I_{sp}(z_1, z_2, e^{ia}) = \sum_{\alpha} \sum_{j \in \mathbb{Z}_{\geq 0}} e^{-ia(\alpha)j} x^{2j+2} \chi_j(x^2) = \frac{t^2}{(1-tx)(1-tx/x)} \sum_{\alpha} e^{-i\alpha(a)},$$

where $\chi_j(x^2)$ is the SU(2) character $\chi_j(x^2) = x^{2j} + \cdots + x^{-2j}$. The $[\lambda, 2]$ modes with $m' = j - 1/2$ contribute to the index by

$$I_{sp}^{(tot)}(z_1, z_2, e^{ia}) = -\sum_{\alpha} \sum_{j \in \mathbb{Z}_{\geq 1}} e^{-ia(\alpha)j} x^{2j} \chi_j(x^2) = \left(1 - \frac{1}{(1-tx)(1-tx/x)}\right) \sum_{\alpha} e^{-i\alpha(a)}.$$  \hspace{1cm} (66)

Combining the bosonic contribution (65), the fermionic contribution (66), and the ghost contribution (12), we obtain the total single-particle index

$$I_{sp}^{(tot)}(z_1, z_2, e^{ia}) = \sum_{\alpha} e^{i\alpha(a)} \left(\delta_{\alpha, 0} - \frac{1-z_1z_2}{(1-z_1)(1-z_2)}\right).$$

Refer to [5] for the result for chiral multiplets.

Before ending this section, we comment on a close relation between the 4d superconformal index and the $S^3$ partition function of 3d supersymmetric field theories. The 3d partition function of $\mathcal{N} = 2$ supersymmetric field theories on round $S^3$ is first calculated in [7] for canonical fields. It is generalized to non-canonical fields in [8,9], and squashed $S^3$ in [10]. (See also [11].) Once we obtain the formula for the 4d superconformal index, it is easy to obtain a general formula for the $S^3$ partition function by taking a small radius limit $\beta \to 0$ [12–14].

4. $\mathcal{N} = 2$ superconformal index in 3d

In this section we consider a 3d $\mathcal{N} = 2$ supersymmetric theory. The 3d superconformal index is defined in [4], and the relation to multiplets of the superconformal algebra is investigated. An important application of the index is a test of AdS$_{3}$/CFT$_{3}$ correspondence. For example, the index of the ABJM model [15] is calculated in [16] for the perturbative sector and the agreement with the corresponding quantity on the gravity side is confirmed. The check including monopole contribution is done in [17], in which the formula of the $\mathcal{N} = 2$ superconformal index for canonical fields (without anomalous dimensions) is derived. The index for chiral multiplets with anomalous dimensions is derived in [18].

Just like the 4d $\mathcal{N} = 1$ case, we have two kinds of multiplets: vector multiplets and chiral multiplets. Again we only consider a vector multiplet, which consists of a vector field $A_{a}$, a scalar field $\sigma$, a gaugino field $\lambda$, and an auxiliary field $D$.

We use different labeling of the coordinates $y^{a}$ from Sect. 2. Instead of $(y^{-1}, y^{0}, y^{1})$, we use $(y^{3}, y^{1}, y^{2})$. For $S^2 \times \mathbb{R}$ we use coordinates $x^{a}$ ($a = 1, 2, 3$), $x^{i}$ ($i = 1, 2$) for $S^2$, and $x^{3}$ for $\mathbb{R}$. We use Pauli matrices as the Dirac matrices: $\gamma_a = \sigma_a$.

We consider an $\mathcal{N} = 2$ supersymmetric theory in $S^2 \times \mathbb{R}$. There are three conserved charges: the Hamiltonian $H = -\partial/\partial x^{3}$, the angular momentum $J_3 = -i\tilde{T}_{12}$, and a $U(1)_{R}$ charge $R$.

An important difference of $S^2$ from $S^3$ discussed in the previous section is the existence of topologically non-trivial gauge field configurations, monopoles. Harmonics describing charged particles coupling to such monopole backgrounds are called monopole harmonics.
4.1. Monopole harmonics

Let us consider harmonics with angular momentum \( j \in \frac{1}{2} \mathbb{Z}_{\geq 0} \). The representation space \( V_j \) is spanned by

\[
E_s, \quad s = -j, -j + 1, \ldots, j - 1, j,
\]
satisfying

\[
\hat{T}_{12} E_s = is E_s.
\]

The dual vectors are defined by \( \tilde{E}_s = E_{-s} = (E_s)^* \). All representations of \( H \) are singlets. Spin \( s \) harmonics are given by

\[
Y^j_{s,m} = (g(y)\tilde{E}_s, E_m) = \rho^j_{s,m}(g(y)),
\]

where \( \rho^j \) is the \( SU(2) \) representation matrix for angular momentum \( j \).

Let us consider a field with \( U(1) \) electric charge \( q \in \mathbb{Z} \) in the monopole background with magnetic charge \( m \in \mathbb{Z} \). We can take the gauge so that the gauge potential is given by \( A = (m/2)\omega_{12} \), and then the covariant derivative for a spin \( s \) field becomes

\[
D = d + is\omega_{12} - iqA = d + i \left( s - \frac{mq}{2} \right) \omega_{12}.
\]

Therefore, we can treat a charged spin \( s \) particle in the magnetic flux as a particle with shifted spin

\[
s_{\text{eff}} = s - \frac{qm}{2}.
\]

Thus we do not have to introduce anything new for dealing with monopole backgrounds. In the case of non-Abelian gauge theory, \( m = \sum_1^i m_{1i} \) becomes a Cartan element of the gauge algebra, and the factor \( qm \) in (72) must be replaced by \( \alpha(m) \) for the components of an adjoint field specified by the weight \( \alpha \).

It is convenient to define \( v_{\pm} \) for a general vector \( v \) by \( v_{\pm} \equiv v_1 \pm iv_2 \). For example, we define

\[
D_{\pm}\equiv D_1 \pm iD_2, \quad \hat{T}_{3\pm}\equiv \hat{T}_{31} \pm i\hat{T}_{32}.
\]

The basis \( E_s \) with \( s = \pm 1 \) are given by \( E_{\pm 1} = f_1 \pm if_2 \). We should note that spin \( \pm 1 \) components of a vector \( v \) are \( \tilde{E}_{\pm 1} \cdot v = v_{\mp} \). The representation matrices \( (T_{3\pm})_{s,s'} = \rho^j_{s,s'}(\hat{T}_{3\pm}) \) have non-vanishing components \( (T_{3\pm})_{s=\pm 1,s=\pm 1} \), and satisfy

\[
(T_{3\pm})_{s,s=\pm 1}(T_{3\mp})_{s'=\pm 1,s} = -[j(j+1) - s(s+1)] = -(j \pm s)(j \mp s + 1).
\]

From the general formula (33) and (35) we obtain

\[
D_{\pm} Y^j_{s,m} = (T_{3\pm})_{s,s'=\pm 1} Y^j_{s'=\pm 1,m}, \quad \Delta Y^j_{s,m} = -[j(j+1) - s^2] Y^j_{s,m}.
\]

4.2. Killing spinors

The parameters of \( \mathcal{N} = 2 \) supersymmetry are two-component spinors \( \epsilon \) and \( \bar{\epsilon} \). In a Euclidean space they are treated as independent spinors. They must satisfy the Killing spinor equations \( D_\mu \epsilon = \gamma_\mu \bar{\epsilon} \) and \( D_\mu \bar{\epsilon} = \gamma_\mu \epsilon \). We can easily show that each component of \( \epsilon \) and \( \bar{\epsilon} \) must be a spinor harmonic with \( j = \frac{1}{2} \). A general form of such spinors are

\[
\epsilon = \begin{pmatrix}
Y_{\frac{1}{2},m}^1 c_m(x^3) \\
Y_{\frac{1}{2},m}^2 c'_m(x^3)
\end{pmatrix}, \quad \bar{\epsilon} = \begin{pmatrix}
Y_{\frac{1}{2},m}^1 \bar{c}_m(x^3) \\
Y_{\frac{1}{2},m}^2 \bar{c}'_m(x^3)
\end{pmatrix},
\]

where \( m \) are summed over \( \pm \frac{1}{2} \). We can determine the \( x^3 \) dependence of the coefficients in the same way as the 4d case, and obtain eight linearly independent Killing spinors. We use the following
specific ones for localization:

\[ \epsilon = e^{-\frac{i}{2} \sigma^3} \left( \begin{array}{cc} Y^\frac{1}{2} & -\frac{1}{2} \\ Y^\frac{1}{2} & -\frac{1}{2} \end{array} \right), \quad \bar{\epsilon} = e^{+\frac{i}{2} \sigma^3} \left( \begin{array}{cc} Y^\frac{1}{2} & \frac{1}{2} \\ -Y^\frac{1}{2} & \frac{1}{2} \end{array} \right). \]  

(77)

\( \epsilon \) and \( \bar{\epsilon} \) carry \((H, J_3, R) = \pm (\frac{1}{2}, -\frac{1}{2}, 1) \). We denote the corresponding supercharges by \( Q \) and \( \bar{Q} \).

Among linear combinations of the three conserved charges \( H, J_3, \) and \( R \), the following two commute with \( Q \) and \( \bar{Q} \):

\[ \{ Q, \bar{Q} \} = H - J_3 - R, \quad 2J_3 + R. \]  

(78)

We define the superconformal index by

\[ I(x, m_I) = \text{tr} \left[ (-1)^F q^{H - J_3 - R} x^{2J_3 + R} \right], \]  

(79)

where \( m_I \) are the background monopole charges.

4.3. Mode analysis

The supersymmetry transformation rules for 3d \( \mathcal{N} = 2 \) vector multiplets in a conformally flat background are

\[ \delta A_a = i (\epsilon \gamma_a \bar{\lambda}) - i (\bar{\epsilon} \gamma_a \lambda), \]
\[ \delta \sigma = (\epsilon \bar{\lambda}) + (\bar{\epsilon} \lambda), \]
\[ \delta \lambda = i \bar{\epsilon} \epsilon - (D\sigma)\epsilon - 2\sigma \bar{\epsilon} + D\epsilon, \]
\[ \delta \bar{\lambda} = -i \epsilon \bar{\epsilon} - (D\sigma) \bar{\epsilon} - 2\sigma \bar{\epsilon} + D\bar{\epsilon}, \]
\[ \delta D = -(\epsilon \gamma^a D_a \lambda) - (\bar{\epsilon} D_a \bar{\lambda}) + (\bar{\epsilon} \lambda) + (\kappa \lambda) + (\epsilon [\sigma, \bar{\lambda}]) - (\bar{\epsilon} [\sigma, \lambda]). \]  

(80)

For localization we use the \( \bar{Q} \)-exact Lagrangian

\[ \mathcal{L} = \frac{1}{(\epsilon \bar{\epsilon})} \delta(\epsilon) \delta(\bar{\epsilon'}) \left( \frac{1}{4} \bar{\lambda} \lambda \right) \]
\[ = \frac{1}{4} F_{ab} F^{ab} - \frac{1}{2} \epsilon^{abc} F_{ab} D_c \sigma + \frac{1}{2} D_a \sigma D^a \sigma + \frac{1}{2} \sigma^2 - \sigma F_{12} + \sigma D^3 \sigma \]
\[ - (\bar{\lambda} \gamma^a D_a \lambda) - \bar{\lambda} [\sigma, \lambda] - \frac{1}{2} (\bar{\lambda} \gamma^3 \lambda) - \frac{1}{2} D^2, \]  

(81)

where \( \bar{\epsilon'} \) is the Killing spinor obtained from \( \bar{\epsilon} \) by replacing \( Y^\frac{1}{2} \) by \( Y^\frac{3}{2}, -\frac{1}{2} \).

The Lagrangian for \( A \) and \( \sigma \) is rewritten in the manifestly positive definite form \( \mathcal{L} = (1/2) V_a V_a \) with

\[ V_a = D_a \sigma + \delta a^3 \sigma - \frac{1}{2} \epsilon_{abc} F_{bc}, \]  

(82)

and saddle points are given by \( V_a = 0 \). This condition is solved, up to gauge transformation, by the monopole backgrounds

\[ A = A^{(0)} = \frac{m}{2} \omega_{12}, \quad \sigma = \sigma^{(0)} = \frac{m}{2}, \quad m = \sum_I m_{1I}, \]  

(83)

where \( m \in \mathbb{Z} \) is the monopole charge. These backgrounds induce the shift of the effective spin

\[ s_{\text{eff}} = s + s_0, \quad s_0 = -\frac{\alpha(m)}{2} = -\alpha(\sigma_0). \]  

(84)
for a field associated with a weight \( \alpha \). We consider fluctuations around the saddle points,

\[
A_a = A_a^{(0)} + \delta A_a, \quad \sigma = \sigma^{(0)} + \delta \sigma. \tag{85}
\]

We fix the gauge symmetry by the gauge-fixing function \( V_{gf} = D_i^{(0)} \delta A_i \equiv \partial_3 \delta A_i - i \left[ A_i^{(0)}, \delta A_i \right] \). The quadratic part of the Lagrangian including the gauge fixing term is

\[
\mathcal{L}_{bos} = \frac{1}{2} V_a V_a + \frac{1}{2} V_{gf} V_{gf}. \tag{86}
\]

We expand the bosonic fields by the monopole harmonics as follows:

\[
\delta A_+ = Y_{s0-1,m}^j a_+, \quad \delta A_- = Y_{s0+1,m}^j a_-, \quad \delta A_3 = Y_{s0,m}^j a_3, \quad \delta \sigma = Y_{s0,m}^j a_4,
\]

\[
V_+ = Y_{s0-1,m}^j a_+, \quad V_- = Y_{s0+1,m}^j a_-, \quad V_3 = Y_{s0,m}^j a_3, \quad V_{gh} = Y_{s0,m}^j v_4. \tag{87}
\]

Unlike the 4d case, harmonics in the four components all have the same \( G \) quantum numbers \( j \) and \( m \), and can mix among them. When \( j \geq |s0| + 1 \), all four components exist, while if \( j \) is equal to or smaller than \( |s0| \) some of them are absent. When all components exist the coefficients are related by

\[
\begin{pmatrix}
0 & +i(T_3^+)_s,s-1 & (T_3^+)_s,s-1 \\
0 & -i s0 + i \partial_3 & -i(T_3^-)_{s,s+1} \\
\frac{i}{2}(T_3^-)_{s-1,s} & -i & \partial_3 + 1 \\
\frac{i}{2}(T_3^-)_{s-1,s} & \frac{i}{2}(T_3^+)_s,s+1 & 0
\end{pmatrix}
\begin{pmatrix}
a_+ \\
a_- \\
an_3 \\
a_4
\end{pmatrix}.
\tag{88}
\]

and the determinant of the \( 4 \times 4 \) matrix is

\[
det = \left[ j(j+1) - s0^2 \right] (\partial_3 - j)(\partial_3 + j + 1), \quad j \geq |s0| + 1. \tag{89}
\]

For smaller \( j \) some of rows and columns are absent, and the determinant is given by

\[
det = \pm i (D_3 + j + 1), \quad (j = |s0| - 1),
\]

\[
det = \pm i \left( j(j+1) - s0^2 \right) (D_3 + j + 1), \quad (j = |s0|). \tag{90}
\]

The factor \( \left[ j(j+1) - s0^2 \right] \) is canceled by the factor arising from the ghost term \( \mathcal{L}_{gh} = \bar{c}' D_i^{(0)} D_i c' \) and does not correspond to any physical modes. The other two factors, \( \partial_3 - j \) and \( \partial_3 + j + 1 \), correspond to physical modes in Table 3.

The fermionic part of the \( \bar{Q} \)-exact Lagrangian is \( \mathcal{L} = \bar{\lambda} D_\lambda \) with

\[
D_\lambda = -\gamma^0 D_a - \sigma - \frac{1}{2} \gamma^3 = \begin{pmatrix} -\partial_3 - \sigma - \frac{1}{2} & -D_- \\ -D_+ & \partial_3 - \sigma + \frac{1}{2} \end{pmatrix}. \tag{91}
\]

We expand the components of \( \lambda \) and \( D_\lambda \) by

\[
\lambda = \begin{pmatrix} cY_{s0+1/2,m}^j \\ dY_{s0-1/2,m}^j \end{pmatrix}, \quad D_\lambda \lambda = \begin{pmatrix} sY_{s0+1/2,m}^j \\ tY_{s0-1/2,m}^j \end{pmatrix}. \tag{92}
\]
Therefore, 

\[ A \]

When \( j = |s_0| + \frac{1}{2} \), the two components are non-vanishing, and the relation among the coefficients is

\[
\begin{pmatrix}
  s \\
  t
\end{pmatrix} = \begin{pmatrix}
  -(\partial_3 + s - \frac{1}{2}) & -(T_{3-})_{s-\frac{1}{2},s+\frac{1}{2}} \\
  -(T_{3+})_{s+\frac{1}{2},s-\frac{1}{2}} & \partial_3 + s + \frac{1}{2}
\end{pmatrix} \begin{pmatrix}
  c \\
  d
\end{pmatrix}. \tag{93}
\]

The determinant of the matrix is

\[ \det = -(\partial_3 + j + 1)(\partial_3 - j), \quad |s_0| + \frac{1}{2} \leq j. \tag{94} \]

When \( j = |s_0| - \frac{1}{2} \) only one of the two components of \( \lambda \) and \( D_3\lambda \) exists, and the matrix element becomes \((\partial_3 + j + 1)\). The physical modes are summarized in Table 4.

Let us compare the bosonic and fermionic modes. By the replacement

\[ j' = j - \frac{1}{2}, \quad j'' = j + \frac{1}{2}, \quad m' = m - \frac{1}{2}, \tag{95} \]

the ranges of \( j, D - J_3 - R \), and \( 2J_3 + R \) completely match between the two tables. However, the ranges of \( m \) are different. The range is \(-j + 1 \leq m \leq j\) for \([\lambda, 1]\), and \(-j \leq m \leq j + 1\) for \([\lambda, 2]\). Therefore, \([\lambda, 1]\) with \( m = -j \) and \([\lambda, 2]\) with \( m = j + 1 \) \((m' = j'')\) contribute to the index. The former contribute

\[
I_{\text{sp}} = \sum_{\alpha} \sum_{j=|s_0|+1}^{\infty} e^{-i\alpha(a)} x^{2j} = \sum_{\alpha} e^{-i\alpha(a)} \frac{x^{\alpha(m) + 2}}{1 - x^2}, \tag{96}
\]

and the latter contribute

\[
I_{\text{sp}} = \sum_{\alpha} \sum_{j=|s_0|-1}^{\infty} e^{i\alpha(a)} x^{2j+2} = \sum_{\alpha} e^{i\alpha(a)} \left( -\frac{x^{\alpha(m)}}{1 - x^2} + \delta_{\alpha(m),0} \right) \tag{97}
\]

to the index.

For the ghost index (12), we must take account of the gauge symmetry breaking due to the monopole background. The ghost zero-modes are present only for unbroken symmetry, and the ghost index becomes

\[
I_{\text{sp}}^{(\text{gh})} = \sum_{\alpha \neq 0, \alpha(m) = 0} e^{-i\alpha(a)} = \sum_{\alpha} e^{i\alpha(a)} (\delta_{\alpha,0} - \delta_{\alpha(m),0}). \tag{98}
\]

By combining (96), (97), and (98), we obtain

\[
I_{\text{sp}}^{(\text{tot})} (x, e^{i\alpha(a)}, m_1) = \sum_{\alpha} e^{i\alpha(a)} (\delta_{\alpha,0} - x^{\alpha(m)}) \tag{99}
\]

Refer to [18] for the chiral multiplet contribution.
5. $\mathcal{N} = (1, 0)$ superconformal index in 6d

The superconformal indices in 6d theories are defined in [4]. The derivation by harmonic expansion in this section is based on [19]. The method used in [19] is essentially the same as that in [20], in which the $S^5$ partition function is calculated by using $\mathbb{C}P^2$ harmonics in [21]. See also [22–24] for the $S^5$ partition function.

Although it is known that instantons make a non-perturbative contribution to the index, it seems in practice impossible to calculate it by harmonic expansion, and we are going to calculate only the perturbative contribution to the index.

5.1. Hopf fibration

In this section we consider a 6d $\mathcal{N} = (1, 0)$ supersymmetric theory in $S^5 \times \mathbb{R}$. As in the 3d and 4d cases, we need to choose a particular supercharge for localization. The choice breaks the rotational symmetry $G = SO(6)$. Unlike the 3d and 4d cases it seems impossible to construct a $\mathcal{Q}$-exact Lagrangian that respects the full $SO(6)$ symmetry. The Lagrangian we use respects only the subgroup $G' = SU(3) \times U(1)$. Fortunately, this is transitive, and harmonic expansion is still efficient. Because of this symmetry breaking, it is natural to regard $S^5$ as a Hopf fibration over $\mathbb{C}P^2$. In this subsection we discuss the Hopf fibration of general odd-dimensional spheres, and define a coordinate system convenient for the following analysis.

Let us consider $S^{2r+1}$ with unit radius defined as the subset of $\mathbb{R}^{2r+2}$ by $y_a y_a = 1$. In this subsection we use indices $a, b, \ldots = -1, 0, 1, \ldots, 2r$, $i, j, \ldots = 0, 1, \ldots, 2r$, and $m, n, \ldots = 1, \ldots, 2r$. The first step to define the Hopf fibration is to specify a complex structure in $R^{2r+2}$. Let $\Pi_{ab}$ be the complex structure in $\mathbb{R}^{2r+2}$ with non-vanishing components

$$\Pi_{(-1)0} = \Pi_{12} = \cdots = \Pi_{(2r-1)(2r)} = 1,$$

and let $U(1)_I$ be the subgroup of $G = SO(2r + 2)$ generated by $\hat{I} = (1/2)\Pi_{ab} F_{ab}$. We define Hopf fibration with $U(1)_I$ orbits, $y \in S^{2r+1}$ can be written as

$$y(\theta, \psi) = e^{-\psi \hat{T}} y_0(\theta),$$

where $0 \leq \psi < 2\pi$ is a coordinate along fibers, and $y_0(\theta)$ is a representative in the fiber that is specified by $\mathbb{C}P^2$ coordinates $\theta = (\theta^1, \theta^2, \theta^3, \theta^4)$. The $S^{2r+1}$ metric is

$$|dy|^2 = (dy + V)^2 + ds_{\mathbb{C}P^r}^2,$$

where $V$ and $ds_{\mathbb{C}P^r}^2$ are defined by

$$V = -dy_0 \cdot \hat{I} y_0 = \Pi_{ab} y^a_0 dy^b_0, \quad ds_{\mathbb{C}P^r}^2 = |dy_0|^2 - (dy_0 \cdot \hat{I} y_0)^2.$$

$ds_{\mathbb{C}P^r}^2$ is the Fubini–Study metric of $\mathbb{C}P^r$.

A convenient choice of the local frame is given by a frame section of the form

$$g(\theta, \psi) = e^{-\psi \hat{T}} g_0(\theta), \quad g_0(\theta) \in SU(r + 1),$$

where $SU(r + 1)$ is the special unitary group that rotates holomorphic vectors $f_{2k-1} + i f_{2k}$ ($k = 0, \ldots, r$). We call this a unitary frame. Because $\partial_\psi y(\theta, \psi) = \xi_0$ the 0 direction points in the fiber direction. The vielbein $e^0$ is given by

$$e^0 = d\psi + V,$$

and the others $e^m$ ($m = 1, 2, \ldots, 2r$) are the pull-back of the vielbein of the base $\mathbb{C}P^r$.  

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The exterior derivative of $V$ is
\[
dV = \mathbb{I}_{ab} dy^a_0 \wedge dy^b_0 = \mathbb{I}_{ab} dy^a \wedge dy^b.
\] (106)

At the second equality we use $|y| = 1$. Clearly $dV$ satisfies $(\partial_y, dV) = 0$, and can be regarded as the pull-back of a two-form in the base $\mathbb{C}P^r$. The unitary transformation $g(\theta, \psi)$ keeps the complex structure intact, and the components of $dV$ in (106) in the unitary frame are essentially the same as $\mathbb{I}_{ab}$ except that the $(-1)$ direction is absent in $S^{2r+1}$. Namely, $dV$ is given by
\[
dV = I_{mn} e^m \wedge e^n,
\] (107)
where $I_{mn}$ is an anti-symmetric $2r \times 2r$ matrix with non-vanishing components
\[
I_{12} = I_{34} = \cdots = I_{(2r-1)(2r)} = 1.
\] (108)

This is nothing but the complex structure in the base $\mathbb{C}P^r$.

From the torsionless condition, we obtain the spin connection
\[
\hat{\omega} = \frac{1}{2} \omega_{ij} \hat{T}_{ij} = -e^0 \hat{T} + e^m I_{mn} \hat{T}_{n0} + \hat{\omega}^{\mathbb{C}P^r},
\] (109)
where $\hat{\omega}^{\mathbb{C}P^r} \equiv (1/2) \omega_{mn} \hat{T}_{mn}$ is the spin connection of $\mathbb{C}P^r$. Let us read off the vielbein $e^i$ and the spin connection $\omega_{ij}$ according to (19) from the Maurer–Cartan form
\[
g^{-1} dg = g_0^{-1}dg_0 - d\psi \hat{\Pi}.
\] (110)

Because $g \in U(r+1)$, $SO(2r+2)$ generators $\hat{T}_{i(-1)}$, whose coefficients are identified with the vielbein $e^i$, always appear in the Maurer–Cartan form through the combinations
\[
\hat{K}_0 = \hat{T}_{0(-1)} + \frac{1}{r} \hat{T}, \quad \hat{K}_m = \hat{T}_{m(-1)} + I_{mn} \hat{T}_{n0}, \quad (m = 1, 2, \ldots, 2r).
\] (111)

Therefore, the components of $\omega_{ij}$ corresponding to $\hat{T}$ and $\hat{T}_{n0}$ are written in terms of $e^i$. We obtain
\[
\hat{\omega} = -e^0 \hat{T} + e^m I_{mn} \hat{T}_{n0} + \hat{\omega}^{SU(r)} + \frac{r+1}{r} V \hat{T},
\] (112)
where $\hat{\omega}^{SU(r)}$ is the $SU(r)$ part of the spin connection acting on the holomorphic vectors $f_{2k-1} + i f_{2k}$ ($k = 1, \ldots, r$). From (109) and (112), we obtain
\[
\hat{\omega}^{\mathbb{C}P^r} = \hat{\omega}^{SU(r)} + \frac{r+1}{r} V \hat{T}.
\] (113)

Under the subgroup $G'$ a $G$ representation $R$ is decomposed into $\bigoplus_i (R'_i, q_i)$ where $R'_i$ and $q_i$ are $SU(r+1)$ representations and $U(1)_3$ charges. Correspondingly, $S^{2r+1}$ harmonics are expanded into Kaluza–Klein modes by
\[
Y^R(\theta, \psi) = \sum_i e^{i q_i} Y^{(R'_i, q_i)}(\theta).
\] (114)

By applying the covariant derivative to this expansion, we obtain
\[
DY^R(\theta, \psi) = \sum_i e^{i q_i} \left( \nabla + e^m I_{mn} \hat{T}_{n0} + e^0 (iq_i - \hat{T}) \right) Y^{(R'_i, q_i)}(\theta).
\]
(115)

$\nabla$ is the covariant derivative on $\mathbb{C}P^r$ defined by
\[
\nabla = d - i Q_V V + \hat{\omega}^{\mathbb{C}P^r} = d - i Q_V V + \frac{r+1}{r} V \hat{T} + \hat{\omega}^{SU(r)},
\] (116)
where $Q_V = \frac{1}{i} \hat{\Pi}$.  

\[
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\]
5.2. Killing spinors

Let us construct Killing spinors in $S^5 \times \mathbb{R}$. We use $6, 5, 1, 2, 3, 4$ instead of the $-1, 0, 1, 2, 3, 4$ used in Sect. 5.1. Namely, 12345, and 6 label $\mathbb{CP}^2$, Hopf fibers, and Euclidean time, respectively. We use the 6d Dirac matrices

$$
\Gamma_7 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad \Gamma_6 = \begin{pmatrix} i \\ -i \end{pmatrix}, \quad \Gamma_i = \begin{pmatrix} \gamma_i \\ \gamma_i \end{pmatrix},
$$

where $\gamma^I$ are the five-dimensional Dirac matrices defined in (37). The charge conjugation matrix in 6d and 5d are

$$
C^{(6)} = \begin{pmatrix} C^{(5)} & \epsilon \end{pmatrix}, \quad C^{(5)} = \begin{pmatrix} \epsilon & -\epsilon \end{pmatrix}.
$$

The supersymmetry transformation parameter $\epsilon^I$ of $\mathcal{N} = (1, 0)$ supersymmetry in an arbitrary conformally flat background is a left-handed symplectic Majorana–Weyl spinor satisfying the Killing spinor equation

$$
D_a \epsilon^I = \Gamma_a \kappa^I, \quad a = 1, \ldots, 6.
$$

$k^I$ is a right-handed symplectic Majorana–Weyl spinor. The R-symmetry is $SU(2)_R$, and both $\epsilon^I$ and $\kappa^I$ have $SU(2)_R$ doublet index $I = 1, 2$.

To write down the Killing spinors in $S^5 \times \mathbb{R}$, we first define Killing spinors in $S^5$. There are two spinor representations of $G = SO(6)$. Let $V^{(L)}$ and $V^{(R)}$ be representation spaces for positive and negative chirality, respectively. We introduce basis vectors $E^{(L)}_{\mu I} (\mu = 1, 2, 3, 4)$ for $V^{(L)}$ and $\tilde{E}^{(R)}_{\mu I} (\mu = 1, 2, 3, 4)$ for $V^{(R)}$. Both these spinor representations are also irreducible as $H$-representations, and we can identify $V_S$ with $V^{(L)}$ and $V^{(R)}$. According to general prescription, we can define eight linearly independent Killing spinors on $S^5$:

$$
\xi^{(L)}_{\alpha I} = \left( g(y) \tilde{E}_{\alpha I}, E^{(L)}_{\mu I} \right), \quad \xi^{(R)}_{\alpha I} = \left( g(y) \tilde{E}_{\alpha I}, \tilde{E}^{(R)}_{\mu I} \right),
$$

where $\tilde{E}_{\alpha}$ are dual basis vectors for $V_S$. By using the general formula (33) and the explicit representation of the Dirac matrices, we obtain

$$
D^I \xi^{(L/R)} = -\frac{i}{2} \Gamma_{\text{iso}} \gamma^I \xi^{(L/R)}, \quad (i = 1, \ldots, 5),
$$

where $\Gamma_{\text{iso}}$ is the chirality operator of the isometry group $G = SO(6)$. Under the subgroup $G' = SU(3) \times U(1)_\psi$, each of $\xi^{(L)}_{\alpha I}$ and $\xi^{(R)}_{\alpha I}$ splits into a singlet and a triplet. This can be seen from the explicit form of $\Gamma$ for the spinor representations:

$$
\rho^{(L)} (\mathbb{I}) = \frac{1}{2} \text{diag} (i, -3i, i, i), \quad \rho^{(R)} (\mathbb{I}) = \frac{1}{2} \text{diag} (3i, -i, -i, -i).
$$

We can see that $\xi^{(L)}_{\alpha I}$ and $\xi^{(R)}_{\alpha I}$ belong respectively to the following $G'$ representations:

$$
\xi^{(L)}_{\alpha I} : (\mathbb{I}, \pm \frac{1}{2}) \oplus (1, -\frac{3}{2}), \quad \xi^{(R)}_{\alpha I} : (1, -\frac{1}{2}) \oplus (1, +\frac{3}{2}).
$$

For the index calculation we use the $SU(3)$ singlet Killing spinors

$$
\epsilon^1 \equiv \xi^{(R)}_{\alpha 1} = e^{+\frac{\psi}{2}} \delta_{\alpha 1}, \quad \epsilon^2 \equiv \xi^{(L)}_{\alpha 2} = e^{-\frac{\psi}{2}} \delta_{\alpha 2}.
$$

We can show that these $\epsilon^I$ are $\nabla$-constant,

$$
\nabla \epsilon^I = 0.
$$

This property drastically simplifies the following calculation.
We define the 6d Killing spinors $\epsilon^I$ in $S^5 \times \mathbb{R}$ by using $S^5$ Killing spinors $\epsilon^I$ by
\[
e^I = e^{-\frac{1}{2} \Gamma_{6\text{iso}}^6} \left( \epsilon^I \right), \quad k^I = -\frac{1}{2} \Gamma_{1\text{iso}}^6 \epsilon^I.
\] (126)

There are five conserved charges: the Hamiltonian $H = -\partial_6$, the angular momenta $J_1 = -i\hat{T}_{12}$, $J_2 = -i\hat{T}_{34}$, $J_3 = -i\hat{T}_{65}$, and the $SU(2)_R$ Cartan generator $R_3$. The supercharges $Q_I$ corresponding to $\epsilon^I$ carry charges $(H, J_1, J_2, J_3, R_3) = \pm \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1 \right)$. The following four commute with $Q \equiv Q_1 + Q_2$:
\[
Q^2 = H - J_1 - J_2 - J_3 - 2R_3, \quad J_1 + \frac{R_3}{2}, \quad J_2 + \frac{R_3}{2}, \quad J_3 + \frac{R_3}{2}.
\] (127)

We define the superconformal index
\[
I(z_1, z_2, z_3) = \text{tr} \left[ (-1)^F q^{H - J_1 - J_2 - J_3 - 2R_3} z_1^{J_1 + \frac{R_3}{2}} z_2^{J_2 + \frac{R_3}{2}} z_3^{J_3 + \frac{R_3}{2}} \right].
\] (128)

Because the supercharges $Q_I$ are $SU(3)$ singlets, we can treat $SU(3)$ as a “flavor” symmetry, and it is convenient to separate $SU(3)$ fugacities from $z_i$. We use variables $x$ and $y_i$ defined by $z_i = x y_i$ ($y_1 y_2 y_3 = 1$) instead of $z_i$. The index is rewritten in terms of these variables by
\[
I(z_1, z_2, z_3) = \text{tr} \left[ (-1)^F q^{H - Q_V - 2R_3} x^{Q_V} z_1^{J_1} z_2^{J_2} z_3^{J_3} \right],
\] (129)
where $Q_V = J_1 + J_2 + J_3 = \frac{1}{2} | \tilde{\tau} |$ and $y^{SU(3)} = y_1 y_2 y_3$.

### 5.3. $\mathbb{C}P^2$ harmonics

According to the general construction in Sect. 2, the scalar spherical harmonics in $S^5$ are characterized by rank $k$ symmetric traceless tensors in $\mathbb{R}^6$. Let $R_k$ denote the tensor representation. Under $G' = SU(3) \times U(1)$, $R_k$ is decomposed into $k + 1$ representations $R_k, q (q = k, k - 2, \ldots, -k)$ where $R_k, q$ is the symmetric traceless part of the product $(3, +1) \otimes (\overline{3}, -1)^{k-q}$. Let $E^{(k,q)}_\mu$ $(\mu = 1, \ldots, \dim R_k, q)$ be basis vectors of $V_{R_k, q}$. By definition $E^{(k,q)}_\mu$ satisfy $Q_V E^{(k,q)}_\mu = q E^{(k,q)}_\mu$. The corresponding scalar harmonics are given by
\[
y^{0(k,q)}_\mu(y) = \left( g(y) N, E^{(k,q)}_\mu \right).
\] (130)

The following relations hold:
\[
\partial_\mu y^{0(k,q)}_\mu = i q y^{0(k,q)}_\mu,
\]
\[
\nabla_m \nabla_n y^{0(k,q)}_\mu = -\lambda_{k,q} y^{0(k,q)}_\mu, \quad \lambda_{k,q} = k(k + 4) - q^2,
\]
\[
[\nabla_m, \nabla_n] y^{0(k,q)}_\mu = -2 i q I_{mn} y^{0(k,q)}_\mu.
\] (131)

We normalize $y^{0(k,q)}_\mu$ by
\[
\int_{S^5} |y^{0(k,q)}_\mu|^2 = 1.
\] (132)

For simplicity of expression we omit the $G'$ indices $\mu$ and $(k, q)$ in the following.
By using the scalar harmonics and the Killing spinors $\epsilon^{\lambda}$ we can define a complete set of spinor harmonics:

$$Y_{1}^{(1)} = \epsilon^{(R)} Y^{0}, \quad Y_{1}^{(2)} = \epsilon^{(L)} Y^{0}, \quad Y_{1}^{(3)} = \gamma^{m} \epsilon^{(R)} \nabla_{m} Y^{0}, \quad Y_{1}^{(4)} = \gamma^{m} \epsilon^{(L)} \nabla_{m} Y^{0}. \quad (133)$$

Spinor harmonics with different $G'$ quantum numbers are orthogonal. The inner products of two harmonics with the same $G'$ quantum numbers are

$$G_{AB}^{1} = \int_{S^{5}} \left( Y_{1}^{(A)} \right)^{\dagger} Y_{1}^{(B)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \lambda_{k,q} - 4q & 0 \\ 0 & 0 & 0 & \lambda_{k,q} + 4q \end{pmatrix}. \quad (134)$$

Among four harmonics $Y_{1}^{(1)}, \ldots, Y_{1}^{(4)}$, some may not exist when $k$ and $q$ take boundary values. For example, the norm of $Y_{1}^{(3)}$, $|Y_{1}^{(3)}|^{2} = k(k + 4) - q(q + 4)$, vanishes when $q = k$, and then $Y_{1}^{(3)}$ does not exist. Similarly, $Y_{1}^{(4)}$ does not exist when $q = -k$, and only $Y_{1}^{(1)}$ and $Y_{1}^{(2)}$ exist when $k = q = 0$.

A complete set of vector harmonics can be defined by

$$Y_{m}^{1(A)} = X_{mn}^{(A)} \nabla_{n} Y^{0}, \quad (A = 1, 2, 3, 4), \quad (135)$$

where $X_{mn}^{(A)}$ are the tensors with the components

$$X_{mn}^{(a)} = \epsilon_{amn} + (\delta_{am} \delta_{4n} - \delta_{an} \delta_{4m}), \quad X_{mn}^{(4)} = \delta_{mn}. \quad (136)$$

$X_{mn}^{(a)} (a = 1, 2, 3)$ can also be defined as the following bilinears of the Killing spinors:

$$X_{mn}^{(3)} = I_{mn} = -i \left( \gamma^{1} \gamma_{mn} \gamma^{1} \right) = i \left( \gamma^{2} \gamma_{mn} \gamma^{2} \right),$$

$$X_{mn}^{(1)} + i X_{mn}^{(2)} = -i \left( \gamma^{1} \gamma_{mn} \gamma^{1} \right),$$

$$X_{mn}^{(1)} - i X_{mn}^{(2)} = -i \left( \gamma^{2} \gamma_{mn} \gamma^{2} \right). \quad (137)$$

These are $\nabla$-constant. The inner products of two vector harmonics with the same $G'$ quantum numbers are

$$G_{AB}^{1} = \int_{S^{5}} \left( Y_{m}^{1(A)} \right)^{\ast} Y_{m}^{1(B)} = \begin{pmatrix} \lambda_{k,q} & 4iq \\ -4iq & \lambda_{k,q} \\ \lambda_{k,q} & -4iq \\ 4iq & \lambda_{k,q} \end{pmatrix}. \quad (138)$$

### 5.4. Mode analysis

A vector multiplet consists of a vector field $A_{a}$, a gaugino $\lambda_{I}$, and an $SU(3)$ triplet auxiliary field $D_{IJ}$. The transformation laws are

$$\delta A_{a} = - \left( \epsilon^{I} \Gamma_{a} \lambda_{I} \right),$$

$$\delta \lambda_{I} = - \xi_{I} \epsilon_{I} + i D_{IJ} \epsilon_{J},$$

$$\delta D_{IJ} = 2i \left( \epsilon_{IJ} \nabla \lambda_{1} \right) - 4i \left( \epsilon_{IJ} \lambda_{1} \right). \quad (139)$$
For index calculation we use the $Q$-exact action

$$
\mathcal{L} = Q \left[ (Q \lambda_I)^\dagger \lambda_I \right] = \frac{1}{2} \kappa^2 + \frac{1}{4} \epsilon^{mnpq} F_{mn} F_{pq} + F_{56}^2 + F_{5m}^2 + F_{6m}^2 - \frac{1}{2} D_{IJ} D^{IJ} \\
- \lambda_I \left( -\gamma^I D_I - i D_6 - \frac{1}{4} I_{mn} \gamma^{mn} - \frac{i}{2} \gamma^5 + \frac{i}{2} \tau_3 \right) \lambda_I,
$$

(140)

where $\tau_3 = \text{diag}(1, -1)$ is the matrix acting on $SU(2)_R$ doublets.

The auxiliary fields $D_{IJ}$ do not give any physical modes.

The gauge field terms in (140) with the gauge-fixing term $(D_I A^I)^2$ added can be rewritten as

$$
\mathcal{L}_V = \frac{1}{2} \sum_{a=1}^{3} \left( X^{(a)}_{mn} F_{mn} \right)^2 + F_{56}^2 + F_{5m}^2 + F_{6m}^2 + \left( D_I A^I \right)^2.
$$

(141)

For the saddle points all terms in (141) must vanish. For the first term to vanish the gauge field in $\mathbb{CP}^2$ must be anti-self-dual. This allows instanton configurations. However, it is difficult to calculate the instanton contribution to the index by means of harmonic expansion. Here we focus only on the perturbative sector with zero instanton number. Then the saddle point is given by the trivial gauge configuration $A_a = 0$ up to gauge transformations. Let us represent the fluctuation of the gauge potential as a column vector $\mathcal{A}$ and expand it by six basis vectors $\mathcal{Y}_1, \ldots, \mathcal{Y}_6$. Their explicit forms are

$$
\mathcal{A} = \begin{pmatrix} A_m \\ A_5 \\ A_6 \end{pmatrix}, \quad \mathcal{Y}_{1,2,3,4} = \begin{pmatrix} Y^{(1,2,3,4)}_m \\ 0 \\ 0 \end{pmatrix}, \quad \mathcal{Y}_5 = \begin{pmatrix} 0 \\ \gamma^0 \end{pmatrix}, \quad \mathcal{Y}_6 = \begin{pmatrix} 0 \\ \gamma^0 \end{pmatrix}.
$$

(142)

We can write the Lagrangian in the form $\mathcal{L} = \mathcal{A}^\dagger D \mathcal{A}$ with a certain differential operator $D$. It is straightforward to calculate the determinant of the differential operator $D$ in each subspace with specific $G'$ quantum numbers $(k, q)$ and $\mu$. For $|q| \leq k - 2$, all six basis vectors are linearly independent, and the determinant is

$$
\det D = \frac{\det \int \mathcal{Y}_A^\dagger D \mathcal{Y}_B}{\det \int \mathcal{Y}_A^\dagger \mathcal{Y}_B} = k^2(k + 4)^2 \left( k^2 - \partial_6^2 \right) \left( (k + 4)^2 - \partial_6^2 \right) \times \left( k^2 + 4k + 9 - 2q - \partial_6^2 \right) \left( k^2 + 4k + 9 + 2q - \partial_6^2 \right).
$$

(143)

The factor $k^2(k + 4)^2$ is canceled by the ghost factor, and the other factors correspond to physical modes. For $|q| = k$, the six vectors $\mathcal{Y}_{1,\ldots,6}$ are not linearly independent, and some factors in (143) are absent. Careful analysis gives the spectrum in Table 5.

The gaugino terms in (140) take the form $\lambda^I D_I \lambda_I$ with the differential operator

$$
D_\lambda = -\gamma^I D_I - i D_6 - \frac{1}{4} I_{mn} \gamma^{mn} - \frac{i}{2} \gamma^5 + \frac{i}{2} \tau_3.
$$

(144)
We define the matrix representation $M_{AB}$ of this operator by $D_\lambda Y_{\frac{1}{2}}^{(A)} = Y_{\frac{1}{2}}^{(B)} M_{BA}$. For $|q| \leq k - 2$ we obtain

$$
M_{AB} = \begin{pmatrix}
-i (q + \frac{7}{2} + \partial_6) & 0 & (\lambda_{k,q} - 4q) & 0 \\
0 & -i (q - \frac{7}{2} + \partial_6) & 0 & (\lambda_{k,q} + 4q) \\
-1 & 0 & i (q + \frac{3}{2} - \partial_6 + \tau_3) & 0 \\
0 & -1 & 0 & i (q - \frac{3}{2} - \partial_6 + \tau_3)
\end{pmatrix},
$$

and the determinant is

$$
\det M_{ij} = \left[ \partial_6 - \tau_3 (k + \frac{7}{2}) \right] \left[ \partial_6 + \tau_3 (k + \frac{1}{2}) \right] \left[ (\partial_6 + \frac{1}{2} \tau_3)^2 - \left( k^2 + 4k + 9 + 2 \tau_3 q \right) \right].
$$

Again, we have to analyze the $|q| = k$ case separately. The physical spectrum is shown in Table 6.

Comparing Table 5 and Table 6, all quantum numbers match except the ranges of $q$. Let us focus on the positive frequency modes because $A_a$ are real and $\lambda_f$ are symplectic Majorana. The only difference in the bosonic and the fermionic spectra is that there are $q = k$ modes in $[\lambda_2]$ and $[\lambda_3]$ but not in $[A_2]$ and $[A_3]$. The modes $[\lambda_2]$ and $[\lambda_3]$ with $q = k$ contribute to the index by

$$
I_{sp}(x, e^{iat}) = \text{tr} \left[ (-1)^F q^{H - Q_V - 2R_3} x^{Q_V + \frac{3}{2} R_3} y^{SU(3)} e^{ia}\tau_3 \right]
$$

$$
= - \sum_a \sum_{k=1}^{\infty} e^{i\alpha} x^k \chi_{(k,k)}(y) - \sum_a \sum_{k=0}^{\infty} e^{i\alpha} x^{k+3} \chi_{(k,k)}(y),
$$

(147)
where $\chi_{(k,q)}(y)$ is the $SU(3)$ character of the representation $R_{k,q}$. By combining this with (12) we obtain

$$I_{sp}^{(tot)}(z_i, e^{i\varphi}) = \sum_{\alpha} e^{i\alpha(a)} \left( \delta_{\alpha,0} - \frac{1 + z_1 z_2 z_3}{(1 - z_1)(1 - z_2)(1 - z_3)} \right),$$

where we used the formula

$$\sum_{k=0}^{\infty} x^k \chi_{(k,k)}(y) = \frac{1}{(1 - z_1)(1 - z_2)(1 - z_3)}$$

for the $SU(3)$ characters.

Refer to [19] for index calculation for hypermultiplets by means of harmonic expansion.

6. Concluding remarks

We reviewed how we calculate 3d, 4d, and 6d superconformal indices by using supersymmetric localization and harmonic expansion. We deformed the theories by introducing $Q$-exact terms, and derived the indices by means of mode analysis in $S^p \times \mathbb{R}$ backgrounds.

The method of harmonic expansion works effectively when the deformed theory has $G = SO(p+1)$ rotational symmetry. This is the case in 3d and 4d. Although it is not the case in 6d and the deformed Lagrangian respects only the subgroup $G' = U(3) \subset SO(6)$, $G'$ is still transitive and harmonic expansion is useful.

In the case of 5d, however, this method does not work. Let $Q$ be a supercharge that we use for localization. In general, $Q^2$ is a linear combination of a $G$ generator $T$ and generators of internal symmetries. Unlike the $S^5$ case we discussed in Sect. 5, in which $T$ generates shifts along Hopf fibers, $S^4$ rotations generated by $T$ always have (at least) two fixed points. These fixed points are often called north and south poles, and the existence of such special points clearly shows that the symmetry of the deformed theory is not transitive on $S^4$. For this reason it is not practical to use harmonic expansion for the index calculation. This is also the case for the $S^4$ partition function of 4d theories.

The 5d superconformal index and the $S^4$ partition function have been calculated by using a more sophisticated technique, called equivariant localization. The $S^4$ partition function is calculated in the seminal work by Pestun [26], and the result is extended to squashed $S^4$ in [27]. The 5d superconformal index is derived in [28,29]. All these works employ the method of equivariant localization.

In the method using equivariant localization, the existence of fixed points is not a problem, but is exactly what is required. The partition function and the index are given as the product of the contribution of each fixed point. This also makes it possible to include the instanton contribution as the contribution from fixed points, each of which is given by the Nekrasov partition function [30,31]. Unfortunately, we have no space to discuss these issues, and the interested reader is referred to the original works cited above.

Acknowledgements

This work was partially supported by Grand-in-Aid for Scientific Research (C) (No. 15K05044), Ministry of Education, Science and Culture, Japan.

Funding

Open Access funding: SCOAP$^3$. 
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