Supersymmetric gauge theory with space-time-dependent couplings

Jaewang Choi\textsuperscript{1,∗}, José J. Fernández-Melgarejo\textsuperscript{1,2,∗}, and Shigeki Sugimoto\textsuperscript{1,3,∗}

\textsuperscript{1}Center for Gravitational Physics, Yukawa Institute for Theoretical Physics, Kyoto University, Kyoto 606-8502, Japan
\textsuperscript{2}Departamento de Física, Universidad de Murcia, Campus de Espinardo, 30100 Murcia, Spain
\textsuperscript{3}Kavli Institute for the Physics and Mathematics of the Universe (WPI), The University of Tokyo, Kashiwanoha, Kashiwa 277-8583, Japan

∗E-mail: jchoi@yukawa.kyoto-u.ac.jp; josejuan@yukawa.kyoto-u.ac.jp; sugimoto@yukawa.kyoto-u.ac.jp

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We study deformations of $\mathcal{N}=4$ supersymmetric Yang–Mills theory with couplings and masses depending on space-time. The conditions to preserve part of the supersymmetry are derived and a lot of solutions of these conditions are found. The main example is the case with $\text{ISO}(1,1) \times \text{SO}(3) \times \text{SO}(3)$ symmetry, in which couplings, as well as masses and the theta parameter, can depend on two spatial coordinates. In the case in which $\text{ISO}(1,1)$ is enhanced to $\text{ISO}(1,2)$, it reproduces the supersymmetric Janus configuration found by Gaiotto and Witten [J. High Energy Phys. \textbf{06}, 097 (2010)]. When $\text{SO}(3) \times \text{SO}(3)$ is enhanced to $\text{SO}(6)$, it agrees with the world-volume theory of D3-branes embedded in F-theory (a background with 7-branes in type IIB string theory). We have also found the general solution of the supersymmetry conditions for the cases with $\text{ISO}(1,1) \times \text{SO}(2) \times \text{SO}(4)$ symmetry. Cases with time-dependent couplings and/or masses are also considered.

Subject Index B10, B16

1. Introduction

Quantum field theories (QFT) usually contain constant parameters such as gauge couplings, Yukawa couplings, theta parameters, mass parameters, etc. We are also familiar with QFT with some parameters depending on space-time. QFT in curved space-time and QFT with background fields or space-time-dependent sources of some operators are such examples. QFT with varying couplings\textsuperscript{1} may be less familiar, probably because there is no clear observational evidence suggesting that such parameters in the Standard Model are not constant.

However, in string theory, they are obtained as the values of background fields and, in this sense, there is no conceptual difference between putting QFT in a curved space-time and making the couplings space-time dependent. In fact, there are a lot of examples. The Newton constant and gauge couplings are related to the value of the dilaton field $\phi$. In type IIB string theory, it is combined with the Ramond–Ramond (RR) 0-form field $C_0$, which corresponds to an analog of the theta parameter, to have a complex coupling $\tau = ie^{-\phi} + C_0$. One way to make it vary in space is to include D7-branes, or more generally $[p,q]$ 7-branes. Regarding $\tau$ as the modulus of a torus, we can think of uplifting the 10D space-time with 7-branes to an elliptic fibered 12D space-time, which is called F-theory [1].

\textsuperscript{1} Here, we regard masses, as well as other parameters in the action, as couplings.
Another interesting example of varying coupling is the so-called Janus configuration, in which the coupling depends on one of the spatial coordinates [2,3].

In this paper, we consider deformations of $\mathcal{N} = 4$ supersymmetric Yang–Mills (SYM) theory with varying couplings as typical examples of QFT with space-time-dependent parameters. In particular, we investigate the conditions to preserve part of the supersymmetry (SUSY). This is a natural extension of the work to find curved space-times preserving SUSY, for which systematic methods using, e.g., topological twist [4] or supergravity [5] have been developed. We are, of course, not the first ones to consider this class of theories. Supersymmetric Janus configurations were studied in Refs. [6–13]. Their generalization to the configurations with couplings depending on more than one direction was also investigated in Refs. [14,15]. SUSY configurations in $\mathcal{N} = 4$ SYM with varying couplings have been studied, for instance, in Refs. [16–22] (see also Refs. [23–25] for earlier works). Time-dependent couplings have also been considered, for instance, in Refs. [26–32].

Our approach here is perhaps the most rudimentary one. We write down all the possible deformations in the action of $\mathcal{N} = 4$ SYM and the SUSY variation, and find the conditions to preserve part of the SUSY by performing the SUSY variation. Although the calculation is a bit tedious, it is straightforward and easy to understand the details. We hope it will provide a useful guide for further analyses and generalizations to other SUSY QFT.

Many of the examples considered in this paper can be realized as the world-volume gauge theory on probe D3-branes embedded in some non-trivial backgrounds in type IIB string theory. This system is related by duality to M5-branes wrapped on a torus with varying modulus in M-theory. This gives a 6D description of the 4D QFT with its complex coupling identified with the modulus of the torus that corresponds to the extra two dimensions, which is analogous to the idea of F-theory mentioned above. The SUSY condition for the deformed $\mathcal{N} = 4$ SYM should be related to the conditions to preserve SUSY for the D3-branes in the supergravity background. In this way, our field-theoretical analysis contains some information on the supergravity background. As we will mention in Sect. 3.3, it is possible to extract part of the equations of motion, including the Einstein equation, of supergravity for the case of an F-theory configuration.

The organization of the paper is as follows. In Sect. 2, after fixing our notations and ansatz for the action and SUSY variation, the conditions to preserve SUSY are derived. The details for the calculation are summarized in Appendix D. It turns out that one of the SUSY conditions only has a trivial solution if we impose certain symmetry. Appendix E provides an explanation for this fact. In Sects. 3 and 4, we demonstrate how our formalism works by examining some explicit examples. We will analyze the case with $ISO(1,1) \times SO(3) \times SO(3)$ symmetry in detail in Sect. 3. We show in Sect. 3.2.1 that the SUSY conditions reduce to two simple equations (3.54) and (3.62). As shown in Sects. 3.2.2 and 3.2.3, large classes of solutions of these equations are found. An F-theory configuration corresponding to the case with $ISO(1,1) \times SO(6)$ symmetry and the supersymmetric Janus configuration found in Ref. [12] are obtained as special solutions. The case with $ISO(1,1) \times SO(2) \times SO(4)$ symmetry is studied in Sect. 4.1, in which the general solution for this case is found. Some examples with time dependence are discussed in Sect. 4.2. In Sect. 5, we summarize the paper and discuss some future directions.

2. SUSY conditions for $\mathcal{N} = 4$ SYM with varying couplings

2.1. Notations and ansatz

In this paper, we consider $\mathcal{N} = 4$ $SU(N)$ supersymmetric Yang–Mills (SYM) theory in a curved background with space-time-dependent gauge coupling $g_{YM}(x^\mu)$ and theta parameter $\theta(x^\mu)$. The
leading-order action is

\[ S_0 = \int d^4x \sqrt{-g} \quad \text{tr} \left\{ \frac{1}{g_{\text{YM}}} \left( -\frac{1}{2} g^{\mu \nu} g^{\rho \sigma} F_{\mu \nu} F_{\rho \sigma} + i \bar{\Psi} \Gamma^I D_I \Psi \right) + \frac{\theta}{32 \pi^2} \epsilon^{\mu \nu \rho \sigma} F_{\mu \nu} F_{\rho \sigma} \right\}, \tag{2.1} \]

where \( I, J = 0, 1, \ldots, 9 \) and

\[
F_{\mu \nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu + i [A_\mu, A_\nu],
\]

\[
F_{\mu A} = -F_{A \mu} \equiv \partial_\mu A_A + i [A_\mu, A_A] \equiv D_\mu A_A,
\]

\[
F_{AB} \equiv i [A_A, A_B],
\]

\[
D_\mu \Psi \equiv \partial_\mu \Psi + i [A_\mu, \Psi] + \frac{1}{4} \omega_\mu ^{\hat{\nu} \hat{\rho}} \Gamma_{\hat{\nu} \hat{\rho}} \Psi, \quad D_A \Psi \equiv i [A_A, \Psi],
\]

with \( \mu, \nu = 0, 1, 2, 3 \) and \( A, B = 4, \ldots, 9 \). Here we are using the 10D notation,\(^2\) in which the gauge field \( A_\mu \) and 6 adjoint scalar fields \( A_A \) are combined into a 10D gauge field \( A_I \), though it depends only on the 4D space-time. The 10D metric \( g_{IJ} \) is assumed to be of the form

\[
ds^2 = g_{IJ} dx^I dx^J = g_{\mu \nu} (x^\rho) dx^\mu dx^\nu + \delta_{AB} dx^A dx^B,
\]

and \( g^{IJ} \) is its inverse.\(^3\) The indices are lowered or raised by this metric and its inverse. \( \epsilon^{\mu \nu \rho \sigma} \) is the 4D epsilon tensor with \( \epsilon^{0123} = 1/\sqrt{-g} \), where \( \sqrt{-g} \equiv \sqrt{-\det(g_{\mu \nu})} = \sqrt{-\det(g_{\mu \nu})} \).

The fermion field \( \Psi \) is written as a 10D negative chirality Majorana–Weyl spinor, which is equivalent to 4 Weyl spinor fields in 4D space-time. It is a real 32-component spinor satisfying

\[
\Gamma^{(10)} \Psi = -\Psi,
\]

where \( \Gamma^{(10)} = \Gamma^{\hat{0}} \Gamma^1 \ldots \Gamma^{\hat{9}} \) is the 10D chirality operator.\(^4\) Here, the gamma matrices \( \Gamma^I (\hat{I} = 0, 1, \ldots, 9) \) are 10D gamma matrices that are realized as 32 \( \times \) 32 real matrices satisfying \( \{ \Gamma^{\hat{I}}, \Gamma^{\hat{J}} \} = 2 \eta^{\hat{I} \hat{J}} \), where \( \eta^{\hat{I} \hat{J}} \equiv \text{diag}(-1, +1, \ldots, +1) \) is the Minkowski metric (see Appendix B for our notations and useful formulas for the gamma matrices). The hatted indices \( (\hat{I}, \hat{J}, \ldots) \) are those for the local Lorentz frame that can be converted to curved indices \( (I, J, \ldots) \) by contracting them with vielbeins \( e^I_J \) as \( \Gamma^I = e^I_J \Gamma^\hat{J} \). The Dirac conjugate \( \bar{\Psi} \) is defined by \( \bar{\Psi} \equiv \Psi^T \Gamma^{\hat{0}} \). Gamma matrices with more than one index are anti-symmetrized products of the gamma matrices defined as

\[
\Gamma^{I_1 I_2 \ldots I_n} = \Gamma^{I_1} \Gamma^{I_2} \ldots \Gamma^{I_n} = \frac{1}{n!} \sum_{\sigma \in S_n} \text{sgn}(\sigma) \Gamma^{I_{\sigma(1)}} \Gamma^{I_{\sigma(2)}} \ldots \Gamma^{I_{\sigma(n)}}.
\]

\( \omega_\mu ^{\hat{\nu} \hat{\rho}} \) in the covariant derivative \( D_\mu \) in Eq. \((2.3)\) is the spin connection:

\[
\omega_\mu ^{\hat{\nu} \hat{\rho}} = \frac{1}{2} \epsilon_{\nu}^{\mu \rho} \left( \partial_\mu e_{\nu}^{\rho} - (\partial_\nu g_{\mu \rho}) e_{\rho}^{\mu} \right) - (\hat{\nu} \leftrightarrow \hat{\rho}). \tag{2.7} \]

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\(^2\) Our notation is similar to (but not exactly the same as) that used in Ref. [12].

\(^3\) One may introduce a non-trivial internal part of the metric \( g_{ab}(x^\mu) \). However, for our purpose, we can set it to be \( g_{ab}(x^\mu) = \delta_{ab} \) without loss of generality. This is because \( g_{ab} \) in the action \((2.8)\) can always be replaced with \( \delta_{ab} \) by redefinition of the scalar fields and the couplings \( a, d^{B A}, m_{AB} \), and \( M \). The off-diagonal components \( g_{\mu A} \) are set to be zero for simplicity.

\(^4\) This should not be confused with \( \Gamma^{10} = \Gamma^1 \Gamma^0 \).
In our notation, $D_\mu$ denotes the covariant derivative including gauge field $A_\mu$, spin connection $\omega^\hat{\nu}_\mu$, and Levi–Civita connection $\Gamma^\rho_\mu_\nu$, depending on the field on which it acts.

When the metric is flat and the couplings ($g_{\text{YM}}$ and $\theta$) are constant, we know that the action (2.1) is invariant under the supersymmetry (SUSY) transformation with 16 independent SUSY parameters.\(^5\) If the metric and/or the couplings are not constant, SUSY is in general completely broken. In order to maintain part of SUSY, we have to add additional terms to the action (2.1) and the SUSY transformation has to be modified accordingly. The action that we consider is

$$S = \int d^4x \sqrt{-g} \text{tr} \left\{ -\frac{1}{2} g^{\mu\nu} g^{\lambda\rho} F_{\mu\lambda} F_{\nu\rho} + i \bar{\Psi} \Gamma^\mu D_\mu \Psi + \frac{c}{4} \epsilon_{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} ight. - d^{IJA} F_{IJA} - \frac{m^{AB}}{2} A_{AAB} - i \bar{\Psi} M \Psi \} \right\}, \quad (2.8)$$

where $a$, $c$, $d^{IJA}$, $m^{AB}$ are real parameters and $M$ is a real $32 \times 32$ matrix that may depend on the space-time coordinates $x^\mu$. $a$ and $c$ are related to $g_{\text{YM}}$ and $\theta$ in Eq. (2.1) by

$$a = \frac{1}{g_{\text{YM}}}^2, \quad c = \frac{g_{\text{YM}}^2 \theta}{8\pi^2}. \quad (2.9)$$

We also use the complex coupling defined by

$$\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g_{\text{YM}}^2} = 4\pi a(c + i). \quad (2.10)$$

We impose the following conditions for the parameters $d^{IJA}$ and $m^{AB}$, which can be done without loss of generality.\(^6\)

$$d^{IJA} = -d^{JIA}, \quad d^{\mu AB} = -d^{\mu BA}, \quad d^{ABC} = d^{[ABC]}, \quad m^{AB} = m^{BA}. \quad (2.11)$$

The second condition in Eq. (2.11) can be imposed because a term with $d^{\mu(AB)} \text{tr}(F_{\mu AA})$ can be converted to a mass term for $A_A$ via integration by parts. We can further assume that the matrix $(m^{AB})$ is diagonal, which can be realized, at least locally, by using local $SO(6)_R$ transformation.

Note that $\bar{\Psi} M \Psi$ is non-vanishing only when $\Gamma^0 M$ is an anti-symmetric matrix that commutes with $\Gamma^{(10)}$. The most general form of such a matrix is

$$M = m_{LJK} \Gamma^{LJK}, \quad (2.12)$$

where $m^{LJK}$ is a real rank-three totally anti-symmetric tensor.

\(^5\) In this paper we will not consider the special conformal supersymmetry.

\(^6\) Square and round brackets on indices indicate anti-symmetrization and symmetrization of the indices, respectively. For example,

$$d^{[ABC]} = \frac{1}{3!} (d^{ABC} + d^{BCA} + d^{CAB} - d^{BAC} - d^{CBA} - d^{BCA}),$$

$$d^{(ABC)} = \frac{1}{3!} (d^{ABC} + d^{BCA} + d^{CAB} + d^{BAC} + d^{CBA} + d^{BCA}).$$
The action (2.8) is constructed by adding operators of dimension\(^7\) less than 4 into the leading-order action (2.1). Although it is not the most general one,\(^8\) it is general enough to cancel the SUSY variations of the leading-order action, as we will see in Sect. 2.2 and Appendix D.

The ansatz for the SUSY transformation is

\[
\delta \epsilon A_I = i \bar{\epsilon} \Gamma_I \Psi, \\
\delta \epsilon \Psi = \frac{1}{2} (F_{IJ} \Gamma^{IJ} + A_A B^A) \epsilon, \\
\delta \epsilon \bar{\Psi} = \frac{1}{2} \bar{\epsilon} (-F_{IJ} \Gamma^{IJ} + A_A \bar{B}^A),
\]

(2.13)

where \(B^A\) are real \(32 \times 32\) matrices acting on the spinor indices that commute with the chirality operator \(\Gamma^{(10)}\), and \(\bar{B}^A \equiv -\Gamma^0 (B^A) \Gamma^0\). The SUSY parameter \(\epsilon\) is a 10D negative chirality Majorana–Weyl spinor. \(BA\) and \(\epsilon\) may also depend on the space-time coordinates.

2.2. SUSY conditions

The goal of this section is to determine the conditions under which the action (2.8) is invariant with respect to the SUSY transformations (2.14) for a non-zero \(\epsilon\). The approach that we follow in this work is straight. We firstly calculate the variation of the deformed action (2.8) with respect to the deformed transformations (2.14). Then, by imposing the vanishing of the variation, we obtain several constraints on the deformation parameters and the SUSY parameter \(\epsilon\).\(^9\) Here, we provide an outline of the derivation and leave the details to Appendix D.

Applying the SUSY variation (2.14) to the action (2.8), we get

\[
\delta \epsilon S = \int d^4 x \sqrt{-g} a \text{ tr} \left\{ \left( i \bar{\epsilon} \Gamma_I \Psi \right) \left[ -2 g^{II'} g^{JJ'} D_{I} F_{J} \epsilon' + \left( -2 a^{-1} D_\mu (a d^{I\mu A}) - m^{I A} \right) A_A \\
+ \left( -2 g^{IJ} g^{\mu K} a^{-1} \partial_\mu a - a^{-1} \partial_\nu (ac) \epsilon^{v IJK} - 3 d^{[IJK]} \right) F_{JK} \right] \\
+ i \bar{\epsilon} \left( -F_{JK} \Gamma^{JK} + A_A \bar{B}^A \right) \left( \Gamma^I D_I \Psi - \tilde{M} \Psi \right) \right\} + (\text{total derivative terms}),
\]

(2.14)

where \(\tilde{M} \equiv M - \frac{1}{2} \Gamma^{I} \partial_\mu \log a\). In this expression, \(d^{IJK}\) and \(m^{I A}\) can be non-zero only if \(I = 4–9\), and \(\epsilon^{IJKL}\) can be non-zero only for \(I, J, K, L = 0–3\).

It can be shown after some calculation that Eq. (2.14) vanishes (up to surface terms) if and only if the parameters satisfy the following two conditions (see Appendix D.1 for more details):

\[
D_\mu \bar{\epsilon} \Gamma^{IJ} \Gamma^{\mu} = \bar{\epsilon} \left( \bar{B}^{IJ} \Gamma^{IJ} \right) - \Gamma^{IJ} a^{-1} \partial_\mu a + (a^{-1} \partial_\nu (ac) \epsilon^{v IJK} + 3 d^{[IJK]}) \Gamma_K - \Gamma^{IJ} \tilde{M},
\]

\[
D_\mu (\bar{\epsilon} \bar{B}^A) \Gamma^{\mu} = \bar{\epsilon} \left( -2 a^{-1} \Gamma_I D_\mu (a d^{I\mu A}) - m^{A B} \Gamma_B - \bar{B}^A \left( M + \frac{1}{2} \Gamma^{\mu} \partial_\mu \log a \right) \right).
\]

(2.15)  (2.16)

Conditions (2.15) and (2.16) correspond to cancellation of terms with dimension \(\frac{7}{2}\) and \(\frac{5}{2}\) operators, respectively. Following Ref. [12], we call them first-order and second-order equations, respectively.

\(^7\) Our analysis in this paper is classical and the anomalous dimensions are not taken into account.

\(^8\) For example, an operator like \(\text{tr}(A_I A_B A_C)\) is not included.

\(^9\) A similar analysis with a more elegant approach using supergravity à la Festuccia–Seiberg [5] has been given in Ref. [19]. However, a detailed comparison of the results still remains.
Further algebra shows that the first-order equation (2.15) is equivalent to the following conditions
(see Appendix D.2 for the derivation):

\[ 0 = \varepsilon e^{j'jK'K'} \Gamma_{K'} \left( P_{I'j'}^{Ij} - \delta_{I'}^{I} \delta_{j'}^{j} \right), \]

\[ 0 = \varepsilon \left( \frac{1}{72} e^{IJK} \Gamma_{IJK} - \frac{1}{2} \Gamma^{\mu} \partial_{\mu} a - \left( \frac{1}{16} e_{\mu KJ} - 3 m_{\mu KJ} \right) \Gamma^{\mu KJ} - M \right), \]

\[ 0 = \varepsilon \left( F \Gamma^{A} + \left( - \frac{1}{4} e^{AJK} + 12 m^{AJK} \right) \Gamma_{JK} \right), \]

\[ \partial_{\mu} \varepsilon = \varepsilon A_{\mu}, \]

where \( F \) is a real \( 32 \times 32 \) matrix acting on the spinor indices (see Eqs. (D.39) or (D.41)), and

\[ \bar{P}_{I'j'}^{Ij} \text{ is a projection, as it satisfies } \]

\[ P_{I'j'}^{Ij} P_{I'j'}^{KL} = P_{Ij}^{KL}. \]

In addition, for an arbitrary \( G^{I} \), we have

\[ G^{[I'} \Gamma^{J']} P_{I'j'}^{Ij} = G^{[I} \Gamma^{I]}, \quad P_{I'j'}^{Ij} G_{I}^{[j} \Gamma_{J]} = G_{[I}^{I'} \Gamma_{J']}. \]

Note that \( \bar{B}^{A}, F, \) and \( A_{\mu} \) always appear in the combination \( \varepsilon \bar{B}^{A}, \varepsilon F, \) and \( \varepsilon A_{\mu}, \) respectively, and hence we only need to determine them up to additions of matrices that vanish when \( \varepsilon \) is multiplied from the left. In particular, the SUSY transformation (2.14) is determined once the right-hand side of Eq. (2.19) is fixed.

Equation (2.20) can be solved when the integrability condition

\[ \varepsilon F_{\mu \nu} = 0 \]

with \( F_{\mu \nu} \equiv \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu} + [A_{\mu}, A_{\nu}] \) is satisfied. Then, Eq. (2.20) can be formally solved as

\[ \varepsilon(x) = \varepsilon_{0} \text{ P exp} \left( \int_{x_{0}}^{x} dx^{\mu} A_{\mu} \right), \]

where \( \varepsilon_{0} \) is a constant spinor, “\( \text{P exp} \)” denotes the path-ordered exponential (the ordering is taken from left to right), and \( x_{0} \) is a fixed position. The integrability condition (2.26) guarantees that Eq. (2.27) is well defined in a neighborhood of \( x_{0} \).

Equation (2.17) has a trivial solution \( e^{IJK} = 0 \). In fact, in all the examples that we consider in the following sections, one can show that \( e^{IJK} = 0 \) is the only solution of Eq. (2.17) that is compatible with the imposed symmetry. When \( e^{IJK} = 0 \), Eq. (2.18) is simplified as

\[ 0 = \varepsilon \left( \frac{1}{2} \Gamma^{\mu} \partial_{\mu} a - 3 m_{\mu KJ} \Gamma^{\mu KJ} + M \right). \]
By definition, $\varepsilon^{IJK} = 0$ is equivalent to
\[
0 = a^{-1} \partial_\mu(ac) \varepsilon^{\mu\nu\rho\sigma} + 24 m^{\nu\rho\sigma},
\tag{2.29}
\]
and
\[
0 = d[IJA] + 8 m^{IJA}. \tag{2.30}
\]
Using Eq. (2.11), the latter is written as
\[
d^{\mu
u} = -24 m^{\mu
u}, \quad d^{\mu AB} = -12 m^{\mu AB}, \quad d^{ABC} = -8 m^{ABC}. \tag{2.31}
\]
Equation (2.29) can have a solution if and only if
\[
\partial_\nu(\sqrt{-g} a m^{\nu\rho\sigma}) = 0 \tag{2.32}
\]
is satisfied.

In summary, we have reduced the conditions for the SUSY invariance of the action (2.8) to Eqs. (2.15) and (2.16), where the former can be split into Eqs. (2.17)–(2.20). In the following section, we are going to elaborate on solutions that preserve different symmetries.

3. Example: ISO(1, 1) × SO(3) × SO(3)

In order to demonstrate how to solve the SUSY conditions obtained in the previous section, we consider the cases with ISO(1, 1) × SO(3) × SO(3) symmetry. Here, ISO(1, 1) is the Poincaré group acting on $x^{0,1}$, and the first and second $SO(3)$ act as rotation of $x^{4,5,6}$ and $x^{7,8,9}$ in the 10D notation, respectively. Although our analysis is purely field theoretical, our motivation is in string theory.

Consider a D3–D5–D7 system in the following table:

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</table>

(3.1)

This configuration preserves part of the supersymmetry as well as the ISO(1, 1) × SO(3) × SO(3) symmetry. If we regard the D5- and D7-branes as a supergravity background and the D3-brane as a probe embedded in it, the low-energy effective theory on the D3-brane is expected to be a deformation of the $\mathcal{N} = 4$ SYM that we have discussed. We may replace the D5-brane and D7-branes with $(p, q)$ 5-brane and $[p, q]$ 7-branes, respectively, to have more complicated configurations preserving SUSY.

The case with D3-branes in the $(p, q)$ 5-brane backgrounds corresponds to the supersymmetric Janus configuration considered in Ref. [12]. D3-branes in the $[p, q]$ 7-brane backgrounds can be generalized to D3-branes in F-theory configurations, which were recently analyzed in Refs. [18,20,21] (see also Refs. [23,25]). These configurations appear as special cases in our example, as we discuss separately in Sects. 3.4 and 3.3, respectively.

3.1. Ansatz and SUSY conditions

We decompose the coordinates in four sectors: $\alpha, \beta = 0, 1; i, j = 2, 3; a, b, c = 4, 5, 6; p, q, r = 7, 8, 9$. The metric and the couplings are assumed to depend only on $x^i$ and preserve the ISO(1, 1) × SO(3) × SO(3) symmetry acting on $\{x^a\} \times \{x^d\} \times \{x^p\}$. The metric (2.4) is of the form
\[
ds^2 = e(x^i) \eta_{\alpha\beta} dx^\alpha dx^\beta + g_{ij}(x^i) dx^i dx^j + \delta_{ab} dx^a dx^b + \delta_{pq} dx^p dx^q. \tag{3.2}
\]
Using the general coordinate transformation and Weyl transformation\(^{10}\) with appropriate rescaling of the fields, we can assume \(e(x^i) = 1\) and \(g_{ij}(x^i) = e^{\Phi(x^i)} \delta_{ij}\) without loss of generality.

The form of \(M\) consistent with the symmetry is
\[
M = 6 \left( m_{01j} \Gamma^{01j} + m_{456} \Gamma^{456} + m_{789} \Gamma^{789} \right) \\
≡ \alpha_i \Gamma^i \Gamma^{01} + \beta \Gamma^{456} + \gamma \Gamma^{789}. \tag{3.3}
\]

The non-zero components of \(d^{IA}\) and \(m^{AB}\) are
\[
d^{abc} = \frac{v}{3} e^{abc}, \quad d^{pqr} = \frac{w}{3} e^{pqr}, \quad m^{ab} = r \delta^{ab}, \quad m^{pq} = \tilde{r} \delta^{pq}. \tag{3.4}
\]

We also define
\[
q_i \equiv \partial_i \log a. \tag{3.5}
\]

The non-trivial components of \(e^{IJK}\) defined in Eq. (2.22) are
\[
e^{01i} = a^{-1} \partial_j (ac) e^{ji} - 4\alpha^i, \quad e^{abc} = (v + 4\beta) e^{abc}, \quad e^{pqr} = (w + 4\gamma) e^{pqr}. \tag{3.6}
\]

In this case, one can show that the condition (2.17) implies \(e^{IJK} = 0\) (see Appendix E.1) and hence
\[
a^{-1} \partial_j (ac) e^{ji} = 4\alpha^i, \quad v = -4\beta, \quad w = -4\gamma. \tag{3.7}
\]

The integrability condition (2.32) for the first equation of Eq. (3.7) is
\[
g^{kj} \partial_k (a\alpha_j) = 0. \tag{3.8}
\]

The non-zero components of the spin connection are
\[
\omega_i^{23} = -\omega_i^{32} = \frac{1}{2} \epsilon_{ij} \partial_j \Phi, \tag{3.9}
\]
and \(A_{\mu}\) defined in Eq. (2.23) is
\[
A_0 = -\frac{1}{4} \left( F \Gamma_0 + 4\alpha_i \Gamma^{0i} \right), \quad A_1 = -\frac{1}{4} \left( F \Gamma_1 - 4\alpha_i \Gamma^{0i} \right), \\
A_i = -\frac{1}{4} \left( F \Gamma_i + 4\alpha_i \Gamma^{0i} - \epsilon_i^j \partial_j \Phi \Gamma^{23} \right). \tag{3.10}
\]

The condition (2.20) with \(\mu = 0, 1\) implies
\[
0 = \bar{\epsilon} (F + 4\alpha_i \Gamma^{0i}). \tag{3.11}
\]

Using this equation, \(A_i\) in Eq. (2.20) with \(\mu = i = 2, 3\) can be replaced with
\[
A_i = \epsilon_i^j \left( -\alpha_j \Gamma^{01} + \frac{1}{4} \partial_j \Phi \right) \Gamma^{23}, \tag{3.12}
\]
and Eqs. (2.28), (2.19), and (2.20) become
\[
0 = \bar{\epsilon} \left( \Gamma^i q_i - 4\alpha_i \Gamma^{01i} + 2\beta \Gamma^{456} + 2\gamma \Gamma^{789} \right), \tag{3.13}
\]

\(^{10}\) See Appendix C.1.
\[ \tau B^a = \tau \left( -4\alpha_i\Gamma^{01a} + 2\beta\epsilon^{abc}\Gamma_{bc} \right), \]
(3.14)

\[ \tau B^p = \tau \left( -4\alpha_i\Gamma^{01p} + 2\gamma\epsilon^{pq\ell}\Gamma_{qr} \right), \]
(3.15)

\[ \partial_j \tau = \tau \epsilon_i^j \left( -\alpha_j\Gamma^{01} + \frac{1}{4}\partial_j\Phi \right) \Gamma^{23}. \]
(3.16)

The integrability condition (2.26) for Eq. (3.16) is

\[ 0 = \tau g^{ij} \left( 4\partial_k\alpha_j\Gamma^{01} - \partial_k\partial_j\Phi \right). \]
(3.17)

The second-order equation (2.16) for this case is

\[ D_i(\tau B^a)\Gamma^i = \tau \left( -\frac{r}{\Gamma^a} - \frac{1}{2}\partial^i q_i \right), \]
(3.18)

\[ D_i(\tau B^p)\Gamma^i = \tau \left( -\frac{\tilde{r}}{\Gamma^p} - \frac{1}{2}\partial^i q_i \right). \]
(3.19)

Using Eqs. (3.14)–(3.16), one can show that Eqs. (3.18) and (3.19) are equivalent to

\[ 0 = r + \tilde{r} + g^{ij} q_i q_j + 2g^{ij}\partial_i\partial_j - 4(\beta^2 + \gamma^2), \]
(3.20)

\[ 0 = \tau \left( r - \tilde{r} + 4\partial_i\beta \Gamma^{i456} - 4\partial_i\gamma \Gamma^{i789} + 8\alpha_i\Gamma^{01i} \left( \beta \Gamma^{456} - \gamma \Gamma^{789} \right) \right). \]
(3.21)

Therefore, the equations that we have to solve are Eqs. (3.8), (3.13), (3.17), and (3.21). Once we find \( a, \alpha_j, \beta, \gamma, \Phi \), and \( r - \tilde{r} \) satisfying these equations, the other parameters can be easily obtained by Eqs. (3.7), (3.14), (3.15), and (3.20). It can be easily checked that these SUSY conditions reduce to those given in Ref. [12] when the symmetry is enhanced to \( ISO(1, 2) \times SO(3) \times SO(3) \).

### 3.2. Solutions of the SUSY conditions

In this section we are going to elaborate on a prescription for the SUSY parameter \( \epsilon \) that simplifies the study of solutions. In addition, we give two examples of solutions, where the latter is a generalization of the former.

#### 3.2.1. More on SUSY conditions

Let us first try to solve Eq. (3.17). Decomposing \( \tau \) as

\[ \tau = \tau_+ + \tau_- \]
(3.22)

with \( \tau_0 \Gamma^{01} = \pm\tau_0 \), Eq. (3.17) can be written as

\[ 0 = \tau_+g^{ij} \left( 4\partial_k\alpha_j - \partial_k\partial_j\Phi \right) - \tau_-g^{ij} \left( 4\partial_k\alpha_j + \partial_k\partial_j\Phi \right). \]
(3.23)

If \( g^{ij}\partial_k\partial_j\Phi = 0 \) and \( g^{ij}\partial_k\alpha_j = 0 \), both \( \tau_+ \) and \( \tau_- \) can be non-zero. However, if \( g^{ij}\partial_k\partial_j\Phi \neq 0 \), it has a non-trivial solution only if

\[ g^{ij}\partial_k\partial_j\Phi \neq 4\alpha_j \]
(3.24)

are satisfied. Therefore, when \( \Phi \) is not a harmonic function, the unbroken SUSY is inevitably chiral in two dimensions. In the following, we impose Eq. (3.24), though we do not assume \( g^{ij}\partial_k\partial_j\Phi \neq 0 \).
The general solution for the case with $g^{kj}\partial_k \partial_j \Phi = 0$ can be easily obtained by taking a linear combination of a solution with $\tau = \tau_+$ and that with $\tau = \tau_-$. Equation (3.24) can be solved (at least locally) if and only if there exists a function $\varphi_\pm$ satisfying

$$\pm \alpha_j - \frac{1}{4} \partial_j \Phi = \epsilon_j \partial_j \varphi_\pm.$$  

(3.25)

Then, the solution of Eq. (3.16) is

$$\tau = \tau_0^\pm e^{\varphi_\pm \Gamma^{33}} = \tau_0^\pm (\cos \varphi_\pm + \Gamma^{33} \sin \varphi_\pm),$$

(3.26)

where $\tau_0^\pm$ is a constant spinor satisfying

$$\tau_0^\pm \Gamma^{01} = \pm \epsilon_0^\pm, \quad \tau_0^\pm \Gamma^{(10)} = \epsilon_0^\pm.$$  

(3.27)

The second condition in Eq. (3.27) follows from the chirality condition (2.5).

This $\epsilon_0^\pm$ belongs to the 8D Majorana–Weyl representation of the $SO(8)$ subgroup of the 10D Lorentz group. Since the operators acting on $\tau$ in Eqs. (3.13) and (3.21) commute with $SO(3) \times SO(3)$ generators $\Gamma^{ab}$ and $\Gamma^{pq}$, it is convenient to decompose it to two $(2, 2)$ representations of the $SO(3) \times SO(3)$ as

$$\tau_0^\pm = \sum_{v=1}^{4} (f_0^v \langle 0; v \mid + f_1^v \langle 1; v \mid),$$

(3.28)

where $v = 1, 2, 3, 4$ is an index labeling the 4D representation $(2, 2)$ representation of $SO(3) \times SO(3)$, and $f_0^v$ and $f_1^v$ $(v = 1, 2, 3, 4)$ are real parameters. The set $\{ \langle 0; v \mid, \langle 1; v \mid | v = 1, 2, 3, 4 \}$ is the basis of the spinors satisfying Eq. (3.27). They can be constructed explicitly as follows. Let us define

$$C_1 = \frac{1}{2}(\Gamma^{2} + \Gamma^{456}), \quad C_2 = \frac{1}{2}(\Gamma^{3} \mp \Gamma^{789}),$$

(3.29)

which satisfy

$$\{ C_s, C_t^\dagger \} = \delta_{st}, \quad \{ C_s, C_t \} = 0, \quad (s, t = 1, 2).$$

(3.30)

Let $\langle 0; v \mid$ be a spinor satisfying

$$\langle 0; v \mid C_s^\dagger = 0, \quad (s = 1, 2)$$

(3.31)

and $\langle 1; v \mid$ is defined as

$$\langle 1; v \mid \equiv \langle 0; v \mid C_1 C_2.$$  

(3.32)

Since all the operators acting on $\tau_0^\pm$ in Eqs. (3.13) and (3.21) do not mix the spaces with different index $v$, we can assume

$$\tau_0^\pm = f_0^v \langle 0; v \mid + f_1^v \langle 1; v \mid$$

(3.33)

with fixed $v$. (The general solution is just a linear combination of this type.) Note that $\Gamma^{23} = (C_1 + C_1^\dagger)(C_2 + C_2^\dagger)$ and

$$\langle 0; v \mid \Gamma^{23} = \langle 1; v \mid, \quad \langle 1; v \mid \Gamma^{23} = - \langle 0; v \mid.$$

(3.34)
Then, Eq. (3.33) can be written as
\[ \tilde{\epsilon}_\pm^0 = \eta \{ 0, \nu | e^{\xi} \Gamma^{\tilde{\epsilon}} \}, \]  
(3.35)
where \( \eta = \sqrt{(f_0^1)^2 + (f_0^1)^2} \) and \( \xi = \arctan(f_0^1/f_0^0) \). As a consequence, the number of remaining SUSY is 4 in general, corresponding to the choice of \( \nu = 1, 2, 3, 4 \). This agrees with what we expect from the brane configuration (3.1). If the phase \( \xi \) can be chosen freely without changing the action, the number of SUSY is enhanced to 8. As mentioned above, if \( \Phi \) satisfies the Laplace equation
\[ \frac{\partial^2}{\partial x^2} \Phi = 0 \quad \text{and} \quad \frac{\partial^2}{\partial y^2} \Phi = 0 \]
(3.21) is equivalent to
\[ \frac{\partial^2}{\partial x^2} (\epsilon \Phi) = \frac{\partial^2}{\partial y^2} (\epsilon \Phi) = 0. \]
and Eq. (3.21) is equivalent to
\[ \frac{\partial^2}{\partial x^2} (\epsilon \Phi) = \frac{\partial^2}{\partial y^2} (\epsilon \Phi) = 0. \]
(3.36)
and Eq. (3.21) is equivalent to
\[ \frac{\partial^2}{\partial x^2} (\epsilon \Phi) = \frac{\partial^2}{\partial y^2} (\epsilon \Phi) = 0. \]
(3.37)
and Eq. (3.21) is equivalent to
\[ \frac{\partial^2}{\partial x^2} (\epsilon \Phi) = \frac{\partial^2}{\partial y^2} (\epsilon \Phi) = 0. \]
(3.38)
and Eq. (3.21) is equivalent to
\[ \frac{\partial^2}{\partial x^2} (\epsilon \Phi) = \frac{\partial^2}{\partial y^2} (\epsilon \Phi) = 0. \]
(3.39)
It is convenient to write these equations using a complex coordinate \( z = \frac{1}{\sqrt{2}} (x^2 + i x^3) \):
\[ \beta \pm i \gamma = \sqrt{2} e^{-\Phi} a^{1/2} \partial_z \left( e^{\frac{1}{2} \Phi + i 2 \varphi_\pm} a^{-1/2} \right), \]
(3.40)
\[ \frac{1}{4} (r - \bar{r}) e^\Phi = \sqrt{2} \partial_z \left( e^{\frac{1}{2} \Phi + i 2 \varphi_\pm} (\beta \pm i \gamma) \right) = 2 \partial_z \left( e^{-\frac{1}{4} \Phi + i 2 \varphi_\pm} \partial_z \left( e^{\frac{1}{2} \Phi + i 2 \varphi_\pm} a^{-1/2} \right) \right), \]
(3.41)
where we have used Eq. (3.40) in the last step. \( \beta, \gamma, \) and \( (r - \bar{r}) \) are determined by Eq. (3.40) and the real part of Eq. (3.41). The imaginary part of Eq. (3.41) gives a non-trivial constraint:
\[ \text{Im} \left[ \partial_z \left( e^{-\frac{1}{4} \Phi + i 2 \varphi_\pm} \partial_z \left( e^{\frac{1}{2} \Phi + i 2 \varphi_\pm} a^{-1/2} \right) \right) \right] = 0. \]
(3.42)
This equation can be solved if there exists a real function \( f(z, \bar{z}) \) satisfying
\[ \partial_z f = e^{-\frac{1}{4} \Phi + i 2 \varphi_\pm} \partial_z \left( e^{\frac{1}{2} \Phi + i 2 \varphi_\pm} a^{-1/2} \right) \]
\[ = e^{i 4 \varphi_\pm} \left( 2 \partial_z \varphi_\pm + \frac{1}{2} \partial_z (\Phi - \log a) \right). \]
(3.43)
This is equivalent to
\[ e^{-i 2 \varphi_\pm + \frac{1}{2} (\Phi - \log a)} \partial_z ef = \partial_z \left( e^{i 2 \varphi_\pm + \frac{1}{2} (\Phi - \log a)} \right) ef. \]
(3.44)
The complex conjugate of this equation is
\[ \partial_z \left( e^{-i 2 \varphi_\pm + \frac{1}{2} (\Phi - \log a)} \right) ef = e^{i 2 \varphi_\pm + \frac{1}{2} (\Phi - \log a)} \partial_z ef. \]
(3.45)
The sum of these two equations gives
\begin{equation}
\partial_z \left( e^{-i\frac{2}{2}z + \frac{1}{2}(\Phi - \log a)} \right) = \partial_z \left( e^{i\frac{2}{2}z + \frac{1}{2}(\Phi - \log a)} \right),
\end{equation}
which is equivalent to
\begin{equation}
\text{Im} \left[ \partial_z \left( e^{i\frac{2}{2}z + \frac{1}{2}(\Phi - \log a)} \right) \right] = 0.
\end{equation}
This equation can be solved if there exists a real function \( g(z, \bar{z}) \) satisfying
\begin{equation}
e^{i\frac{2}{2}z + \frac{1}{2}(\Phi - \log a)} = e^{-f} \partial_z g.
\end{equation}
Inserting this into Eq. (3.44), we obtain
\begin{equation}
\partial_z g \partial_z f + \partial_z f \partial_z g = \partial_z \partial_z g.
\end{equation}
If we are able to find real functions \( g \) and \( f \) satisfying this relation, \( \varphi_{\pm} \) and \( (\Phi - \log a) \) are obtained by Eq. (3.48). Equations (3.40) and (3.41) are
\begin{align}
\beta \pm i\gamma &= \sqrt{2} e^{-\Phi} a^{1/2} \partial_z e^{-f} \partial_z g = \sqrt{2} e^{-\Phi} a^{1/2} e^{-f} \partial_z g \partial_z f, \\
r - \tilde{r} &= 8 e^{-\Phi} \partial_z \partial_z f.
\end{align}
Note that Eq. (3.49) can also be written as
\begin{equation}
\text{Re} \left[ \partial_z (e^{-2f} \partial_z g) \right] = 0.
\end{equation}
This equation can be solved if there exists a real function \( h(z, \bar{z}) \) satisfying
\begin{equation}
e^{-2f} \partial_z g = i \partial_z h,
\end{equation}
which implies
\begin{equation}
\partial_z g \partial_z h + \partial_z g \partial_z h = 0.
\end{equation}
This shows that the gradients of \( g \) and \( h \) are orthogonal to each other. Conversely, if we are able to find real functions \( g \) and \( h \) satisfying Eq. (3.54), \( f \) is obtained as
\begin{equation}
e^{-2f} = i \frac{\partial_z h}{\partial_z g} = -i \frac{\partial_z h}{\partial_z g}.
\end{equation}
Equation (3.48) can also be written as
\begin{equation}
e^{i\frac{2}{2}z + \Phi - \log a} = -i \partial_z h \partial_z g.
\end{equation}
In addition to these equations, we should also solve Eq. (3.8). Using Eq. (3.25), Eq. (3.8) can be written as
\begin{equation}
g^{ij} \partial_j (a \partial_j \Phi) + 4 e^{ik} \partial_j a \partial_k \varphi_{\pm} = 0,
\end{equation}
which is equivalent to
\begin{equation}
\text{Re} \left[ \partial_z (a \partial_z (\Phi + 4i \varphi_{\pm})) \right] = 0.
\end{equation}
This equation can be solved if there exists a real function \( k(z, \bar{z}) \) satisfying
\[
a \partial_z (\Phi - \log a + 4i \varphi_{\pm}) + \partial_{\bar{z}} a = a \partial_{\bar{z}} (\Phi + 4i \varphi_{\pm}) = i \partial_z k. \tag{3.59}
\]
From the first equation of Eqs. (3.7) and (3.25), we see that this \( k \) is proportional to the theta parameter as
\[
k = \mp ac = \mp \frac{\theta}{8\pi \mathcal{V}}, \tag{3.60}
\]
up to an additive constant.

Using Eq. (3.48), Eq. (3.59) becomes
\[
2a \partial_z (-f + \log \partial_{\bar{z}} g) + \partial_{\bar{z}} a = i \partial_z k, \tag{3.61}
\]
which can also be written as
\[
\partial_z (a \partial_{\bar{z}} h \partial_{\bar{z}} g) = i \partial_z h \partial_{\bar{z}} g \partial_{\bar{z}} k. \tag{3.62}
\]

In summary, the SUSY conditions are now reduced to a problem of finding real functions \( h, g, a, \) and \( k \) satisfying Eqs. (3.54) and (3.62). Then, the real function \( f \) is obtained by Eq. (3.55) and other parameters are determined by Eqs. (3.48) (or (3.56)), (3.50), (3.51), and (3.60). Although we have not been able to find the general solution of the SUSY conditions (3.54) and (3.62), a lot of non-trivial solutions have been found. In the following subsections, we show some of the explicit solutions.

3.2.2. Solution 1
First, we introduce new coordinates \((y^1, y^2)\) defined as
\[
y^1 \equiv \bar{l}(\bar{z}) + l(z), \quad y^2 \equiv i(\bar{l}(\bar{z}) - l(z)), \tag{3.63}
\]
where \( l(z) \in \mathbb{C} \) is a holomorphic function of \( z = \frac{1}{\sqrt{2}}(x^2 + i x^3) \) and \( \bar{l}(\bar{z}) \) is its complex conjugate. They are related to the original coordinates \((x^2, x^3)\) by a conformal transformation on the 2D plane. Note that our ansatz explained in Sect. 3.1 is compatible with the conformal transformation\(^{11}\) and hence our results in the previous subsection are valid in the coordinates \((y^1, y^2)\) as well. In fact, it is easy to see that Eqs. (3.54) and (3.62) are invariant under the conformal transformation. Once one finds a solution, one can generate new solutions by the conformal transformations. In order to emphasize this point, we write down the solutions that work for any choice of the holomorphic function \( l(z) \), rather than using this degree of freedom to simplify the equations.

Since the condition (3.54) is equivalent to the statement that the gradients of \( g \) and \( h \) are orthogonal to each other, it is clear that it can be solved when \( g \) and \( h \) are of the form:
\[
g(z, \bar{z}) = G_1(y^1), \quad h(z, \bar{z}) = H_2(y^2), \tag{3.64}
\]

\(^{11}\) Because we set \( g_{ij}(x^i) = e^{\Phi(x^i)} \delta_{ij} \), the general coordinate transformation of \((x^2, x^3)\) is not compatible with our ansatz. The conformal transformation on the \((x^2, x^3)\)-plane keeps this form invariant.
where $G_1$ and $H_2$ are real functions. The subscripts of these functions suggest which of the coordinates ($y^1$ or $y^2$) they depend on. Inserting these into Eq. (3.62), we obtain

$$\partial_z (a H'_2 G'_1) = i H'_2 G'_1 \partial_z k,$$

where the prime denotes the derivative, e.g., $G'_1 \equiv \partial_{y^1} G_1(y^1)$.

One can check that the following ansatz gives a solution of Eq. (3.65):

$$a(z, \bar{z}) = \frac{L(y^1, y^2)}{G'_1(y^1) H'_2(y^2)}, \quad k(z, \bar{z}) = K_1(y^1) + K_2(y^2),$$

where $K_1$ and $K_2$ are real functions satisfying

$$G'_1 K'_1 = \kappa_1, \quad H'_2 K'_2 = \kappa_2,$$

with real constants $\kappa_i$ ($i = 1, 2$), and $A$ is a real function defined as

$$L(y^1, y^2) \equiv \kappa_2 G_1(y^1) - \kappa_1 H_2(y^2).$$

Then, using Eqs. (3.20), (3.50), (3.51), (3.56), and (3.60), we obtain the rest of the parameters:

$$c = \mp \frac{G'_1 H'_2}{L} (K_1 + K_2),$$

$$\Phi = \log \left( L \partial_z \partial_{\bar{z}} \right),$$

$$\varphi_{\pm} = \frac{1}{2} \text{Im} \left[ \log \partial_{\bar{z}} \right],$$

$$\beta \pm i \gamma = \frac{1}{\sqrt{2L}} \left( (\log G'_1)' - i (\log H'_2)' \right),$$

$$r - \tilde{r} = \frac{4}{L} \left( (\log G'_1)'' - (\log H'_2)'' \right),$$

$$r + \tilde{r} = \frac{4}{L} \left( (\log G'_1)'' + (\log H'_2)'' \right) + \frac{2}{L^2} (\kappa_2^2 G_1^2 + \kappa_1^2 H_2^2).$$

### 3.2.3. Solution 2

Let us generalize the solutions in the previous subsection by considering the following ansatz:

$$g(z, \bar{z}) = G_1(y^1) + G_2(y^2), \quad h(z, \bar{z}) = H_1(y^1) + H_2(y^2),$$

where $G_i$ and $H_i$ ($i = 1, 2$) are real functions. Then, Eq. (3.54) can be solved when these functions satisfy

$$G'_1 H'_1 = -c_0, \quad G'_2 H'_2 = c_0,$$

where $c_0$ is a real constant. Note that Eq. (3.55) implies

$$e^{-2f} = \frac{H'_2}{G'_1},$$

which makes sense when $H'_2 G'_1 > 0.$
Then, Eq. (3.62) can be written as
\[ e^{i\sigma} \partial_z A = \partial_z B, \quad (3.78) \]
where we have defined
\[ e^{i\sigma} \equiv \frac{W + 2c_0i}{V}, \quad (3.79) \]
\[ W \equiv G_1'H_2' + G_2'H_1' = G_1'H_2' - \frac{c_0^2}{G_1'H_2'}, \quad (3.80) \]
\[ V \equiv \sqrt{W^2 + 4c_0^2} = G_1'H_2' + \frac{c_0^2}{G_1'H_2'}, \quad (3.81) \]
\[ A \equiv aW + 2c_0k, \quad (3.82) \]
\[ B \equiv aV. \quad (3.83) \]

Some useful identities are
\[ G_1'H_2' = c_0 \cot \left( \frac{\sigma}{2} \right), \quad W = 2c_0 \cot \sigma, \quad V = \frac{2c_0}{\sin \sigma}. \quad (3.84) \]

Note that in the Taylor expansion of Eq. (3.78) with respect to \( c_0 \), the leading term is trivially satisfied and the \( O(c_0) \) term reproduces Eq. (3.65).

Here, we try to solve Eq. (3.78) using the following ansatz:
\[ A(y^1, y^2) = A_1(y^1) + A_2(y^2), \quad B(y^1, y^2) = B_1(y^1) + B_2(y^2), \quad (3.85) \]
where \( A_i \) and \( B_i \) are real functions. Inserting this ansatz into Eq. (3.78), we obtain
\[ G_1'H_2'(A_1' - B_1') - \frac{c_0^2}{G_1'H_2'}(A_1' + B_1') = -2c_0A_2', \quad (3.86) \]
\[ G_1'H_2'(A_2' - B_2') - \frac{c_0^2}{G_1'H_2'}(A_2' + B_2') = 2c_0A_1'. \quad (3.87) \]

These equations can be solved when
\[ A_1' + B_1' = 2\kappa_2 G_1', \quad A_1' - B_1' = \frac{2c_0\kappa_1}{G_1'}, \quad A_2' + B_2' = -2\kappa_1 H_2', \quad A_2' - B_2' = \frac{2c_0\kappa_2}{H_2'}, \quad (3.88) \]
where \( \kappa_i \) (\( i = 1, 2 \)) are real constants. Then, we obtain
\[ A = \kappa_2 G_1 - \kappa_1 H_2 + c_0(K_1 + K_2) + a_0, \quad (3.89) \]
\[ B = \kappa_2 G_1 - \kappa_1 H_2 - c_0(K_1 + K_2) + b_0, \quad (3.90) \]
where \( K_i = K_i(y^i) \) (\( i = 1, 2 \)) are real functions satisfying Eq. (3.67), and \( a_0 \) and \( b_0 \) are real constants. (We can set \( a_0 = b_0 = 0 \) by absorbing them in the constant parts of \( G_1, H_2, \) and \( K_i \), but we will keep them for convenience).
Then, by the definition of $A$ and $B$, we get
\[
a = \frac{B}{V} = \frac{B}{2c_0} \sin \sigma, \quad (3.91)
\]
\[
k = \frac{1}{2c_0} \left( \frac{A}{2} - B \frac{W}{V} \right) = \frac{1}{2c_0} (A - B \cos \sigma). \quad (3.92)
\]

Other parameters are obtained by using Eqs. (3.20), (3.50), (3.51), (3.56), and (3.60):
\[
c = \pm \frac{1}{2c_0} \left( W - V \frac{A}{B} \right) = \pm \left( \cot \sigma - \frac{A}{B \sin \sigma} \right), \quad (3.93)
\]
\[
\Phi = \log \left( B \partial_2 \partial_3 \right), \quad (3.94)
\]
\[
\varphi_{\pm} = \frac{\sigma}{4} + \frac{1}{2} \text{Im} \left[ \log \partial_2 \partial_3 \right], \quad (3.95)
\]
\[
\beta \pm i\gamma = \frac{e^{-\frac{\varphi}{2}i}}{\sqrt{2B}} \left( (\log G_1')' - i(\log H_2')' \right), \quad (3.96)
\]
\[
r - \tilde{r} = \frac{4}{B} (\log G_1'') - (\log H_2''), \quad (3.97)
\]
\[
r + \tilde{r} = \frac{4}{B} \left( \log G_1'' + (\log H_2'')' + \frac{2}{B^3} (\kappa_2^2 G_1^2 + \kappa_1^2 H_2^2 + c_0^2 (K_1^2 + K_2^2)) \right)
+ \frac{24}{B^2} \left( (\log G_1')' + (\log H_2')'' \right)
+ \frac{4}{B^2} \left( \frac{W}{V} - 1 \right) (\kappa_2^2 G_1' - \kappa_1^2 H_2'') - \frac{4c_0}{B^2} \left( \frac{W}{V} + 1 \right) (K_1'' + K_2'')
= \frac{4}{B} \cos \sigma \left( (\log G_1')'' + (\log H_2')'' \right) + \frac{2}{B^3} \left( \kappa_2^2 G_1^2 + \kappa_1^2 H_2^2 + c_0^2 (K_1^2 + K_2^2) \right)
+ \frac{6}{B} \sin^2 \sigma \left( (\log G_1')' + (\log H_2')'' \right)
- \frac{8}{B^2} \sin^2 \frac{\sigma}{2} (\kappa_2^2 G_1' - \kappa_1^2 H_2'') - \frac{8c_0}{B^2} \sin^2 \frac{\sigma}{2} (K_1'' + K_2''). \quad (3.98)
\]

One can check that this solution reduces to the one given in the previous subsection in the $c_0 \to 0$, $\sigma \to 0$ limit.

### 3.3. F-theory configuration: ISO(1, 1) $\times$ SO(6)

As an explicit example of the solutions obtained in Sect. 3.2.2, let us consider

\[
G_1 = y^1, \quad H_2 = y^2, \quad K_1 = \kappa_1 y^1, \quad K_2 = \kappa_2 y^2. \quad (3.99)
\]

In this case, Eq. (3.66) gives

\[
a = \kappa_2 y^1 - \kappa_1 y^2, \quad k = \kappa_1 y^1 + \kappa_2 y^2, \quad (3.100)
\]

and hence

\[
a + ik = 2(\kappa_2 + i\kappa_1)l(z) \quad (3.101)
\]
Another immediate consequence is that the combination \( \Phi - \log a \propto \text{Re}(\log \partial_z I) \) is a harmonic function satisfying the Laplace equation in two dimensions:

\[
\partial_z \partial_{\bar{z}} (\Phi - \log a) = 0.
\] (3.102)

Actually, in this case, Eqs. (3.69)–(3.74) imply \( \beta = \gamma = 0 \) and \( r = \bar{r} \), and the symmetry \( SO(3) \times SO(3) \) is enhanced to \( SO(6) \).

Though this solution is a special solution, the properties (3.101) and (3.102) hold for general solutions with \( ISO(1,1) \times SO(6) \) symmetry. In fact, it is not difficult to find general solutions for the cases with \( ISO(1,1) \times SO(6) \) symmetry. If we impose \( \beta = \gamma = 0 \), Eq. (3.50) implies \( g = \) constant or \( f = \) constant. However, in order to have a non-singular solution of Eq. (3.48), \( g \) cannot be a constant. Then, \( f \) has to be a constant and Eq. (3.49) implies that \( g \) is a harmonic function satisfying \( \partial_z \partial_{\bar{z}} g = 0 \). Then, Eqs. (3.48) and (3.61) imply that

\[
\Phi - \log a - 4i \varphi_{\pm} = 0
\] (3.103)

and \( a + ik \) are holomorphic functions. Therefore, the general solution can be written as

\[
a = \tilde{I}(\bar{z}) + l(z), \quad k = i(\tilde{l}(\bar{z}) - l(z)),
\] (3.104)

\[
\Phi = \text{log}(\tilde{I}(\bar{z}) + l(z)) + \text{Im}(\tilde{m}(\bar{z}) + m(z)), \quad \varphi_{\pm} = -\frac{i}{4}(\text{Im}(\bar{z}) - m(z))
\] (3.105)

with holomorphic functions \( l(z) \) and \( m(z) \).

Note that all the parameters in the Lagrangian are invariant under a constant shift of the imaginary part of \( m(z) \), but \( \varphi_{\pm} \) is shifted. This shift induces a shift of \( \xi \) in Eq. (3.35) through Eq. (3.26) and hence we can choose any value for \( \xi \) without changing the action. Therefore, as explained below Eq. (3.35), the number of preserved SUSY is enhanced to 8.

This solution corresponds to the D3-brane probes in the F-theory configurations (background with 7-branes in type IIB string theory) [23–25] and Eq. (3.102) is interpreted as the Einstein equation [33]. To see this explicitly, note that the type IIB supergravity action for the dilaton \( \phi \), RR 0-form \( C_0 \), and gravity is given by

\[
S_{\text{IIB}} = \frac{1}{2\kappa_0^2} \int d^{10}x \sqrt{-g} \left( R - \frac{\partial_{\mu} \tau \partial^{\mu} \tau}{2(\text{Im} \tau)^2} + \cdots \right),
\] (3.106)

where \( \tau \equiv C_0 + ie^{-\phi} \). Assuming that \( \tau \) only depends on \( x^2 \) and \( x^3 \), and using the ansatz (2.4) for the metric, the Einstein equation and the equation of motion for \( \tau \) become

\[
- \partial_z \partial_{\bar{z}} \Phi + \frac{\partial_z \tau \partial_{\bar{z}} \bar{\tau} + \partial_{\bar{z}} \tau \partial_z \bar{\tau}}{(\tau - \bar{\tau})^2} = 0,
\] (3.107)

\[
\partial_z \bar{\tau} = 0,
\] (3.108)

\[
\partial_z \partial_{\bar{z}} \tau + 2 \frac{\partial_z \bar{\tau} \partial_{\bar{z}} \bar{\tau}}{\tau - \bar{\tau}} = 0.
\] (3.109)

Then, it is easy to see that \( \tau = \tau(z) \) (a holomorphic function of \( z \)) solves the equations of motion for \( \tau \) and then Eq. (3.107) becomes

\[
\partial_z \partial_{\bar{z}} (\Phi - \log \text{Im} \tau) = 0,
\] (3.110)

which agrees with Eq. (3.102) with the identification \( \tau \propto i(a + ik) \).
3.4. Gaiotto–Witten solution: ISO(1, 2) × SO(3) × SO(3)

The supersymmetric Janus configuration found by Gaiotto–Witten in Ref. [12] can be obtained as a special solution of the solutions obtained in Sect. 3.2.3. It is a solution with ISO(1, 2) × SO(3) × SO(3) symmetry and all the deformation parameters depend only on one coordinate \( x^3 \).

To see this, let us consider the case with \( \kappa_i = 0, K_i = 0 \) (\( i = 1, 2 \)), and \( H_2 = y^2 \). In this case, \( \Phi \) is a harmonic function and, as discussed in Sect. 3.2.1, we may have solutions with both \( \varepsilon_+ \) and \( \varepsilon_- \) being non-zero. To obtain such solutions, we should make sure that the parameters in the Lagrangian are consistent with the SUSY conditions for both \( \varepsilon_+ \) and \( \varepsilon_- \) simultaneously. To this end, we choose

\[
c_0 = \pm \frac{1}{D}, \quad \sigma = \pm 2 \psi
\]

for the solution in Eqs. (3.94)–(3.98). Then, we have the relation \( \cot \psi = D \eta_1 \) (\( y^1 \)) and

\[
a = \frac{b_0 D}{2} \sin(2\psi),
\]

\[
c = \cot(2\psi) - \frac{a_0}{b_0 \sin(2\psi)},
\]

\[
\Phi = \log \left( b_0 \partial_z l \partial_{\tilde{z}} l \right),
\]

\[
\varphi_\pm = \pm \frac{\psi}{2} + \frac{1}{2} \Im \left[ \log \partial_{\tilde{z}} l \right],
\]

\[
\beta \pm i \gamma = e^{\mp i \psi} \frac{\sqrt{2} b_0}{(\log \cot \psi)^{\prime}} (\log \cot \psi)^{\prime},
\]

\[
r - \tilde{r} = \frac{4}{b_0} (\log \cot \psi)^{\prime \prime},
\]

\[
r + \tilde{r} = \frac{4}{b_0} \cos(2\psi)(\log \cot \psi)^{\prime \prime} + \frac{6}{b_0} \sin^2(2\psi)(\log \cot \psi)^{\prime 2}.
\]

This agrees with the supersymmetric Janus solution in Ref. [12] when \( l(z) = -iz/\sqrt{2} \) and \( b_0 = 2 \). In this configuration, 8 supersymmetries are preserved.

4. Other examples

4.1. ISO(1, 1) × SO(2) × SO(4)

Similar to the case with ISO(1, 1) × SO(3) × SO(3) symmetry considered in the previous section, we expect to have SUSY-preserving configurations with ISO(1, 1) × SO(2) × SO(4) symmetry, as it can be realized in a D3–D3(–D7) system in the following table:

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>D3</td>
<td>o</td>
<td>o</td>
<td>o</td>
<td>o</td>
<td></td>
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<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>D3</td>
<td>o</td>
<td>o</td>
<td>o</td>
<td>o</td>
<td></td>
<td></td>
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<td></td>
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<td></td>
</tr>
<tr>
<td>D7</td>
<td>o</td>
<td>o</td>
<td>o</td>
<td>o</td>
<td>o</td>
<td>o</td>
<td>o</td>
<td>o</td>
<td>o</td>
<td>o</td>
</tr>
</tbody>
</table>

In this case, we regard the D3-brane extended along \( x^{0–3} \) directions as a probe embedded in the supergravity background corresponding to the other D3- and D7-branes. The 4D gauge theory with varying couplings is realized on the world-volume of the probe D3-brane.
4.1.1. Ansatz for deformation

First, we set the metric and other deformation parameters consistent with the global symmetry. The ansatz for the metric (2.4) is the same as that used in Sect. 3.1:

\[ ds^2 = \eta_{\alpha \beta} dx^\alpha dx^\beta + e^{\Phi(x')} \delta_{ij} dx^i dx^j + \delta_{ab} dx^a dx^b + \delta_{pq} dx^p dx^q, \]

(4.2)

where the indices are \( \alpha, \beta = 0, 1; i, j = 2, 3; a, b = 4, 5; \) and \( p, q = 6, 7, 8, 9. \) The non-trivial components of the spin connection are given in Eq. (3.9). The parameters \( M, d^{IJA}, m^{AB} \) consistent with the symmetry are of the form

\[ M = 6 \left( m_{01} \Gamma^{01i} + m_{i45} \Gamma^{i45} \right) \equiv \alpha_i \Gamma^i 01 + \beta_i \Gamma^i 45, \]

(4.3)

\[ d^{iab} = -d^{aib} = \nu^i \epsilon^{ab}, \quad m^{ab} = r \delta^{ab}, \quad m^{pq} = \tilde{r} \delta^{pq}, \]

(4.4)

where \( \epsilon^{ab} \) is the epsilon tensor satisfying \( \epsilon^{45} = -\epsilon^{54} = 1. \) Here, \( \alpha_i, \beta_i, \nu^i, r, \) and \( \tilde{r} \) are functions of \( x^i. \)

4.1.2. Solutions of the SUSY conditions

First, let us apply the above ansatz to the first-order equations (2.17)–(2.20). It is easy to show that all the components of \( e_{IJK} \) vanish. The components of \( e_{IJK} \) that are allowed by the symmetry are

\[ e^{01i} = a^{-1} \partial_j (ac) e^{ji} - 4 \alpha^i, \]

(4.5)

\[ e^{i45} = 4 \beta^i + 2 \nu^i. \]

(4.6)

Equation (2.17) for \((I, J) = (6, 7)\) implies

\[ 0 = \tilde{\epsilon} e^{J' K' I'} \Gamma_{K'} \Gamma_{I' J'} = 6 \tilde{\epsilon} (e^{01i} \Gamma_{01i} + e^{i45} \Gamma_{i45}). \]

(4.7)

From this equation, together with Eq. (2.17) for \((I, J) = (4, 6)\), we obtain

\[ \tilde{\epsilon} e^{01i} \Gamma_i = \tilde{\epsilon} e^{i45} \Gamma_i = 0. \]

(4.8)

Because \( \Gamma^{23} \) does not have a real eigenvalue, we conclude that \( e^{01i} = e^{i45} = 0 \) to have non-zero \( \tilde{\epsilon}. \) Therefore, we obtain

\[ -2 \beta^i = \nu^i, \quad a^{-1} \partial_j (ac) e^{ji} = 4 \alpha^i. \]

(4.9)

In this case, Eq. (2.28) gives

\[ 0 = \tilde{\epsilon} (\Gamma^i q_i - 4 \alpha_i \Gamma^{01i}), \]

(4.10)

where \( q_i \equiv \partial_i \log a. \) When we decompose \( \tilde{\epsilon} \) as \( \tilde{\epsilon} = \tilde{\epsilon}_+ + \tilde{\epsilon}_- \) with \( \tilde{\epsilon}_\pm \Gamma^{01} = \pm \tilde{\epsilon}_\pm, \) Eq. (4.10) becomes

\[ 0 = \tilde{\epsilon}_+ (q_i - 4 \alpha_i) \Gamma^i + \tilde{\epsilon}_- (q_i + 4 \alpha_i) \Gamma^i. \]

(4.11)

This is satisfied if and only if

\[ q_i = \pm 4 \alpha_i \quad \text{and} \quad \tilde{\epsilon}_\pm = 0, \]

(4.12)

or

\[ q_i = \alpha_i = 0. \]

(4.13)
Note that the second equation of Eq. (4.9) implies that the complex coupling (2.10) satisfies\(^{12}\)

\[
\partial_z \tau = 4\pi a i (4\alpha_z + q_z), \quad \partial_{\bar{z}} \tau = 4\pi a i (-4\alpha_{\bar{z}} + q_{\bar{z}}).
\]  (4.14)

From this we find that the complex coupling \(\tau\) is holomorphic or anti-holomorphic when \(q_i = 4\alpha_i\) or \(q_i = -4\alpha_i\), respectively. This is the same situation as that observed in Sect. 3.3. For the case with Eq. (4.13), the gauge coupling and theta parameter are constant.

From Eq. (2.23), we obtain the components

\[
A_0 = -\frac{1}{4} \left(F \Gamma_0 + 4\alpha_i \Gamma^{1i}\right), \quad A_1 = -\frac{1}{4} \left(F \Gamma_1 - 4\alpha_i \Gamma^{0j}\right),
\]

\[
A_i = -\frac{1}{4} \left(F \Gamma_i + 4\alpha_i \Gamma^{01} + 4\beta_i \Gamma^{45} - \epsilon_i^j \delta_j \Phi \Gamma_2^3\right). \quad (4.15)
\]

We find that Eq. (3.11) is also valid in this case. Then, Eq. (2.19) implies

\[
\epsilon_{Ba} = -\frac{1}{4} \epsilon \left(\alpha_i \Gamma^{01} - \beta_i \Gamma^{45}\right) / F, \quad \epsilon_{Bp} = -\frac{1}{4} \epsilon \alpha_i \Gamma^{01} / F, \quad (4.16)
\]

and Eq. (2.20) with \(\mu = i = 2, 3\) can be written as

\[
\partial_i \epsilon = \frac{1}{4} \partial_j \Phi \Gamma_i^j + \alpha_i \Gamma^{01} \Gamma_i^j - \beta_i \Gamma^{45}. \quad (4.17)
\]

The integrability condition (2.26) for Eq. (4.17) is equivalent to

\[
0 = \frac{1}{4} \partial_j \Phi \Gamma_i^j + \alpha_i \Gamma^{01} \Gamma_i^j - \beta_i \Gamma^{45} \quad (4.18)
\]

In order to solve this equation, we decompose \(\epsilon\) into the eigenspaces of \(\Gamma^{01}\) and \(\Gamma^{2345}\) as

\[
\epsilon = \epsilon^s_+ + \epsilon^s_+ + \epsilon^- + \epsilon^t_-, \quad (4.19)
\]

where \(\epsilon^s_t (s = \pm, t = \pm)\) satisfy

\[
\epsilon^s_i \Gamma^{01} = t \epsilon^s_i, \quad \epsilon^s_i \Gamma^{2345} = s \epsilon^s_i, \quad \epsilon^s_i \Gamma^{(10)} = \epsilon^s_i. \quad (4.20)
\]

Equation (4.18) implies that \(\epsilon^s_i\) can be non-zero only if

\[
0 = -\frac{1}{4} \epsilon \partial_j \Phi + t \epsilon \alpha_j + s \epsilon \beta_j \quad (4.21)
\]

is satisfied. This equation can be solved when there exists a function \(\varphi^s_i\) satisfying

\[
-\frac{1}{4} \partial_j \Phi + t \alpha_j + s \beta_j = \epsilon \partial_i \varphi^s_i. \quad (4.22)
\]

Then, the solution of Eq. (4.17) is given by

\[
\epsilon^s_i = \epsilon^{0s}_i e^s_i \Gamma^{23}, \quad (4.23)
\]

where \(\epsilon^{0s}_i\) is a constant spinor satisfying the conditions in Eq. (4.20).

\(^{12}\) The notation for complex coordinates is given in Appendix A.
Then we distinguish the following four cases:

- **(C1)**: $\alpha_j \neq 0$ and $\epsilon_{ij} \partial_i \beta_j \neq 0$.
  
  Equations (4.12) and (4.21) imply that only one combination of the signs $\{t, s\}$ can have non-zero $\tau_i^\pm$.

- **(C2)**: $\alpha_j \neq 0$ and $\epsilon_{ij} \partial_i \beta_j = 0$.
  
  Equation (4.12) implies that only one sign for $s$ is allowed, but both $\epsilon^s_t$ and $\epsilon^{-s}_t$ can be non-zero.

- **(C3)**: $\alpha_j = 0$ and $\epsilon_{ij} \partial_i \beta_j \neq 0$.
  
  Equation (4.21) implies that only one sign for $s$ is allowed, but both $\epsilon^s_t$ and $\epsilon^{-s}_t$ can be non-zero.

- **(C4)**: $\alpha_j = 0$ and $\epsilon_{ij} \partial_i \beta_j = 0$.
  
  All the sixteen components of $\tau$ can be non-zero. Equation (4.21) implies $g^{ij} \partial_i \partial_j (\Phi - \log a) = 0$.

Next, let us consider the second-order equation (2.16). In this case, it can be written as

\[
D_i (\tau B^a) \Gamma^i = \varepsilon \left( -2 (D_i \nu^i + q_i \nu^i) e^{ab} \Gamma_b - r \Gamma^a - \bar{B}^a \left( M + \frac{1}{2} \Gamma_i q_i \right) \right),
\]

and

\[
D_i (\tau B^p) \Gamma^i = \varepsilon \left( -\bar{r} \Gamma^a - \bar{B}^p \left( M + \frac{1}{2} \Gamma^i q_i \right) \right).
\]

Inserting Eq. (4.16) into these equations and using Eq. (4.17), we obtain

\[
0 = \varepsilon \left( r + \frac{1}{2} g^{ij} q_i q_j + g^{ij} \partial_i q_j - 8 g^{ij} \beta_i \beta_j + 4 \partial_i \beta_j \epsilon^{ij} \hat{\Gamma}_{2345} \right),
\]

\[
\tilde{r} = -\frac{1}{2} g^{ij} q_i q_j - g^{ij} \partial_i q_j,
\]

where we have used the relation in Eq. (4.9),

\[
0 = \varepsilon \left( q_i + 4 \alpha_i \Gamma^{01} \right),
\]

that is valid for both cases (4.12) and (4.13). Then, from Eq. (4.28) we obtain

\[
r = -\frac{1}{2} g^{ij} q_i q_j - g^{ij} \partial_i q_j + 8 g^{ij} \beta_i \beta_j - 4 s \partial_i \beta_j \epsilon^{ij},
\]

where we have used Eq. (4.20) in order to have non-zero $\tau_i^s$.

In summary, we can construct a generic solution of the SUSY conditions by the following steps. First, pick a holomorphic (or anti-holomorphic) function $\tau(z)$ such that $\text{Im} \tau > 0$. Then, $a$ and $c$ are obtained from Eq. (2.10) and $\alpha_i$ is given by Eq. (4.12). Next, choose arbitrary real functions $\Phi$
and $\varphi^2$. Then, $\beta_i$ is determined by Eq. (4.22). If one wants to find a solution with $\epsilon_{ij} \partial_i \beta_j = 0$ (case (C2) above), $\Phi$ is determined by solving Eq. (4.24). More explicitly, the solutions of Eq. (4.24) are obtained by choosing a holomorphic function $l(z)$ and setting

$$\Phi = \log a + l + \bar{l}.$$  \hspace{1cm} (4.32)

$\varphi_1$ in Eq. (4.25) is given by

$$\varphi_1 = \frac{i}{4} (l - \bar{l}).$$ \hspace{1cm} (4.33)

Other parameters $\nu$, $r$, and $\tilde{r}$ are obtained by solving Eqs. (4.9), (4.31), and (4.29), respectively.

The number of unbroken SUSY is 4 for the generic case (C1), 8 for cases (C2) and (C3), and 16 for (C4). Case (C4) is a trivial solution that is related to the undeformed $\mathcal{N} = 4$ SYM by a coordinate transformation and a field redefinition. In fact, because $\Phi$ is harmonic, it can be eliminated by a conformal transformation on the $z$-plane. $\beta_i$ is a pure gauge configuration and it can be eliminated by a local $SO(6)$ rotation (see Appendix C.2). One can guess that cases (C1), (C2), and (C3) correspond to the D3–D3–D7, D3–D7, and D3–D3 systems, respectively (J. Choi et al., manuscript in preparation).

4.2. Time-dependent solutions

4.2.1. $ISO(3)$

Let us consider the cases in which the couplings depend on time. First, as a trial, let us assume that spatial translational and rotational symmetry $ISO(3)$ are preserved. It turns out that the time-dependent solutions of the SUSY conditions with this symmetry can always be mapped to a system with a constant gauge coupling and theta parameter by general coordinate transformations and field redefinitions.\(^{13}\)

Note that the metric (2.4) can be chosen to be flat. This is because the general form of a metric preserving $ISO(3)$ symmetry (a flat FLRW metric) can be written as

$$ds^2 = e^{\Phi(\eta)} (-d\eta^2 + \delta_{ij} dx^i dx^j) + \delta_{AB} dx^A dx^B,$$ \hspace{1cm} (4.34)

where $i,j = 1,2,3$, $\eta$ is the conformal time defined by $d\eta = e^{-\Phi/2} dt$, and the overall factor $e^\Phi$ in the 4D metric can be eliminated by the Weyl transformation\(^{14}\) ($g_{\mu\nu} \rightarrow e^{-\Phi} g_{\mu\nu}$) without loss of generality.

Then, it is particularly easy to show that the gauge coupling cannot depend on time to preserve SUSY. To see this, note that Eq. (2.18) may be written as

$$\bar{\epsilon}(\partial_0 \log a) = \bar{\epsilon} \left( \frac{1}{36} \epsilon^{JK} \Gamma_{JK} \Gamma_0 - \left( \frac{1}{8} \epsilon_{\mu JK} - 6 m_{\mu JK} \right) \Gamma^\mu JK \Gamma_0 - 2 M \Gamma_0 \right).$$ \hspace{1cm} (4.35)

Because $\Gamma_{JK} \Gamma_0$ is an anti-symmetric matrix for all $I,J,K = 0,\ldots,9$, the right-hand side of this equation is $\bar{\epsilon}$ times a real anti-symmetric matrix. However, since a real anti-symmetric matrix cannot

\(^{13}\) We have not been able to exclude the possibility that mass parameters have non-trivial time dependence, despite the gauge coupling and theta parameter being constant.

\(^{14}\) See Appendix C.1.
have a real non-zero eigenvalue, the only possibility is that both the left- and right-hand sides are zero:

$$\partial_0 a = 0, \quad \epsilon \left( \frac{1}{36} e^{IJK} \Gamma_{IJK} - \frac{1}{8} e_{\mu JK} - 6 m_{\mu JK} \right) \Gamma^{\mu JK} - 2M = 0. \quad (4.36)$$

Therefore, the gauge coupling has to be time independent.

As explained in Appendix E.2, all the components of $e^{IJK}$ vanish for the cases with ISO(3) symmetry. Because non-trivial components of $m^{IJK}$ are $m^{ABC}$, $m^{01B}$, and $m^{123}$, the second equation of Eq. (4.36) becomes

$$\epsilon \left( -12 m^{123} \right) = \epsilon m_{ABC} \Gamma^{ABC} \Gamma^{123}, \quad (4.37)$$

which means that $m^{123}$ is proportional to the eigenvalue of $m_{ABC} \Gamma^{ABC} \Gamma^{123}$. However, since $m_{ABC} \Gamma^{ABC} \Gamma^{123}$ is a real anti-symmetric matrix, the eigenvalue cannot take a non-zero real value. The only possibility is

$$m^{123} = 0, \quad \epsilon m_{ABC} \Gamma^{ABC} = 0. \quad (4.38)$$

Then, Eq. (2.29) implies $\partial_0 (ac) = 0$ and, as a consequence, the theta parameter (2.9) is also a constant.

Furthermore, Eq. (2.20) with $\mu = 1, 2, 3$ and Eq. (2.23) implies $0 = \epsilon A_i \propto \epsilon F_i$ and we have

$$\partial_0 \epsilon = \epsilon A_0 = -3 \epsilon m_{0AB} \Gamma^{AB}. \quad (4.39)$$

We can set $m_{0AB} = 0$ by the local SO(6) transformation$^{15}$ and then the SUSY parameter $\epsilon$ is also constant.

### 4.2.2. ISO(2) × SO(6)

To get a solution with time-dependent gauge coupling, we consider the cases in which the couplings can depend on $x^0$ and $x^1$. We impose the ISO(2) symmetry that acts on the $x^2, 3$ directions and SO(6)$_R$ symmetry to simplify the analysis. The ansatz for the metric is

$$ds^2 = e^{\Phi(x^a)} \eta_{\alpha \beta} dx^\alpha dx^\beta + \delta_{ij} dx^i dx^j + \delta_{AB} dx^A dx^B, \quad (4.40)$$

where $\alpha, \beta = 0, 1; \ i, j = 2, 3; \ A, B = 4, 5, \ldots, 9$. The non-trivial components of the spin connection are

$$\omega_{0\hat{0}i} = -\frac{\partial_i \Phi}{2}, \quad \omega_{1\hat{0}i} = -\frac{\partial_0 \Phi}{2}. \quad (4.41)$$

Because of the ISO(2) × SO(6) symmetry, all the components of $d^{IJK}$ are zero and the form of $M$ consistent with the symmetry is

$$M = 6m_{a23} \Gamma^a_{23} \equiv \alpha_a \Gamma^a_{23}. \quad (4.42)$$

Then, $e^{\mu A} \ (\mu = 0, \ldots, 3$ and $A = 4, \ldots, 9)$ are all zero and we can use the argument given in Appendix E.1 to conclude that all the components of $e^{IJK}$ are zero. In addition, Eqs. (2.29) and (2.32) imply

$$a^{-1} \partial_0 (ac) = -4\alpha_1, \quad a^{-1} \partial_1 (ac) = -4\alpha_0, \quad g^{a\beta} \partial_a (aa \beta) = 0, \quad (4.43)$$

$^{15}$ See Appendix C.2.
and Eq. (2.28) becomes

\[ 0 = \bar{\epsilon} (q_\alpha \Gamma^\alpha - 4 \alpha_0 \Gamma^{23}), \]  

(4.44)

where \( q_\alpha = \partial_\alpha \log a \). Equation (4.44) is equivalent to

\[ 0 = \bar{\epsilon} \left( (q_0 \pm q_1) - 4(\alpha_0 \pm \alpha_1)\Gamma^{23} \right), \]  

(4.45)

where \( \bar{\epsilon} \) is decomposed as \( \bar{\epsilon} = \bar{\epsilon}^+ + \bar{\epsilon}^- \). Since \( \Gamma^{23} \) does not have a real eigenvalue, this equation implies

\[ q_0 = \mp q_1, \quad \alpha_0 = \mp \alpha_1, \quad \text{and} \quad \tau = 0, \]  

(4.46)

or a trivial solution:

\[ q_\alpha = \alpha_\alpha = 0. \]  

(4.47)

In the following we focus on the case in Eq. (4.46). Equations (4.46) and (4.43) imply

\[ (\partial_0 \pm \partial_1)a = 0, \quad (\partial_0 \pm \partial_1)(ac) = 0, \quad (\partial_0 \pm \partial_1)\alpha_\alpha = 0, \]  

(4.48)

and the solution can be written as

\[ \alpha_0 = \mp \alpha_1 = h^\prime_\mp(x^\mp), \quad ac = k_\mp(x^\mp), \quad a = \pm \frac{k'_\mp(x^\mp)}{4h'_\mp(x^\mp)}, \]  

(4.49)

or

\[ \alpha_\alpha = 0, \quad c = 0, \quad \text{and} \quad a = a_\mp(x^\mp), \]  

(4.50)

where \( h_\mp, k_\mp, \) and \( a_\mp \) are arbitrary real functions of \( x^\mp \equiv x^0 \mp x^1 \) and prime denotes the derivative.\(^{16}\)

From Eq. (2.23), \( A_\mu \) is given by

\[ A_0 = -\frac{1}{4} \left( F \Gamma_0 + 4\alpha_0 \Gamma^{23} + \partial_1 \Phi \Gamma^{0\bar{1}} \right), \]  

(4.51)

\[ A_1 = -\frac{1}{4} \left( F \Gamma_1 + 4\alpha_1 \Gamma^{23} + \partial_0 \Phi \Gamma^{0\bar{1}} \right), \]  

(4.52)

\[ A_i = -\frac{1}{4} (F \Gamma_i + \epsilon_{ij} 4\alpha_\alpha \Gamma^{i\bar{a}}). \]  

(4.53)

For \( \mu = i \), the condition (2.20) is \( 0 = \partial_i \bar{\epsilon} = \bar{\epsilon} A_i \) and implies

\[ \bar{\epsilon} F = -4 \bar{\epsilon} \alpha_\alpha \Gamma^{23} = 0, \]  

(4.54)

where we have used Eq. (4.46). Then, from Eqs. (2.19) and (2.20) we obtain, respectively,

\[ \bar{\epsilon} B^4 = 0, \]  

(4.55)

and

\[ \partial_0 \bar{\epsilon} = -\bar{\epsilon} \left( \alpha_0 \Gamma^{23} + \frac{1}{4} \partial_1 \Phi \Gamma^{0\bar{1}} \right), \quad \partial_1 \bar{\epsilon} = -\bar{\epsilon} \left( \alpha_1 \Gamma^{23} + \frac{1}{4} \partial_0 \Phi \Gamma^{0\bar{1}} \right). \]  

(4.56)

\(^{16}\) The solution (4.50) was discussed in Ref. [14].
The integrability condition (2.26) for Eq. (4.56) is

\[ 0 = \bar{\epsilon} \left( (\partial_1 \alpha_0 - \partial_0 \alpha_1) \Gamma^{23} + \frac{1}{4} \eta^{\alpha \beta} \partial_\alpha \partial_\beta \Phi \Gamma^{0 \dot{1}} \right), \]  

which implies

\[ \partial_1 \alpha_0 - \partial_0 \alpha_1 = 0, \quad g^{\alpha \beta} \partial_\alpha \partial_\beta \Phi = 0. \]  

The solutions (4.49) and (4.50) satisfy the first equation of Eq. (4.58) and the general solution of the second equation is

\[ \Phi = f_+(x^+) + f_-(x^-), \]  

where \( f_\pm \) are arbitrary real functions of \( x^\pm \). Then, Eqs. (4.56) can be integrated as

\[ \bar{\epsilon}_\pm = \bar{\epsilon}_\pm^0 e^{\pm \frac{1}{4} (f_+ - f_-) x_+ - h_+ \Gamma^{23}}, \]  

where \( \bar{\epsilon}_\pm^0 \) is a constant spinor satisfying \( \bar{\epsilon}_\pm \Gamma^{0 \dot{1}} = \pm \bar{\epsilon}_\pm \). Using the above results, the second-order equation (2.16) becomes simply

\[ 0 = \tau m^{AB} \Gamma_B. \]  

Multiplying this equation by \( m_{AC} \Gamma^C \), we find \( m_{AB} = 0 \).

5. Conclusion and outlook

We have studied the deformations of \( \mathcal{N} = 4 \) SYM that preserve SUSY with space-time-dependent couplings. A lot of explicit solutions of the SUSY conditions have been found. For example, we have found wide classes of solutions for the cases with \( ISO(1, 1) \times SO(3) \times SO(3) \) symmetry and the general solutions for the cases with \( ISO(1, 1) \times SO(6) \) and \( ISO(1, 1) \times SO(2) \times SO(4) \) symmetries. Time-dependent cases with \( ISO(3) \) and \( ISO(2) \times SO(6) \) symmetries have also been analyzed.

As we mentioned in the introduction, it is a commonly faced situation that gauge couplings are not constant when one tries to engineer a gauge theory using D-branes in string theory. Therefore, it is natural to ask whether there are string theory realizations of the solutions that we found in this paper. We hope to address this question in our forthcoming paper (J. Choi et al., manuscript in preparation).

Our analysis in this paper is classical and it would be important to take into account the quantum effects. Since these theories preserve SUSY, we may be able to use some techniques, such as localization, to calculate some physical quantities exactly. Since our system can be realized as D3-branes embedded in type IIB string theory, we may be able to study S-duality and holographic dual. It would also be interesting to consider how the BPS solitonic objects, such as dyons or instantons, behave when the couplings depend on space-time.

As mentioned in Sect. 2.1, our ansatz for the action in Eq. (2.8) is not completely general, even if we restrict the deformation to be of dimension less than 4. As an obvious extension of the analysis, one may try to include all the terms that are compatible with renormalizability. It would also be interesting to consider the case with \( U(N) \) gauge group and include the terms like \( \text{tr} F_{IJ} \), \( \text{tr} A_4 \). The \( U(1) \) part will be important to consider the configurations of D3-branes, when the system is embedded in string theory.
theory. Since our strategy should work for arbitrary supersymmetric theory, further generalization would also be possible.

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Appendix A. Complex coordinates

We use complex coordinates to parametrize the 2–3 plane. Our convention used in Sects. 3 and 4.1 is as follows:

\[
\begin{align*}
  z &= \frac{1}{\sqrt{2}}(x^2 + ix^3), \quad \bar{z} = \frac{1}{\sqrt{2}}(x^2 - ix^3), \\
  \partial_z &= \frac{1}{\sqrt{2}}(\partial_2 - i\partial_3), \quad \partial_{\bar{z}} = \frac{1}{\sqrt{2}}(\partial_2 + i\partial_3), \\
  g_{z\bar{z}} &= g_{\bar{z}z} = e^{\Phi_1}/2, \quad g^{z\bar{z}} = g^{\bar{z}z} = e^{-\Phi_1}, \\
  \epsilon_{z\bar{z}} &= -\epsilon_{\bar{z}z} = -ie^{\Phi_1}/2, \quad \epsilon_{z\bar{z}} = \epsilon_{\bar{z}z} = ie^{\Phi_1}/2, \\
  \epsilon_{z\bar{z}} &= -\epsilon_{\bar{z}z} = i, \quad \epsilon_{z\bar{z}} = \epsilon_{\bar{z}z} = -i.
\end{align*}
\]

\[
\begin{align*}
  \partial_z \tau &= 4\pi a i(4\alpha z + q_z), \quad \partial_{\bar{z}} \tau = 4\pi a i(-4\alpha z + q_{\bar{z}}), \\
  \partial_z \bar{\tau} &= 4\pi a i(4\alpha z - q_z), \quad \partial_{\bar{z}} \bar{\tau} = 4\pi a i(-4\alpha z - q_{\bar{z}}), \\
  4\alpha_z &= -ia^{-1}\partial_z(ac), \quad 4\alpha_{\bar{z}} = ia^{-1}\partial_{\bar{z}}(ac).
\end{align*}
\]

Appendix B. Gamma matrices

In this paper, we have chosen the $SO(1, 9)$ gamma matrices $\Gamma^I$ to be real $32 \times 32$ matrices.\(^{17}\) They can be written as

\[
\Gamma^0 = i\sigma_2 \otimes 1_{16}, \quad \Gamma^i = \sigma_1 \otimes \gamma^i_{SO(8)}, \quad (i = 1, 2, \ldots, 9)
\]

where $\gamma^i_{SO(8)}$ are $SO(8)$ gamma matrices and $\gamma^9_{SO(8)}$ is the chirality operator.\(^{18}\) $\Gamma^0$ is an anti-symmetric matrix and $\Gamma^{1–9}$ are symmetric matrices. $\Gamma^0\Gamma^{12...n}$ are symmetric for $n = 1, 2 \mod 4$ and anti-symmetric for $n = 0, 3 \mod 4$.

\(^{17}\) In this appendix, we assume that the metric is flat.

\(^{18}\) See, e.g., Appendix 5.B of Ref. [34] for an explicit realization.
The following formulas are useful:

\[ \Gamma^I \Gamma^J = \Gamma^{IJ} + \eta^{IJ}, \]  
(\text{B.2})

\[ \Gamma^I \Gamma^K = \Gamma^{IK} + \eta^{IK} \Gamma^I - \eta^{JK} \Gamma^J, \]  
(\text{B.3})

\[ \Gamma^{LK} \Gamma^L = \Gamma^{LK} + \eta^{LK} \Gamma^L - \eta^{JL} \Gamma^J + \eta^{KL} \Gamma^K, \]  
(\text{B.4})

\[ \Gamma^{IJ} \Gamma_{KL} = \Gamma^{IJ} \Gamma_{KL} - 4 \delta^{[I}_{[K} \Gamma^{J]}_L] - 2 \delta^{[I}_{[K} \delta^{J]}_L], \]  
(\text{B.5})

\[ \Gamma^{JK} \Gamma_{LM} = \Gamma^{JK} \Gamma_{LM} + 6 \delta^{[I}_{[L} \Gamma^{JK]}_M] - 6 \delta^{[I}_L \delta^{J}_M \Gamma^{K}], \]  
(\text{B.6})

\[ \Gamma^{LM} \Gamma^{JK} = \Gamma^{LM} \Gamma^{JK} - 6 \delta^{[I}_L \Gamma^{JK]}_M] - 6 \delta^{[I}_L \delta^{J}_M \Gamma^{K}], \]  
(\text{B.7})

\[ \Gamma^{JK}_L \Gamma^{LMN} = \Gamma^{JK}_L \Gamma^{LMN} + 9 \delta^{[I}_L \Gamma^{JK]}_M \Gamma^N] - 18 \delta^{[I}_L \delta^{J}_M \Gamma^{K]}_N] - 6 \delta^{[I}_L \delta^{J}_M \delta^{K}}_N]. \]  
(\text{B.8})

\[ \Gamma^I \Gamma^J \Gamma^J \Gamma^J = (\text{B.11}) \]  
(\text{B.11})

\[ \Gamma^I \Gamma^J \Gamma^J \Gamma^J = (\text{B.11}) \]  
(\text{B.11})

where \( D \) is the number of dimensions. These formulas work for any \( D \), though we are mainly interested in the case \( D = 10 \).

**Appendix C. Useful local transformations**

**C.1. Weyl transformation**

Let us consider the transformations

\[ g_{\mu \nu} \rightarrow e^{-2 \omega} g_{\mu \nu}, \]  
\[ e^\mu_{\tilde{\mu}} \rightarrow e^{-\omega} e^\mu_{\tilde{\mu}}, \]  
\[ \Gamma^\mu \rightarrow e^{\omega} \Gamma^\mu, \]  
\[ A_\mu \rightarrow A_\mu, \]  
\[ A_A \rightarrow e^{\omega} A_A, \]  
\[ \Psi \rightarrow e^{\frac{1}{2} \omega} \Psi, \]  
(\text{C.1})

where \( \omega = \omega(x^\mu) \) is a real function of \( x^\mu \).

Then the following quantities entering the action transform as:

\[ D_{\mu} A^A \rightarrow e^{\omega} (D_{\mu} A^A + \partial_{\mu} \omega A^A), \]  
(\text{C.2})

\[ g^{\mu \nu} \text{tr}(D_{\mu} A^A D_{\mu} A_A) \rightarrow e^{4 \omega} g^{\mu \nu} \text{tr}(D_{\mu} A^A D_{\mu} A_A + 2 \partial_{\mu} \omega (D_{\mu} A^A) A_A + \partial_{\mu} \omega \partial_{\mu} \omega A^A A_A) \]  
\[ = e^{4 \omega} g^{\mu \nu} \text{tr}(D_{\mu} A^A D_{\mu} A_A + \partial_{\mu} \omega D_{\mu} (A^A A_A) + \partial_{\mu} \omega \partial_{\mu} \omega A^A A_A). \]  
(\text{C.3})
According to the definition

\[ D_\mu \Psi = \partial_\mu \Psi + i [A_\mu, \Psi] + \frac{1}{4} \omega_{\mu \nu} \Gamma^{\nu} \hat{\rho} \Psi, \]  

(C.4)

where the spin connection is

\[ \omega_{\mu \nu} = \frac{1}{2} e_\nu (\partial_\mu e_\rho - \partial_\rho e_\mu) - (\hat{\nu} \leftrightarrow \hat{\rho}), \]  

(C.5)

these objects transform as follows:

\[ \omega_{\mu \nu} \rightarrow \omega_{\mu \nu} + (e_\mu e_\nu' - e_\nu e_\mu') \partial_\nu \omega, \]  

(C.6)

\[ D_\mu \Psi \rightarrow e^{\frac{1}{2} \omega} \left( D_\mu + \frac{3}{2} \partial_\mu \omega + \frac{1}{4} \partial_\nu \omega (e_\mu e_\nu' - e_\nu e_\mu') \Gamma^{\nu} \hat{\rho} \right) \Psi, \]  

\[ = e^{\frac{1}{2} \omega} \left( D_\mu + \frac{1}{2} \partial_\nu \omega (\Gamma^{\nu} \mu + 3 \delta^{\nu} \mu) \right) \Psi. \]  

(C.7)

Therefore

\[ \Gamma^{\mu} D_\mu \Psi \rightarrow e^{\frac{5}{2} \omega} \Gamma^{\mu} D_\mu \Psi, \]  

(C.8)

and

\[ \Psi \Gamma^{\mu} D_\mu \Psi \rightarrow e^{4 \omega} \Psi \Gamma^{\mu} D_\mu \Psi. \]  

(C.9)

The action (2.8) is invariant under the Weyl transformation (C.1), if we also transform the couplings as

\[ d^{\mu AV} \rightarrow e^{3 \omega} d^{\mu AV}, \]  

\[ d^{\mu BA} \rightarrow e^{2 \omega} d^{\mu BA}, \]  

\[ d^{BCA} \rightarrow e^{\omega} d^{BCA}, \]  

\[ m^{AB} \rightarrow e^{2 \omega} m^{AB} + 2 e^{2 \omega} g^{\mu \nu} (- \partial_\mu \omega \partial_\nu \omega + a^{-1} D_\mu (a \partial_\mu \omega)) \delta^{AB}, \]  

\[ M \rightarrow e^{\omega} M. \]  

(C.10)

C.2. \( SO(6)_R \) transformation

Let us consider a local \( SO(6)_R \) transformation

\[ A_A \rightarrow O_A^B A_B, \]  

(C.11)

\[ \Psi \rightarrow O \Psi, \]  

(C.12)

where \( O = (O_A^B) \in SO(6) \) and \( O \) is the corresponding \( SO(6) \) element in the spinor representation of \( SO(1, 9) \) acting on the fermion. This implies

\[ D_\mu A_A \rightarrow O_A^B (D_\mu A_C + (O^{-1} \partial_\mu O)_B^C A_C), \]  

(C.13)

\[ g^{\mu \nu} \text{tr}(D_\mu A_A D_\mu A_A) \rightarrow g^{\mu \nu} \text{tr}(D_\mu A_A D_\mu A_A + 2 (O^{-1} \partial_\mu O)^{AC} (D_\mu A_A) A_C}. \]
\[ - (O^{-1} \partial_{\mu'} OO^{-1} \partial_{\mu'} O)^{AB} A_A A_B), \quad \text{(C.14)} \]
\[ D_\mu \Psi \rightarrow \mathcal{O}(D_\mu \Psi + O^{-1} \partial_{\mu} O\Psi). \quad \text{(C.15)} \]

Then, the action (2.8) is invariant if we also transform the couplings as

\[ d^{\mu \nu A} \rightarrow d^{\mu \nu B}(O^{-1})_B^A, \quad \text{(C.16)} \]
\[ d_{\mu B}^A \rightarrow O_B^D d_{\mu B}^{A'} (O^{-1})_D^{A'} + (O \partial_{\mu} O^{-1})_B^A, \quad \text{(C.17)} \]
\[ d^{ABC} \rightarrow d^{A'B'C'} (O^{-1})_C^A (O^{-1})_D^B (O^{-1})_E^C, \quad \text{(C.18)} \]
\[ m_A^B \rightarrow O_A^D m_{A'}^B (O^{-1})_D^B - 2g^{\mu \nu} (O \partial_{\mu} O^{-1} O \partial_{\nu} O^{-1})_A^B \]
\[ - 2g^{\mu \nu} (O_A^{A'} d_{\mu B}^{A'} \partial_{\nu} (O^{-1})_B^B - \partial_{\mu} O_A^{A'} d_{\nu B}^{A'} (O^{-1})_B^B), \quad \text{(C.19)} \]
\[ M \rightarrow \mathcal{O} M O^{-1} - \Gamma^{\mu} O \partial_{\mu} O^{-1}. \quad \text{(C.20)} \]

Since \( d_{\mu A}^B \) behaves as the gauge field of the \( SO(6) \) symmetry, it is useful to define the covariant derivative including the \( SO(6) \) gauge field:

\[ \hat{D}_\mu A_A = D_\mu A_A + d_{\mu A}^B A_B, \quad \text{(C.21)} \]
\[ \hat{D}_\mu \Psi = D_\mu \Psi + \frac{1}{4} d_{\mu AB} \Gamma_{AB}, \quad \text{(C.22)} \]

which transform as

\[ \hat{D}_\mu A_A \rightarrow O_B^D \hat{D}_\mu A_B, \quad \text{(C.23)} \]
\[ \hat{D}_\mu \Psi \rightarrow \mathcal{O} \hat{D}_\mu \Psi. \quad \text{(C.24)} \]

The action (2.8) can be written as

\[ S = \int d^4x \sqrt{-g} a \text{tr} \left\{ - \frac{1}{2} F_{\mu \nu} F^{\mu \nu} - (\hat{D}_\mu A_A)^2 + \frac{1}{2} [A_A, A_B]^2 + i(\overline{\Psi} \gamma_\mu \hat{D}_\mu \Psi + \overline{\Psi} \gamma^A i[\Psi, A_A]) \right. \]
\[ + \left. C \frac{4}{4} \epsilon^{\mu \nu \rho \sigma} F_{\mu \nu} F_{\rho \sigma} - d^{\mu \nu A} F_{\mu \nu} A_A - d^{BCA} i[A_B, A_C] A_A - \frac{\hat{m}_{AB}}{2} A_A A_B - i \overline{\Psi} \hat{M} \Psi \right\}, \quad \text{(C.25)} \]

where

\[ \hat{m}_{AB} = m_{AB} + 2g^{\mu \nu} d_{\mu}^{A'A'} d_{\nu}^{B' B}, \quad \text{(C.26)} \]
\[ \hat{M} = M + \frac{1}{4} \Gamma^{\mu AB} d_{\mu AB}. \quad \text{(C.27)} \]

\( \hat{m}_{AB} \) and \( \hat{M} \) transform as

\[ \hat{m}_A^B \rightarrow (O \hat{m} O^{-1})_A^B, \quad \text{(C.28)} \]
\[ \hat{M} \rightarrow \mathcal{O} \hat{M} O^{-1}. \quad \text{(C.29)} \]
Appendix D. Derivation of the SUSY conditions

D.1. Derivation of Eqs. (2.15) and (2.16)

First, the variation of the action (2.1) is given by

\[\delta S = \int d^4x \sqrt{-g} \, a \text{ tr} \left\{ \delta A_I \left[ -2g^{II}g^{JJ'}D_JF_{J'} + (-2a^{-1}D_\mu(ad^\mu A) - m^A) A_A \right. \right. \\
\left. \left. + \left( -2g^{II}g^{\mu K}a^{-1}\partial_\mu a - a^{-1}\partial_\nu(ac) e^{\nu JK} - 3d^{[IJK]} \right) F_{JK} \right] + 2i \delta \overline{\Psi} \left( \Gamma^I D_I \Psi - \tilde{M} \Psi \right) - \overline{\Psi} \Gamma^I \left[ \delta A_I, \Psi \right] \right\} + (\text{total derivative terms}), \]

(D.1)

with the understanding that \(d^{JKI}\) and \(m^A\) can be non-zero only if \(I = 4–9\), and \(e^{IJK}\) can be non-zero only for \(I, J, K, L = 0–3\). Here, we have used

\[\delta F_{IJ} = D_I \delta A_J - D_J \delta A_I, \quad \epsilon^{\mu \nu \rho \sigma} D_\nu F_{\rho \sigma} = 0, \quad \overline{\Psi} \Gamma^I \delta \Psi = -\delta \overline{\Psi} \Gamma^I \Psi, \]

and defined

\[d^{[IJK]} \equiv \frac{1}{3} (d^{JKI} + d^{IKJ} + d^{KJI}), \]

(D.3)

\[\tilde{M} \equiv M - \frac{1}{2} \Gamma^\mu \partial_\mu \log a. \]

(D.4)

The SUSY variation of the action with respect to the transformations (2.14) is given by

\[\delta_\epsilon S = \int d^4x \sqrt{-g} \, a \text{ tr} \left\{ (i \epsilon \Gamma_I \Psi) \left[ -2g^{II}g^{JJ'}D_JF_{J'} + (-2a^{-1}D_\mu(ad^\mu A) - m^A) A_A \right. \right. \\
\left. \left. + \left( -2g^{II}g^{\mu K}a^{-1}\partial_\mu a - a^{-1}\partial_\nu(ac) e^{\nu JK} - 3d^{[IJK]} \right) F_{JK} \right] + i \epsilon (-F_{JK} \Gamma^{JK} + A_A B^A) \left( \Gamma^I D_I \Psi - \tilde{M} \Psi \right) \right\} + (\text{total deriv. terms}), \]

(D.5)

where we have used the identity\(^{19}\)

\[\text{tr}(\overline{\Psi} \Gamma^I [\delta_\epsilon A_I, \Psi]) = 0. \]

(D.6)

We would like to find a condition under which the integrand of Eq. (D.5) is a total derivative. Our ansatz for the total derivative is

\[S_{\text{der}} = \int d^4x \sqrt{-g} \, a \text{ tr} \left\{ D_I (i \epsilon A^{IJK} \Psi F_{JK}) + D_I (i \epsilon B^{JA} \Psi A_A) \right\}, \]

(D.7)

\(^{19}\) See Appendix 4.A of Ref. [34] for a proof of this identity.
where $A^{IJK}$ and $B^{IA}$ are $32 \times 32$ matrices with $A^{IJK} = -A^{JKI}$ to be determined. This total derivative term can be expanded as

$$
S_{\text{der}} = \int d^4x \sqrt{-g} \left\{ i\tilde{\epsilon} A^{IJK} (\Psi(D_I F_{JK}) + (D_I \Psi) F_{JK}) + iD_I (\tilde{\epsilon} B^{I\bar{A}}) \Psi A_{\bar{A}} 
+ i \left( \tilde{\epsilon} B^{[IJK]} + D_I (\tilde{\epsilon} A^{IJK}) \right) \Psi F_{JK} + i\tilde{\epsilon} B^{I\bar{A}} (D_I \Psi) A_{\bar{A}} \right\}.
$$

(D.8)

Comparing Eqs. (D.5) and (D.8), we obtain the following conditions:

$$
\tilde{\epsilon} \Gamma^I_l (-g^{IJ} g^{JK} + g^{LK} g^{JI}) a = \tilde{\epsilon} (A^{IJ} + E^{IJ}) = \tilde{\epsilon} A^{IJK},
$$

(D.9)

$$
\tilde{\epsilon} \Gamma^I_l \left( (-g^{IJ} g^{\mu K} + g^{KJ} g^{IJ}) \partial_\mu a + \partial_\nu (ac) \epsilon^{I\nu JK} - 3a d^{IJK}\right) + a \tilde{\epsilon} \Gamma^J K \tilde{M} = \tilde{\epsilon} B^{IJK} + D_I (\tilde{\epsilon} A^{IJK}),
$$

(D.10)

$$
\tilde{\epsilon} \Gamma^I_l \left( -\tilde{\epsilon} B^{I\bar{A}} \Gamma^\bar{A} \right) = \tilde{\epsilon} B^{IA},
$$

(D.11)

$$
\tilde{\epsilon} \Gamma^I_l \left( -2D_J (a d^{IJA}) - a m^A \right) - a \tilde{\epsilon} \tilde{B}^{IA} \tilde{M} = D_I (\tilde{\epsilon} A^{IJK}),
$$

(D.12)

In Eq. (D.9), $E^{IJK}$ is a $32 \times 32$ matrix that is totally anti-symmetric with respect to $I, J, K$, which can be added because of the Bianchi identity:

$$
D_I F_{JK} + D_J F_{KI} + D_K F_{IJ} = 0.
$$

(D.13)

Equations (D.10) and (D.12) determine $\tilde{\epsilon} A^{IJK}$ and $\tilde{\epsilon} B^{IA}$, respectively. It is easy to see that Eq. (D.9) is satisfied with $E^{IJK} = a \Gamma^{IJK}$, using the identity

$$
\Gamma^{IJK} \Gamma^I = \Gamma^{IJK} + g^{JK} \Gamma^I - g^{IJ} \Gamma^K.
$$

(D.14)

Then, using Eqs. (D.10) and (D.12), Eq. (D.11) becomes

$$
\tilde{\epsilon} \left( (-\Gamma^J g^{\mu K} + \Gamma^K g^{IJ}) \partial_\mu a + \partial_\nu (ac) \epsilon^{I\nu JK} - 3a d^{IJK}\right) + a \tilde{\epsilon} \Gamma^J K \tilde{M}
= \tilde{\epsilon} \Gamma^I_l - D_I (\epsilon (\Gamma^{IJK} \Gamma^I)),
$$

(D.15)

which is equivalent to

$$
(D_K \tilde{\epsilon}) \Gamma^{IJK} = \tilde{\epsilon} \left( \tilde{B}^{I\bar{A}} \Gamma^{\bar{A}} \Gamma^I - \Gamma^{IJ} a^{-1} \partial_\mu a + (a^{-1} \partial_\nu (ac) \epsilon^{I\nu JK} + 3 a d^{IJK}) \Gamma^K - \Gamma^{IJ} \tilde{M} \right).
$$

(D.16)

This is the condition (2.15). Similarly, Eq. (D.13) becomes

$$
\tilde{\epsilon} \Gamma^I_l \left( -2D_J (a d^{IJA}) - a m^A \right) - a \tilde{\epsilon} \tilde{B}^{IA} \tilde{M} = D_I (a \tilde{\epsilon} \tilde{B}^{I\bar{A}}) \Gamma^I,
$$

(D.17)

which is equivalent to

$$
a^{-1} D_I (a \tilde{\epsilon} \tilde{B}^{I\bar{A}}) \Gamma^I = \tilde{\epsilon} \left( -2a^{-1} \Gamma_I D_J (a d^{IJA}) - m^{AB} \Gamma_B - \tilde{B}^{IA} \tilde{M} \right).
$$

(D.18)

This gives Eq. (2.16).
D.2. Derivation of Eqs. (2.17)–(2.20)

Using the identity

$$\Gamma^U \Gamma^K \left( \frac{1}{16} \Gamma_{[ulg]L} - \frac{7}{16 \times 54} \Gamma_{UL} \Gamma_L \right) = \delta^K_L,$$

(D.20)

we rewrite Eq. (2.15) as

$$D_L \tilde{\varepsilon} = \tilde{\varepsilon} C^{IJ} \left( \frac{1}{16} \Gamma_{[ulg]L} - \frac{7}{16 \times 54} \Gamma_{UL} \Gamma_L \right),$$

(D.21)

where

$$C^{IJ} = B^{[IJ]} - \Gamma^{IJ} \mu^{-1} \partial_\mu a + (a^{-1} \partial_\nu (ac) \epsilon^{\nu ljk} + 3 d^{[ljk]}) \Gamma^K - \Gamma^{IJK} \tilde{M}.$$  

(D.22)

Inserting Eq. (D.21) back into Eq. (2.15), we obtain

$$\tilde{\varepsilon} C^{IJ} = \tilde{\varepsilon} C^{I'I'J'} P_{I'J'}^{IJ},$$

(D.23)

where we have defined

$$P_{I'J'}^{IJ} = \left( \frac{1}{16} \Gamma_{[I'g]J']K} - \frac{7}{16 \times 54} \Gamma_{I'J'} \Gamma_K \right) \Gamma^I \Gamma^K \left( \frac{1}{16} \Gamma gJ - \frac{7}{16 \times 54} \Gamma_{IJ} \Gamma_L \right)$$

$$= \frac{1}{72} \Gamma_{I'J'} \Gamma^K + \frac{1}{4} \Gamma [I'gJ].$$

(D.24)

One can check that $P_{I'J'}^{IJ}$ is a projection operator satisfying

$$P_{IJ} P_{I'J'}^{KL} = P_{IJ}^{KL},$$

(D.25)

and

$$G^{[IJ} \Gamma^{J]} P_{I'J'}^{IJ} = G^{[IJ} \Gamma^{J]}, \quad P_{I'J'}^{IJ} G_{[IJ]} = G_{[IJ]} \Gamma^{J]},$$

(D.26)

with arbitrary $G^I$. Therefore, if $\tilde{\varepsilon} C^{IJ}$ can be written as

$$\tilde{\varepsilon} C^{IJ} = \tilde{\varepsilon} G^{[IJ]},$$

(D.27)

with some $G^I$, Eq. (D.23) is satisfied. Conversely, if Eq. (D.23) is satisfied,

$$G^I = \frac{1}{72} (C^{I'I'J'} \Gamma_{I'J'}) \Gamma^I + \frac{1}{4} C^{IJ} \Gamma_I,$$

(D.28)

satisfies Eq. (D.27). It can also be shown, using Eq. (D.26), that $\nu^{I'J'} P_{I'J'}^{IJ} = 0$ is equivalent to $\nu^{IJ} \Gamma_J = 0$.

Then, using the property (D.26), one can easily show

$$C^{I'I'J'} P_{I'J'}^{IJ} - C^{IJ} = \left( \frac{a^{-1} \partial_\nu (ac) \epsilon^{\nu ljk} + 3 d^{[ljk]}}{\Gamma^{IJK}} \Gamma_{JK} - \Gamma^{IJK} \tilde{M} \right) \left( P_{I'J'}^{IJ} - \delta^I_{I'} \delta^J_{J'} \right).$$

(D.29)
Therefore, Eq. (D.23) can be written as

\[ 0 = \tau \left( (a^{-1} \partial_v (ac) e^{\nu J K'}) + 3 d^{[IJ]} \Gamma_K' - \Gamma_{IJ}^J \tilde{M} \right) \left( P_{I'J'}^{IJ} - \delta_{I'}^I \delta_{J'}^J \right). \]  

(D.30)

In this equation, \( \tilde{M} \) can be replaced with \( M \), because of the relation (D.26).

When \( M \) is expanded as

\[ M = \frac{m_{IJK}}{\Gamma_{IJK}}, \]  

(D.31)

it satisfies

\[ \Gamma^{I'J'M} = \frac{M I'J'}{6 m_{IJK}} \left( g^{I'K'} \Gamma_I'J' - g^{J'K'} \Gamma_I'J' + 4 g^{I'J'} g^{J'K'} \Gamma_K' \right). \]  

(D.32)

Therefore, Eq. (D.30) is equivalent to

\[ 0 = \tau e^{I'J'K'} \Gamma_K' \left( P_{I'J'}^{IJ} - \delta_{I'}^I \delta_{J'}^J \right), \]  

(D.33)

where

\[ e^{IJK} \equiv a^{-1} \partial_v (ac) e^{IJK} + 3 d^{[IJK]} + 24 m^{IJK}. \]  

(D.34)

This is Eq. (2.17).

Next, we write Eq. (D.21) as

\[ DJ \bar{\epsilon} = \bar{\epsilon} E_J, \]  

(D.35)

where

\[ E_J \equiv \bar{B}^I \left( \frac{1}{4} g_{IJ} - \frac{1}{24} \Gamma^I_A \Gamma_J^A \right) - \frac{1}{12} \Gamma^{\mu} \Gamma_I^J \partial_\mu \log a \\
+ e^{I'J'K'} \left( \frac{1}{16} g_{IJ} \Gamma_{I'J'} - \frac{7}{16 \times 54} \Gamma_{I'J'K'} \Gamma_J^A \right) - m^{I'J'K'} \left( 3 g_{I'J'} \Gamma_K' \Gamma_A^I - \frac{1}{3} \Gamma_{I'J'K'} \Gamma_J^A \right), \]  

(D.36)

Equation (D.35) with \( J = A \) implies

\[ 0 = \tau \left( \bar{B}^I \left( \frac{1}{4} \delta_A^I - \frac{1}{24} \Gamma_A^I \Gamma^A \right) - \frac{1}{12} \Gamma^{\mu} \Gamma_I^A \partial_\mu \log a \\
+ e^{I'J'K'} \left( \frac{1}{16} \delta_A^I \Gamma_{I'J'} - \frac{7}{16 \times 54} \Gamma_{I'J'K'} \Gamma^A \right) - m^{I'J'K'} \left( 3 \delta_A^I \Gamma_K' \Gamma_A^I - \frac{1}{3} \Gamma_{I'J'K'} \Gamma^A \right) \right). \]  

(D.37)

This equation can be written as

\[ \tau \left( \bar{B}^A + \left( \frac{1}{4} e^{AK'} - 12 m^{AK'} \right) \Gamma_K' \right) = \tau F^A, \]  

(D.38)

where

\[ F^A \equiv \frac{1}{6} \bar{B}^A \Gamma_A^I + \frac{7}{4 \times 54} e^{I'J'K'} \Gamma_{I'J'K'} - \frac{4}{3} m^{I'J'K'} \Gamma_{I'J'K'} + \frac{1}{3} \Gamma^{\mu} \partial_\mu \log a. \]  

(D.39)
This is Eq. (2.19). Multiplying Eq. (D.37) by $\Gamma_A$, we obtain

$$0 = \varepsilon \left( \frac{1}{72} e^{IJK} \Gamma_{IJK} - \frac{1}{2} \gamma^\mu \partial_\mu \log a - \left( \frac{1}{16} e_{\mu IJK} - 3m_{\mu IJK} \right) \Gamma^\mu_{IJK} - M \right), \tag{D.40}$$

which is Eq. (2.18).

With Eq. (D.40), the explicit form of $F$ in Eq. (D.39) is not needed anymore, because Eq. (D.40) implies that Eq. (D.39) can be replaced with

$$F = \frac{1}{6} \left( \bar{B}^4 + \left( \frac{1}{4} e^{AK'I'} - 12m^{AK'I'} \right) \Gamma_{K'I'} \right) \Gamma_A, \tag{D.41}$$

which can be obtained by contracting Eq. (D.38) with $\Gamma_A$.

Equation (D.35) with $J = \mu$ is written as

$$D_\mu \varepsilon = \varepsilon \left( -\frac{1}{4} F \Gamma_\mu + \left( \frac{1}{16} e_{\mu IJ} - 3m_{\mu IJ} \right) \Gamma^{IJ} \right). \tag{D.42}$$

This equation is equivalent to Eq. (2.20).

**Appendix E. On $e^{IJK}$**

**E.1.** $e^{I\alpha a} = 0 \Rightarrow e^{IJK} = 0$

In this section, we split the 10D gamma matrices $\{\Gamma^I\}$ into two sectors $\{\Gamma^\alpha\}$ and $\{\Gamma^a\}$, where $I, J, K = 0, 1, \ldots, 9$; $\alpha, \beta, \gamma = 0, 1, \ldots, d - 1$; and $a, b, c = d, d + 1, \ldots, 9$, with $0 < d < 10$. For simplicity, the 10D metric is assumed to be the flat Minkowski metric, though generalization to a curved metric is straightforward.

Here, we prove the following statement: Suppose that $e^{IJK}$ with mixed indices such as $e^{\alpha\beta a}$ and $e^{a\alpha b}$ are all zero. Then, the condition (2.17) implies that all the components of $e^{IJK}$ vanish when $d \neq 3, 7$. This also holds for the case with $d = 7$, which will be shown separately in Appendix E.2.

The condition (2.17) is equivalent to

$$0 = \varepsilon \left( e^{IJ'K'} \Gamma_{IJ'K'} \Gamma^{IJ} + 9 e^{K'I'J} \Gamma_{K'I'} \Gamma^J - 9 e^{K'IJ} \Gamma_{K'J} \Gamma^I - 72 e^{IJK'} \Gamma_{K'} \right). \tag{E.1}$$

When $e^{\alpha\beta a} = 0$ and $e^{a\alpha b} = 0$, this condition for $(I, J) = (\alpha, \beta)$, $(I, J) = (a, b)$, and $(I, J) = (a, a)$ becomes, respectively,

$$0 = \varepsilon \left( e^{IJ'K'} \Gamma_{IJ'K'} \Gamma^{IJ} + 9 e^{\alpha\beta a} \Gamma_{\gamma a} - 9 e^{\beta\gamma a} \Gamma_{\gamma a} - 72 e^{\alpha\beta a} \Gamma_{\gamma a} \right), \tag{E.2}$$

$$0 = \varepsilon \left( e^{IJ'K'} \Gamma_{IJ'K'} \Gamma^{IJ} + 9 e^{b\alpha c} \Gamma_{b\gamma a} - 9 e^{b\gamma a} \Gamma_{b\gamma a} - 72 e^{b\alpha c} \Gamma_{b\gamma a} \right), \tag{E.3}$$

$$0 = \varepsilon \left( e^{IJ'K'} \Gamma_{IJ'K'} \Gamma^{IJ} + 9 e^{bc a} \Gamma_{b\gamma a} - 9 e^{b\gamma a} \Gamma_{b\gamma a} - 72 e^{bc a} \Gamma_{b\gamma a} \right), \tag{E.4}$$

respectively. Multiplying Eq. (E.2) by $\Gamma_{\alpha\beta}$, we obtain

$$0 = \varepsilon \left( d(d - 1) e^{abc} \Gamma_{abc} + (9 - d)(10 - d) e^{\alpha\beta a} \Gamma_{\alpha\beta a} \right), \tag{E.5}$$

and we have

$$\varepsilon e^{IJK} \Gamma_{IJK} = \frac{18(d - 5)}{d(d - 1)} \varepsilon e^{\alpha\beta a} \Gamma_{\alpha\beta a} = \frac{18(5 - d)}{(9 - d)(10 - d)} \varepsilon e^{abc} \Gamma_{abc}. \tag{E.6}$$
Inserting this equation into Eqs. (E.2) and (E.3), we obtain

\[ 0 = \bar{\epsilon} \left( \frac{2(5-d)}{(9-d)(10-d)} e^{a'b'c'} \Gamma_{a'b'c'}^{ab} + e^{b'c'} \Gamma_{b'c'}^{a} \Gamma^{a} - e^{b'c'} \Gamma_{b'c'}^{a} \Gamma^{b} - 8 e^{abc'} \Gamma^{c'} \right). \] (E.7)

Multiplying Eq. (E.7) by \(\Gamma_{b}^{b}\), we obtain

\[ 0 = \bar{\epsilon} \left( e^{a'b'c'} \Gamma_{a'b'c'}^{ab} + (10-d) e^{b'c'} \Gamma_{b'c'}^{a} \Gamma^{a} - (10-d) e^{b'c'} \Gamma_{b'c'}^{a} \Gamma^{b} \right). \] (E.8)

which implies

\[ 0 = \bar{\epsilon} \left( 2 e^{a'b'c'} \Gamma_{a'b'c'}^{ab} + (10-d) e^{b'c'} \Gamma_{b'c'}^{a} \Gamma^{a} - 2(9-d) e^{b'c'} \Gamma^{c'} \right). \] (E.9)

Inserting this back into Eq. (E.7), we get

\[ 0 = \bar{\epsilon} \left( e^{b'c'} \Gamma_{b'c'}^{a} - e^{b'c'} \Gamma_{b'c'}^{a} \Gamma^{b} - 2(9-d) e^{b'c'} \Gamma^{c'} \right), \] (E.10)

From these equations, it is easy to show

\[ 0 = \bar{\epsilon} \left( e^{abc} e_{abc} \Gamma_{bc}^{b'c'} + (9-d) e^{abc} e_{abc} \right), \] (E.12)

\[ 0 = \bar{\epsilon} \left( (e^{a'b'c'} \Gamma_{a'b'c'}^{ab})^2 + (9-d)(10-d) e^{abc} e_{abc} \right). \] (E.13)

Using the identities

\[ \{\Gamma^{ab}, \Gamma_{a'b'}^{b'}\} = 2 \left( \Gamma_{a'b'}^{ab} - 2 \delta_{a'b'}^{[a} \delta_{b']}^{b} \right), \] (E.14)

\[ \{\Gamma^{abc}, \Gamma_{a'b'c'}^{b'}\} = 2 \left( 9 \Gamma_{[a}^{[ab} \delta_{c']}^{c] - 6 \delta_{a}^{[a} \delta_{b}^{b} \delta_{c}^{c]} \right). \] (E.15)

Eqs. (E.12) and (E.13) can be written as

\[ 0 = \bar{\epsilon} \left( e^{abc} e_{abc} \Gamma_{bc}^{b'c'} + (7-d) e^{abc} e_{abc} \right), \] (E.16)

\[ 0 = \bar{\epsilon} \left( 9 e^{abc} e_{abc} \Gamma_{bc}^{b'c'} + (12-d)(7-d) e^{abc} e_{abc} \right). \] (E.17)

These equations imply

\[ 0 = (3-d)(7-d) \bar{\epsilon} e^{abc} e_{abc}, \] (E.18)

and hence we have \(e^{abc} = 0\) for \(d \neq 3, 7\).

Then, Eqs. (E.2)–(E.4) with \(e^{abc} = 0\) imply

\[ \bar{\epsilon} e^{\alpha\gamma} \Gamma_{\alpha\gamma} = 0, \quad \bar{\epsilon} e^{\alpha\gamma} \Gamma_{\beta\gamma} = 0, \quad \bar{\epsilon} e^{\alpha\gamma} \Gamma_{\gamma} = 0. \] (E.19)

Multiplying the last equation by \(e^{\alpha\beta'\gamma'} \Gamma_{\gamma'}\) without summing over \(\alpha\) and \(\beta\), we obtain

\[ - (e^{\alpha\beta})^2 + \sum_{\gamma=1}^{d-1} (e^{\alpha\beta\gamma})^2 = 0 \] (E.20)

for all \(\alpha, \beta = 0, 1, \ldots, d-1\) to have non-zero \(\bar{\epsilon}\). Choosing \(\alpha = 0\), this equation implies \(e^{0\beta\gamma} = 0\) and then again using this equation, we see that all the components \(e^{\alpha\beta\gamma}\) must vanish.
E.2. Proof of $e^{IJK} = 0$ for the ISO(3) symmetric case

Here, we show that $e^{IJK} = 0$ is the only solution of Eq. (2.17) with $\tau \neq 0$ for the cases with ISO(3) symmetry considered in Sect. 4.2.1. Because of the ISO(3) symmetry, the allowed components of $e^{IJK}$ are $e^{ABC}$, $e^{0AB}$, and $e^{123}$, where $A, B, C = 4, \ldots, 9$. This is the case with $d = 7$ considered in Appendix E.1 by assigning the indices as $\alpha, \beta, \gamma = 0, 4, \ldots, 9$ and $a, b, c = 1, 2, 3$

The condition (2.17) with $(I, J) = (1, 2)$ implies

$$0 = \tau \left( \frac{1}{72} e^{J'K'} I_{J'K'} + \frac{1}{2} e^{123} \Gamma_{123} \right).$$

(E.21)

Using this equation, the condition (2.17) with $(I, J) = (0, 1)$ implies

$$0 = \tau \left( \frac{3}{4} e^{123} \Gamma_{123} + \frac{1}{8} e^{0AB} \Gamma_{0AB} \right).$$

(E.22)

Multiplying the condition (2.17) with $(I, J) = (1, C)$ by $\Gamma_{1C}$ and summing over $C = 4, \ldots, 9$, we obtain

$$0 = \tau \left( 3 e^{123} \Gamma_{123} + \frac{1}{8} e^{ABC} \Gamma_{ABC} \right).$$

(E.23)

Because $\Gamma_{ABC} \Gamma_{123}$ is an anti-symmetric matrix, this equation implies that $e^{123}$ is an eigenvalue of a real anti-symmetric matrix. This is possible only when

$$e^{123} = 0, \quad \tau e^{ABC} \Gamma_{ABC} = 0.$$  

(E.24)

These equations, together with Eqs. (E.21) and (E.22), imply

$$\tau e^{IJK} \Gamma_{IJK} = 0, \quad \tau e^{0AB} \Gamma_{AB} = 0.$$  

(E.25)

Using these results, the condition (2.17) with $(I, J) = (0, C)$ becomes

$$0 = \tau \left( -\frac{3}{4} e^{0BC} \Gamma_{0B} + \frac{1}{8} e^{ABC} \Gamma_{AB} \right).$$

(E.26)

Multiplying the condition (2.17) with $(I, J) = (B, C)$ by $\Gamma_B$ and summing over $B$, we obtain

$$\tau e^{ABC} \Gamma_{AB} = 0.$$  

(E.27)

This equation and Eq. (E.26) imply

$$\tau e^{0AB} \Gamma_B = 0.$$  

(E.28)

Multiplying this equation by $e^{0AC} \Gamma_C$, without summing over the index $A$, we obtain

$$0 = \tau \left( \sum_{B=4}^{6} (e^{0AB})^2 \right).$$

(E.29)

Therefore, we get $e^{0AB} = 0$ for all $A, B = 4, \ldots, 9$. Using this result and Eq. (E.27), the condition (2.17) with $(I, J) = (B, C)$ implies

$$\tau e^{ABC} \Gamma_A = 0.$$  

(E.30)

Again, multiplying this equation by $e^{A'BC} \Gamma_{A'}$, we obtain $e^{ABC} = 0$. In conclusion, all the components of $e^{IJK}$ are zero for this case.
References


