On the role of caustics in solar gravitational lens imaging

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We consider scattering of electromagnetic waves from a distant point source by the gravitational field of the sun, taking the field oblateness due to the quadrupole moment of the sun into account. The effects of field oblateness can play an important role in high-resolution solar gravitational lens imaging in the sub-micrometer wavelength range of the electromagnetic spectrum.

Subject Index A01, A13, E00, F10

1. Introduction

The idea of using the sun as a powerful telescope goes back to Eshleman [1]: The gravitational field of the sun acts as a spherical lens to magnify the intensity of radiation from distant objects along a semi-infinite focal line with the nearest point of observations being about 550 AU (for a general introduction, see, e.g., Refs. [3–5]). For example, the intensity from a distant point source of electromagnetic (EM) radiation at $\lambda = 1 \mu m$ wavelength can be pre-magnified by the solar gravitational lens up to $\mu \approx 10^{11}$ times. Depending on the observation device, the resolvable angle between two point sources at this wavelength could be as small as $10^{-10}$ arcsec.

Recently, the properties of the solar gravitational lens have attracted attention due to both the discovery of numerous exo- (and possibly earth-like) planets and the success of the Voyager-1 spacecraft, presently operating at about 140 AU. The possibilities of mega-pixel imaging of such planets from the focal line of the solar gravitational lens are now being discussed.

In the present work we consider the effects of the oblateness of the gravitational field and of the rotation of the sun on the image formation and the diffraction pattern of the lens. Although the quadrupole moment of the sun is very small, the effects of oblateness nevertheless turn out to be important: The focal line caustic unfolds and can be several hundred meters in cross section at distances up to several thousand AU from the sun. Moreover, for wavelengths in the micrometer range, the diffraction pattern of the point monochromatic source changes significantly and the maximum of the amplification of the EM energy flux radiated by such a source can decrease up to several orders of magnitude, depending on the direction of observation and the distance between the sun and an observer.

We stress that the last statement concerns the maximal magnification of density of flux radiated by a point source, and not the intensity magnification for a realistic extended object. Since the “focal blur” of the gravitational lens is comparable with the size of the whole image, there will be no...
significant difference in the flux magnification for realistic objects. For instance, for visible light from exo-planets of interest, the maximal pre-magnification of the flux by the solar gravitation is about $10^5 - 10^6$ [3], compared to $\mu \sim 10^{11}$ for a point source (see, e.g., Refs. [10,11]), and it is determined mainly by the geometrical optics. However, the effects caused by the field oblateness should be accounted for in the deconvolution of images of objects of interest: In the case of exo-planets, such effects can already be important at the hecto-pixel level of resolution of deconvolution (see Sect. 6).

Lensing by oblate objects has been extensively studied in the literature in general and (to a lesser extent) in connection with the solar gravitational lens. For instance, in Ref. [1] some heuristic considerations of effects caused by the field oblateness were presented. In Ref. [7] some estimates were also made that led to the conclusion that the oblateness of the sun has a negligible effect. This conclusion was drawn from the computation of differences in deflection angles in the sun’s equatorial and polar planes, which are based on a heuristic model of the gravitational field of the sun[^2]. Rigorous estimates of the deflection corrections due to the quadrupole moment of the sun have been done earlier, for instance, by Epstein and Shapiro [6]. The correct expression for the size of the aberration/caustic was derived by, e.g., Eshleman et al. in Ref. [2]. Our estimates are based on direct computations using methods of the uniform (caustic) expansions in the geometrical theory of diffraction (for a detailed introduction to these methods see, e.g., Ref. [17]), rather than on empirical approaches.

It is also worth noting that detailed studies of the wave-optical aspects of the solar gravitational lensing in the sub-micrometer diapason of the EM spectrum have been done mainly for the spherically symmetric case [8–11].

This work is organized as follows: In the next section we introduce notations and review geometrical optics of a spheroidal gravitational lens with small quadrupole moment. In Sect. 3 we consider the wave effects, deriving the gravitational point spread function in the form of a 1D integral and making a numerical evaluation of it. Three limiting cases when this integral can be computed analytically (the cases of observations in directions (1) close to the sun’s polar axis, (2) close to the sun’s equatorial plane, and (3) at the central axis of the caustic) are considered in Sect. 4. A compound system consisting of the gravitational lens and a telescope is considered in Sect. 5. A discussion of the role of caustic structure/size in prospective observations as well as suggestions for further studies are made in the concluding section. The main text of the paper is supplemented with two appendices: In Appendix A we double-check our results with the help of well known algorithms from the geometrical theory of diffraction and in Appendix B we consider the effects of refraction in the solar atmosphere (corona).

Concluding this section, we would like to mention that the article contains several examples where we estimate our results at the distance 550 AU from the sun. Although, due to the brightness of the solar corona, observations will be quite possible at distances $> \approx 1000$ AU rather than at 550 AU, our estimates are scalable: one gets the same estimate for bigger distance and proportionally smaller wavelength.

[^2]: In Ref. [7], a model of two spheres of half of the density offset by the distance comparable with that determined by the oblateness is used. This leads to an underestimate in the corrections to deflections by about three orders of magnitude in comparison with, e.g., computations made by Epstein and Shapiro [6]. The latter are based on a model with a correctly estimated quadrupole moment. As a consequence, the transverse aberration was also underestimated by about three orders of magnitude in Ref. [7]. The correct expression for the size of the aberration/caustic was derived by, e.g., Eshleman et al. in Ref. [2].
2. Geometrical optics problem

We are interested in a description of the diffraction pattern of an EM wave scattered by the gravitational field of the sun. For this we first review the geometrical optics counterpart of this problem: namely the deflection of initially parallel light rays coming from an infinitely distant point source.

The trajectory of light in the gravitational field of the sun can be found using post-Newtonian approximations for the null geodesics of the post-Minkowski metric element (see, e.g., Refs. [12–16]):

\[ ds^2 = \left( 1 + \frac{2\Phi}{c^2} \right) (c dt)^2 - \left( 1 - \frac{2\Phi}{c^2} \right) d\vec{r}^2 - \frac{8}{c^2} (\vec{A} \cdot d\vec{r}) dt, \]

where \( \Phi \) is the scalar (Newtonian) gravitational potential and \( \vec{A} \) is the gravitomagnetic vector potential.

In the asymptotically Cartesian heliocentric coordinate system where parallel beams are incoming from \( z = -\infty \), the post-Newtonian deflection angle \( \vec{\alpha} \), which is the difference between the incoming and outgoing beam direction vectors, equals the 2D gradient of the potential \( \Psi \):

\[ \vec{\alpha} = \nabla \Psi, \quad \nabla = (\partial_x, \partial_y), \quad \Psi = -\frac{2}{c^2} \int_{Z_{source} \rightarrow -\infty}^{Z} \left( \Phi(x, y, z) - \frac{2}{c^2} A_z(x, y, z) \right) dz, \tag{1} \]

where \( Z \) is the \( z \)-coordinate of an observer. The condition \( Z \gg \sqrt{x^2 + y^2} \gg r_g \), where \( r_g \approx 3 \times 10^3 \) m is the gravitational radius of the sun, is also imposed on Eq. (1). In this limit the 2D gradient of \( \Psi \) is independent of \( Z \): The potential \( \Psi \) is a sum of the \( x, y \)-dependent (\( Z \)-independent) and the \( Z \)-dependent (\( x, y \)-independent) terms (see below), so it is essentially a 2D potential.

Here, one can apply the thin-lens approximation, which leads to the following picture (see Fig. 1): a light ray is incoming from \( z = -\infty \) and hits the \( z = 0 \) “lens plane” at \((x, y)\). At this plane the ray is deflected by the angle given by the \( Z \rightarrow \infty \) limit in Eq. (1). Then, it follows that the outgoing ray intersects the observer plane \( z = Z > 0 \) at the point whose position \((X, Y)\) is determined by the extremum of the Fermat potential \( S \) (the “lens equation”):

\[ \partial_x S = 0, \quad \partial_y S = 0, \tag{2} \]

where

\[ S = \frac{(X - x)^2 + (Y - y)^2}{2Z} - \Psi(x, y). \tag{3} \]

This equation is a manifestation of the Fermat principle for the beam delay time \( S/c \).

For a compact lens, a combined contribution to the 2D potential \( \Psi \) in Eq. (1) from the dipole terms of \( \Phi \) and \( \vec{A} \) can be canceled by a translation. Since the gravitomagnetic field of the sun, produced by its rotation, is a dipole field, without loss of generality we can set \( \vec{A} = 0 \) (for more details see, e.g., Ref. [16]).

The exterior Newtonian potential of the sun can be approximated by that of the quadrupole:

\[ \Phi(\vec{r}) = -\frac{r_g c^2}{2r} \left[ 1 - \frac{I_2}{2} \left( \frac{R_0}{r} \right)^2 \left( \frac{3(\vec{n} \cdot \vec{r})^2}{r^2} - 1 \right) \right], \tag{4} \]

where \( \vec{n} \) is a unit vector in the direction of the polar axis of the sun, \( R_0 \approx 7 \times 10^8 \) m is the sun’s radius and \( I_2 \approx 2 \times 10^{-7} \) is its dimensionless quadrupole moment.
Fig. 1. Diagram of the geometrical optics problem. A section of the caustic surface by the observer $z = Z$ plane ($X$, $Y$ plane) is schematically shown on the right, while the corresponding critical line in the lens $z = 0$ plane ($x$, $y$ plane) is schematically shown on the left (see Eqs. (13), (14)).

Without loss of generality we select the coordinate system where

$$\vec{n} = (0, \sin \beta, \cos \beta)$$

(5)

with $\beta$ being the angle between the polar axis of the sun and the beams incoming from $z = -\infty$ (see Fig. 1).

Introducing the polar coordinates $(r_\perp, \phi)$ in the lens $z = 0$ plane,

$$x = r_\perp \cos \phi, \quad y = r_\perp \sin \phi, \quad r = \sqrt{r_\perp^2 + z^2},$$

and taking Eqs. (1), (4) and the fact that $Z \gg r_\perp$ into account we get the 2D potential $\Psi$:

$$\Psi = 2r_g \left( \log \frac{r_\perp}{r_g} - \frac{I_2 R_0^2 \sin^2 \beta}{2r_\perp^2} \cos 2\phi \right) + cT(Z).$$

(6)

The term $cT(Z) = r_g \log(-4Z_{source}/r_g^2)$ can be dropped without loss of generality: It does not affect the geometrical optical values since it is independent of $x, y$.

In order not to carry numerous constants through the computations, we rescale both the lens plane and the observer plane lengths with the scaling length parameter $b$:

$$b \equiv \sqrt{2r_g Z}.$$  

(7)

The new dimensionless polar coordinates $(\rho, \phi)$ in the lens plane and the dimensionless Cartesian coordinates $(\xi, \eta)$ in the observer plane then read

$$r_\perp = b \rho, \quad (x, y) = (b \rho \cos \phi, b \rho \sin \phi), \quad (X, Y) = (b \xi, b \eta).$$

(8)

3 For details of this simple computation one can also refer to, e.g., Ref. [15].

4 The above term can also be dropped in the wave-optical computations of the field intensity, which is the square of the absolute value of the complex field amplitude, since it contributes only to the common phase factor to the amplitude (see the next section).

5 We would like to stress that $b$ is not an impact parameter. The latter will be introduced further.
In these coordinates,

\[ \Psi = 2r_g \psi, \quad \psi = \log(\rho) - \frac{\epsilon}{\rho^2} \cos 2\phi, \quad (9) \]

where

\[ \epsilon = \frac{I_2 R_0^2 \sin^2 \beta}{4r_g Z} = \frac{I_2 R_0^2 \sin^2 \beta}{2b^2}. \quad (10) \]

In the case of the sun \( \epsilon \leq \) about \( 10^{-7} \). The small parameter \( \epsilon \) is maximal when \( \beta = \pi/2 \), i.e., when the source is placed in the sun’s equatorial plane. It decreases as the source is displaced towards the sun’s polar axis, on which it vanishes (at \( \beta = 0 \)): The light radiated by a source from the polar axis is deflected as if the sun were spherically symmetric. So, we will refer to both situations of \( I_2 = 0 \) and \( \beta = 0 \) as the “spherically symmetric”, “degenerate”, or “monopole” case. The parameter \( \epsilon \) also decreases when the observer plane moves away from the sun (i.e., as \( Z \) increases).

It is worth mentioning that one can also account for the light refraction in the solar plasma by adding a corresponding correction term to the potential \( \psi \). However, this contribution can be discarded for the sub-micrometer diapason of wavelengths. Evaluation of this contribution is given in Appendix B.

It follows from Eqs. (2), (3), (8), (9) that the coordinates \((\rho, \phi)\) of the images of the point \((\xi, \eta)\) are solutions of the lens equation, which has the following form in complex notation:

\[ \xi + i\eta = \left( \rho - \frac{1}{\rho} \right) e^{i\phi} - \frac{2\epsilon}{\rho^3} e^{3i\phi}. \quad (11) \]

We recall that the solution of this equation gives the “impact parameter” \( b \rho(\xi, \eta; \epsilon) \) and the corresponding polar angle \( \phi(\xi, \eta; \epsilon) \) in the lens plane for the ray(s) arriving at the observer plane at \( X = b\xi, Y = b\eta \). Both the scaling factor \( b \) and a small parameter \( \epsilon \) depend on the distance \( Z \) between the planes.

In the geometrical optics, the inverse intensity magnification equals the ratio of the corresponding surface elements on the observer and the lens planes (the Jacobian of transformation (11) or the Hessian of the Fermat potential \( S \)):

\[ \mu^{-1} = \left| \frac{\partial(X, Y)}{\partial(x, y)} \right| = \frac{1}{\rho} \left| \frac{\partial(\xi, \eta)}{\partial(\rho, \phi)} \right| = 1 - \frac{1}{\rho^4} - \frac{12\epsilon \cos 2\phi}{\rho^6} + \mathcal{O}(\epsilon^2). \quad (12) \]

The magnification \( \mu \) diverges at the critical line (see Fig. 1):

\[ \rho = \rho_c(\phi) = 1 + 3\epsilon \cos 2\phi + \mathcal{O}(\epsilon^2) \quad (13) \]

or, according to Eq. (11), at the astroid (tetracuspid) caustic\(^6\) in the observer plane:

\[ (\xi, \eta) = (\xi_c(\phi), \eta_c(\phi)) = 4\epsilon (\cos^3 \phi_s - \sin^3 \phi). \quad (14) \]

Here we recall the standard procedure of solving the lens equation (11).

\(^6\) According to the recent data the dimensionless octopole moment of the sun \( \sim 10^{-9} \). Since \( \epsilon \) is at most \( \approx 10^{-7} \), the caustic cross section can be considered as a pure astroid for our purposes.
Fig. 2. Top: Observer plane. Bottom: Lens plane. Arrows on the tangents indicate directions of the coming rays. A: Four point images of a point source viewed by an observer from the interior of the astroid. B: Observer is outside the astroid (two point images). C: Spans of the (“limb”) images of a small (centered at the z-axis) disk source viewed by an observer at the z-axis (“Einstein cross”). D: Spans of the strong and weak images of the same small disk source viewed by an observer near the cusp: The maximal angular span of the “strong limb” is approximately proportional to the cubic root of the ratio between the size of the heliocentric projection of the source to the observer plane and the size of the astroid (while the span of the “weak limb”, as well as the limbs of C, is linearly proportional to the above ratio). Note that the image of the disk source does not form a full ring if the apparent size of the disk is smaller than $d_{\text{astroid}}/2Z$.

When the deviation of the observer from the z-axis is much smaller than b, i.e., when $|\xi| \ll 1$ and $|\eta| \ll 1$, we have

$$\rho = 1 + \delta, \quad \delta \ll 1.$$ 

Then, from Eq. (11) it follows that $\xi + i\eta = 2\delta e^{i\phi} - 2\epsilon e^{3i\phi}$. Eliminating $\delta$ from the last expression we get

$$\xi \sin \phi - \eta \cos \phi = 2\epsilon \sin 2\phi.$$ (15)

The solutions $\phi$ of Eq. (15) are the “Einstein ring” coordinates of images of the point ($\xi, \eta$). It is not difficult to see that these are angles between the $\xi$-axis and the tangents to the astroid (14) drawn from the point ($\xi, \eta$) (see Fig. 2). When the point ($\xi, \eta$) is inside the astroid, Eq. (15) has four solutions. Otherwise it has two solutions. The magnification of the $j$th image $\mu_j$ is inversely proportional to the distance from the observer to the corresponding tangency point on the astroid (see also Appendix A).

In the $\epsilon = 0$ case the lens equation has two solutions when $\xi^2 + \eta^2 \neq 0$ or an infinite number of solutions forming a unit circle when $\xi = \eta = 0$. Since the number of solutions does not exceed four in the non-degenerate case, the image of a source of small (with respect to the astroid) apparent size never forms a ring\(^7\) if $\epsilon \neq 0$ (see Fig. 2).

\(^7\) Obviously, this does not imply that images of bigger sources, such as, e.g., exo-planets of interest, do not form rings.
Returning to the $\epsilon = 0$ case we note that the caustic (14) degenerates to the focal line $\xi = 0, \eta = 0$ and the deflected beams converge towards the $z$-axis at the “Einstein” angles $|\alpha| = \alpha_E(Z)$:

$$\alpha_E(Z) = \sqrt{\frac{2r_g}{Z}}. \tag{16}$$

The “on-axis” observer “sees” the whole critical line. In other words, the rays are coming from the circle of radius $b$ in the lens plane towards an observer at $X = Y = 0$ in the $z = Z$ plane. Therefore, $b > R_0$, where $R_0$ is the sun radius. The distance from the sun to the closest focal point is determined by the condition $b = R_0$ and equals $Z_{\text{min}} = R_0^2/(2r_g) \approx 550$ AU.

In the wave optics, the maximal spatial resolution of the spherical lens in the neighborhood of the focal line is restricted by the radius of diffraction (radius of the Airy disk), which is of the order of $\lambda/\alpha_E$, where $\lambda$ is the light wavelength [8–11]: The circularly symmetric diffraction pattern of a point source oscillates in the radial direction and its intensity reaches a maximum at the $z$-axis. The spatial scale of the oscillations is of the order of the diffraction radius. For $\lambda = 1 \mu$m, at the position of the closest observation $Z \approx 550$ AU, this radius is on the decimeter scale.

On the other hand, from Eqs. (7), (8), (10), (14) it follows that the non-spherical model produces an astroid caustic of the diameter:

$$d_{\text{astroid}} = 8\epsilon b = d_{\text{max}}\sqrt{\frac{Z_{\text{min}}}{Z}} \sin^2 \beta, \quad d_{\text{max}} = 4I_2R_0, \tag{17}$$

which reaches up to $d_{\text{max}} \approx 5.6 \times 10^2$ m when $\beta \to \pi/2$ and $Z \to Z_{\text{min}}$. Thus, the effects of oblateness clearly lead to significant changes in the diffraction pattern of the point source when it moves from the sun’s polar axis to the equatorial plane. Indeed, the maximum of magnification is now reached in the neighborhood of the astroid, where the geometrical optics magnification diverges, thousands of diffraction radii away from the $z$-axis, which is now non-singular.

We note that, as follows from Eq. (17), the size of the astroid is proportional to $Z^{-1/2}$, i.e., the size varies slowly with distance from the sun. For example, the maximal astroid diameter is about 400 m at 1000 AU, while the sizes of the heliocentric projection of possible objects of observation are about several kilometers across.

Before going to the detailed wave optics computations in the next section, it is worth mentioning some heuristic arguments explaining significant differences in the maximal EM energy flux amplification in the $\beta = 0$ and $\beta = \pi/2$ cases for point source and small wavelengths: In the $\epsilon = 0$ case (spherical lens) the lens plane image produced by a small distant source consists either of one single ring or two opposite arc-shaped “limbs” (one limb inside and another outside the critical curve $\rho = \rho_c = 1$). Whether the image is a ring or limbs, as well as the size of the limbs, depends on the source size and the observer position $X, Y$ in the $z = Z$ plane (the further away the observer is from the $z$-axis, the smaller the limbs are).

In contrast to the symmetric case, in the case when the lens caustic is the astroid (14) the image of a small source never forms the whole ring (see Fig. 2). Such an image consists of two to four disjoint small limbs, some of them being weak and some strong, depending on the observer’s position with

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8 For a beam grazing the edge of the sun $\alpha_\text{E} \approx 0.85 \times 10^{-5}$ rad $\approx 1.75$ arcsec.

9 Also note that $\epsilon = \epsilon_{\text{max}} \sin^2 (\beta)Z_{\text{min}}/Z$, where for the sun $\epsilon_{\text{max}} \approx 10^{-7}$.

10 By “small” we mean a source whose heliocentric projection to the observer plane has dimensions that are much smaller than the size of the astroid.
Fig. 3. Schematic illustration of the contraction of the principal Fresnel zones (gray) towards geometrical optics images of the point source as \( \lambda \to 0 \). The images are shown by the black circle for \( \epsilon = 0, X = Y = 0 \) (left) or by black dots for \( \epsilon \neq 0 \) (right). Directions of contraction are indicated by arrows. The observer is denoted by the cross. The critical line and caustic are schematically drawn by dashed lines. In the symmetric case (left) the maximal zone corresponds to the on-axis position of the observer. The zone is an annulus that contracts towards the circle as \( \lambda \to 0 \). In the \( \epsilon \neq 0, \lambda \ll \epsilon r_g \) case (right) the maximal zones correspond to positions near cusps. In this case the zones contract both radially and tangentially towards point images as \( \lambda \to 0 \). The angular span of the maximal zone in the \( \epsilon \neq 0, \lambda \ll \epsilon r_g \) case is proportional to \( (\lambda/\epsilon r_g) ^{1/4} \), since, e.g., near the cusp corresponding to \( \phi = 0, \phi = \delta \phi, \rho = \rho_c(0) + \delta \rho \) and the variation of the Fermat potential for an observer at the cusp equals \( \delta S \approx 2 r_g \delta \rho^2 + \epsilon r_g (3 \delta \rho \delta \phi^2 - 3 \delta \rho^2 \delta \phi^2 + \delta \phi^4/2) \). The radial dimension of zones is proportional to \( (\lambda Z)^{1/2} \).

respect to the caustic. When the source size goes to zero, the limbs become a set of (two to four) points. Considering the principal Fresnel zones\(^{11}\) around these points, one can explain the decrease of maximal magnification in the wave optics.

In more detail (see Fig. 3): the above zones have the form of “limbs”, whose dimensions depend on the size of the caustic \( \sim \epsilon b \), on the diffraction radius \( \sim \lambda/\alpha E \), and on the position of the observer: It is easy to see that the thickness (i.e., radial dimension) of the limb is approximately the same in the symmetric and non-symmetric cases when \( \epsilon \) is small.

However, in the spherically symmetric case even a point source can produce a circular geometrical optics image, and the maximal zone spans the whole circle even if \( \lambda \to 0 \) (the thickness of the maximal zone goes to zero while its angular span always equals \( 2\pi \)). In contrast to the symmetric case, in the \( \epsilon \neq 0 \) case the geometrical optics image of a point is always a point and therefore the maximal principal Fresnel zone contracts to a point as \( \lambda \to 0 \). This leads to decrease of the maximal EM energy flux amplification in comparison with the symmetric case\(^{12}\) (since the thicknesses of zones have practically the same dependence on \( \lambda \) when \( \epsilon \) is small).

\(^{11}\) Since \( S \) (modulo a \( Z \)-dependent term) equals the optical path, point \((x, y)\) belongs to the principal zone if \( |S(X, Y; x, y) - S(X, Y; x_i, y_i)| < \lambda/2 \), where \((x_i, y_i)\) are coordinates of an image.

\(^{12}\) An estimate of the maximal magnification, based on an evaluation of the dimensions of the Fresnel zone corresponding to the cusp of the astroid (14), was presented in Ref. [1] by analogy with focusing by atmosphere of an oblate planet [2]. There, the angular span (“horizontal dimension” in the terminology of Ref. [2]) of the maximal Fresnel zone was estimated by using the following assumption (Sect. 6 of Ref. [2]): The variation, along the half-span of the zone, of distance between the cusp and the involute of the astroid equals the “vertical” dimension (i.e., thickness) of the zone. From this it follows that the angular span of the zone is proportional
3. Diffraction optics and gravitational PSF

The geometrical theory of diffraction provides algorithms for finding the near-caustic intensity from its geometric optics asymptotics [13,17]. For caustics of the type (14) the near-field is expressed through the Airy function for caustic folds (see, e.g., Refs. [12,13,19]), and through the Pearcey integral near caustic cusps (see, e.g., Refs. [17,20]). In cases of focal lines the near-field is expressed through Bessel functions [17,19]. Application of these algorithms to our problem is presented in Appendix A.

Below, we derive the gravitational point spread function as a single 1D integral having all the above-mentioned limits.

The change of polarization angles of light in weak gravitomagnetic fields is of a post-post-Newtonian order and can be neglected (see, e.g., Refs. [14,18] and references therein). The deflection angles are also small and space is asymptotically flat, so we can apply the scalar Huygens–Fresnel principle (the Fresnel–Kirchhoff diffraction formula): In the thin-lens short-wavelength approximation ($Z \gg R_0 \gg r_g, \lambda \ll r_g, Z_{\text{min}}/r_g \approx 3 \times 10^{10}$) the diffraction field in the vicinity of the $z$-axis is a sum of contributions by spherical waves propagating from the lens plane $z = 0$ with the phase delays corresponding to the sum of the gravitational and geometric delays. This total time delay equals $S/c$, where $S$ is given by Eq. (3). Then, the complex amplitude of the electromagnetic (EM) field at the observer position equals (up to the phase factor $e^{ikZ}$)

$$u = \frac{k}{2\pi Z} \int e^{iS} dx dy, \quad k = 2\pi/\lambda. \quad (18)$$

The intensity magnification, i.e., amplification of the EM energy flux at the point $X, Y$ of the observer plane, equals the square of the absolute value of the amplitude\footnote{Dropping common phase factors, such as $e^{ikZ}$ in Eq. (18) etc., does not affect the value of the above gravitational magnification. Therefore, for simplicity, we will perform our computations of $u$ modulo common phase factors. Note that factor $e^{ikZ}$ cannot be dropped in Eq. (18) if one considers a combination of the gravitational lens with an optical device (see Sect. 5).}

$$\mu = |u|^2. \quad (19)$$

This function of $X, Y$ ($\mu$ also depends on $Z, \epsilon$, and $k$) is a point spread function (PSF) of a gravitational lens only, so we call it the gravitational PSF or GPSF (the PSF of a combination of a gravitational lens and a telescope is discussed in Sect. 5).

For a detailed derivation and justification of the Fresnel–Kirchhoff diffraction integral (18) see, e.g., Ref. [12] or Ref. [19]. In Eq. (18), the geometrical optics magnification (12) is recovered as $\lim_{k \to \infty} |u|^2$. When $kr_g \gg 1$ and the point on the observer $X, Y$ plane is far away from the caustic, the integral (18) can be expressed in a simple manner through the geometrical optics data as (see, e.g., Refs. [12,19])

$$u(X, Y) = \sum_j \sqrt{|\mu_j|} e^{i(kS_j - \pi n_j/2)}. \quad (19)$$

Here the sum is taken over the number of images in the lens plane, $\mu_j$ is the geometrical optics magnification of the $j$th image, $S_j = S(X, Y; x_j, y_j)$ is the extremal value of the Fermat potential (3) to $(\lambda/\epsilon^2 r_g)^{1/8}$ as $\lambda \to 0$, while from direct evaluation of the variation of the Fermat potential $S$ it follows that the span $\sim (\lambda/\epsilon r_g)^{1/4}$. 

for the $j$th image, and $n_j = 0, 1, 2$ corresponds to $x_j, y_j$ being the minimum, saddle, and maximum point respectively. The above approximation breaks down in the neighborhood of the caustic, the case that we are mainly interested in. So, one has to either evaluate Eq. (18) exactly or apply suitable asymptotic methods.

Up to a common $(X, Y, Z$-dependent) phase factor

$$u(\xi, \eta) = \frac{q}{2\pi} \int_{R_0/b}^{\infty} d\rho \int_0^{2\pi} \rho e^{iqV} d\phi, \quad V = \frac{\rho^2}{2} - \rho (\xi \cos \phi + \eta \sin \phi) - \psi(\rho, \phi),$$

(20)

where $q$ is the dimensionless wavenumber

$$q = 2kr_g = \frac{4\pi r_g}{\lambda}. \quad (21)$$

For $\lambda = 10^{-6}$ m, $q \approx 3.7 \times 10^{10}$.

The main purpose of this section is the direct numerical evaluation of the 2D integral (20). Before presenting the numerical results we would like to make several remarks:

Remark 1. It is worth noting that, in the $\epsilon = 0$ case, the exact 2D integration in Eq. (20) is possible:

$$u = \frac{q}{2\pi} \int \rho d\rho \int_0^{2\pi} e^{iq\left[\frac{\rho^2}{2} - \log(\rho) - \rho (\xi \cos \phi + \eta \sin \phi)\right]} d\phi = q \int \rho d\rho e^{iq\left[\frac{\rho^2}{2} - \log(\rho)\right]}J_0\left(q\rho\sqrt{\xi^2 + \eta^2}\right).$$

(22)

where $J_0$ is the zero-order Bessel function. After integration in $\rho$ one can express $\mu$ in terms of the confluent hypergeometric function (see, e.g., Ref. [19]):

$$\mu = |u|^2 = \frac{\pi q}{1 - e^{-\pi q}} |\text{1F1}(iq/2, 1; iq(\xi^2 + \eta^2)/2)|^2.$$

In the short-wavelength limit $q \gg 1$ and when the argument $iq(\xi^2 + \eta^2)/2$ of $\text{1F1}$ is small, i.e.,

$$\sqrt{\xi^2 + \eta^2} \ll \frac{1}{\sqrt{q}}, \quad (23)$$

the hypergeometric function $\text{1F1}$ degenerates to the zero-order Bessel function (see, e.g., Refs. [11,19]):

$$\mu = \pi q J_0^2\left(q\sqrt{\xi^2 + \eta^2}\right).$$

The maximum $\mu = \mu_0$ of the GPSF is reached at the focal line $\xi = \eta = 0$ and equals

$$\mu_0 = \pi q = \frac{4\pi^2 r_g}{\lambda}. \quad (24)$$

Remark 2. Condition (23) is, in fact, a condition of validity of the stationary phase integration at $\rho = 1$ in the last integral in Eq. (22). Indeed, the stationary phase approximation can be applied in Eq. (22) when the width of the stationary phase region $\delta \rho \sim 1/\sqrt{q}$ is much smaller than the scale of oscillations of the Bessel function $\delta \rho \sim 1/(q\sqrt{\xi^2 + \eta^2})$, which leads to Eq. (23).
Remark 3. Condition (23) will be encountered in the next section, when a similar type of the stationary phase integration will be performed for the general case $\epsilon \neq 0$: As follows from Eq. (20), in the general situation

$$u = q \int \rho e^{i q \left(\frac{\rho^2}{2} - \log(\rho)\right)} F(e q / \rho^2, q \xi \rho, q \eta \rho) d \rho, \quad (25)$$

where the function of three variables $F(\cdot, \cdot, \cdot)$ is defined as follows$^{14}$:

$$F(\chi, \kappa, v) = \frac{1}{2\pi} \int_0^{2\pi} e^{\chi \cos 2\phi - \kappa \cos \phi \sin \phi} d\phi.$$  

Like the $\epsilon = 0$ case, the stationary phase integration can be performed at $\rho = 1$ when all the arguments of $F$ in Eq. (25) are much smaller than $\sqrt{q}$, i.e., when

$$\epsilon \ll \frac{1}{\sqrt{q}} \quad (26)$$

and Eq. (23) holds. This will be demonstrated in detail in the next section.

Consider now the general case, i.e., the one when Eq. (23) is not necessarily true. Below we will proceed with the main subject of the present section, estimating the 2D integral (20) numerically without any assumptions.

Since $q \gg 1$ (e.g., $q \approx 3.7 \times 10^{10}$ for $\lambda = 10^{-6}$ m) one can reduce Eq. (20) to a 1D integral in $\phi$ by the stationary phase integration$^{15}$ in $\rho$ at fixed $\phi$: The stationary phase integration in $\rho$ produces a relative $O(1/q)$ error, which is negligible for wavelengths of interest.

First, one should find the “stationary phase line” $\rho = \rho_{st}(\xi, \eta; \phi)$ such that$^{16}$

$$\left(\frac{\partial V}{\partial \rho}\right)_{\rho=\rho_{st}} = \rho_{st} - \frac{1}{\rho_{st}} - \frac{\xi \cos \phi - \eta \sin \phi}{\rho_{st}^3} = 0. \quad (27)$$

Then, up to a common phase factor

$$u = \sqrt{\frac{q}{2\pi}} \int_0^{2\pi} \left(\frac{\partial^2 V}{\partial \rho^2}\right)^{-1/2}_{\rho=\rho_{st}(\phi)} \rho_{st}(\phi) \exp \left[i q V(\rho_{st}(\phi), \phi)\right] d\phi. \quad (28)$$

From Eq. (27) we obtain$^{17}$ that up to $O(\epsilon^2)$

$$\rho_{st}(\xi, \eta; \phi) = \frac{\sqrt{4 + \tau^2} + \tau}{2} + \frac{8 \epsilon \cos 2\phi}{(\sqrt{4 + \tau^2} + \tau)^2(\sqrt{4 + \tau^2} + \tau) \sqrt{4 + \tau^2}}; \quad \tau = \tau(\xi, \eta; \phi) = \xi \cos \phi + \eta \sin \phi. \quad (29)$$

---

$^{14}$ $F(\chi, \kappa, v)$ degenerates to $J_0$ in two special cases: 1) $\chi = 0$ (see Remark 1) and 2) $\kappa = \nu = 0$ (see Eq. (54)).

$^{15}$ For $q \gg 1$, $\int e^{iqf(x)} dx = x_i \sqrt{\frac{2\pi}{q f'(x_i)}} e^{iqf(x_i)} \left(1 + O \left(\frac{1}{q}\right)\right)$, where $f'(x_i) = 0$.

$^{16}$ It is not difficult to see that solutions of the lens equation (11) lie on this line. When the observer is far away from the caustic, the stationary phase integration in $\phi$ can also be performed around these points together with the above integration in $\rho$. Such a double stationary phase integration results in Eq. (19). This is not the case when the observer is in the neighborhood of the caustic since the second tangential derivative of $V$ vanishes on the critical line and one has to apply other methods for computing the integral in $\phi$.

$^{17}$ Equation (27) has two solutions: $\rho_1(\phi) = \rho_{st}(\phi)$ and $\rho_2(\phi) = -\rho_{st}(\phi + \pi)$. A positive solution has to be chosen, since the integration in $\rho$ in Eq. (20) is performed for $\rho > 0$. Note, however, that permutation of the solutions only reverses the sign of integral (28) since both solutions parametrize the same curve differently.
Fig. 4. Left: Normalized diffraction patterns for $q = 3.7 \times 10^{10}, \epsilon q = 10,$ and $\epsilon q = 100.$ The former image ($\epsilon q = 10$) is $3 \times$ zoomed in with respect to the latter. Right: A log–log plot of $\mu/\mu_0$ as a function of $\epsilon q.$ The numerical evaluation is shown by the dotted plot. High-$\epsilon q$ asymptotics (32) is shown by the dashed line. The low-$\epsilon q$ approximation (see Eq. (54)) is shown by the gray solid line. The point corresponding to observations in the equatorial plane of the sun at $Z = 550$ AU and $\lambda = 10^{-6}$ m is marked with a cross. However, the numerical results (as well as the analysis in the next section) show that $\mu/\mu_0$ approximately depends only on $\epsilon q$ when Eq. (26) holds.

Therefore the GPSF (intensity magnification) is expressed through the 1D integral in $\phi$:

$$\mu = \left| u \right|^2 = \pi q |F|^2 = \mu_0 |F|^2,$$

where two functions of $\phi$, $\tau = \tau(\xi, \eta; \phi)$ and $\rho_{st} = \rho_{st}(\xi, \eta; \phi)$, are given in Eq. (29).

The above integral in $\phi$ is particularly easy to take when $\epsilon = 0$, $\xi = \eta = 0$, giving the value of $\mu_0$, obtained earlier (24). In the general case, the integral (31) should be taken numerically.

Now, we present the results of the direct numerical computation of the GPSF (30), (31): Fig. 4 shows the ratio of the maximal intensity $\mu_\epsilon$ at $\epsilon \neq 0$,

$$\mu_\epsilon := \max_{X,Y} \mu(X,Y;q,\epsilon),$$

to that of the symmetric case $\mu_0 = \pi q$ for different values of the parameter $q\epsilon$. The computations presented in Fig. 4 are performed at fixed $q \approx 3.7 \times 10^{10}$ (which corresponds to $\lambda \approx 1 \mu$m) for different values of $\epsilon$. However, the numerical results (as well as the analysis in the next section) show that $\mu_\epsilon/\mu_0$ approximately depends only on $\epsilon q$ when Eq. (26) holds.

As seen from Fig. 4 one naturally recovers the unit ratio for $q\epsilon \to 0$:

$$\frac{\mu_\epsilon}{\mu_0} \to 1, \quad q\epsilon \to 0.$$
(More precisely $\mu_\epsilon / \mu_0 \approx J_0^2 (\epsilon q)$ when $\epsilon q \ll 1.4$. See Eq. (54) and the end of this section.)

On the other hand,

$$\frac{\mu_\epsilon}{\mu_0} \approx \frac{0.25}{\sqrt{\epsilon q}}$$  \hspace{1cm} (32)

when $q \epsilon \gg 1$. In this case four equal maxima of the intensity magnification are symmetrically situated on the $X$, $Y$-axes: two at the $X$-axis and another two at the $Y$-axis in the caustic interior close to four cusps $(X, Y) = (\pm 4b \epsilon, 0), (X, Y) = (0, \pm 4b \epsilon)$. When $q \epsilon$ is large, the distance between the cusp and the neighboring maximum is proportional to $\epsilon b / \sqrt{\epsilon q}$, while the value of $\mu$ at the cusp is proportional to $\mu_\epsilon$ (i.e., $\mu_{\text{cusp}} \approx 0.5 \mu_\epsilon$).

The GPSF oscillates and the amplitude of oscillations grows as one moves towards the neighborhood of the caustic folds/cusps. The amplitude falls off similarly to the $\epsilon = 0$ case and the pattern becomes more and more radially symmetric, as one moves far away from the caustic in the exterior direction (which is in agreement with Eq. (19)).

As $q \epsilon$ becomes smaller the four global maxima approach (discontinuously, see below) the center $X = Y = 0$. The diffraction pattern becomes circularly symmetric at $q \epsilon = 0$.

It is useful to introduce the parameter $\chi = \epsilon q$:

$$\chi = \epsilon q = \frac{\lambda_0}{\lambda} \frac{Z_{\text{min}}}{Z} \sin^2 \beta, \quad \lambda_0 = 2\pi I_2 r_g.$$  \hspace{1cm} (33)

As follows from Eq. (32),

$$\frac{\mu_\epsilon}{\mu_0} \approx \frac{0.25}{\sqrt{\chi}} = \frac{0.25}{|\sin \beta|} \left( \frac{\lambda}{\lambda_0} \right) \frac{Z}{Z_{\text{min}}}, \quad \text{when} \quad \chi \gg 1.$$  \hspace{1cm} (34)

In the case of the sun $\lambda_0 \approx 3.7 \times 10^{-3}$ m, i.e., the effects due to quadrupole moments of the sun are already noticeable in the far-infrared part of the EM spectrum. For $\lambda = 10^{-6}$ m, the parameter $\chi$ can be as large as $\approx 3.7 \times 10^3$. For these wavelengths the maximum magnification of the energy flux from the point source can decrease up to several orders of magnitude when the source goes from the polar axis of the sun $\beta = 0$ towards its equatorial plane $\beta = \pi/2$.

Concluding this section, we would like to mention that (as can be seen from Fig. 4) $\mu_\epsilon / \mu_0$ is a non-smooth function of $\chi$: its derivative with respect to $\chi$ jumps at certain points (e.g., the first jump occurs at $\chi \approx 1.4$, the second at $\chi \approx 2.5$ etc.). A jump takes place every time some of the local maxima of $\mu(X, Y)$ become global ones. For example, the extremum at the center $X = Y = 0$ ($\mu|_{X=0,Y=0} \approx \mu_0 J_0^2 (\chi)$; see Eq. (54)) is the global maximum when $\chi \ll 1.4$. At $\chi \approx 1.4$ this maximum becomes smaller than four maxima at the distance $\approx 0.54 \lambda / a_{\text{E}}$ from the center. As $\chi$ increases, the positions of new global maxima change continuously until the next jump at $\chi \approx 2.5$ etc.

In the next section we give an analytic explanation of the numerical results obtained.

### 4. Diffraction optics: limiting cases

For an analytic description of the above numerical result we expand the argument of the exponential in Eq. (31) in $\tau$ and $\epsilon$ (we recall that $\tau = \frac{X}{b} \cos \phi + \frac{Y}{b} \sin \phi$; see Eq. (29)):

$$F = \frac{1}{2\pi} \int_0^{2\pi} G e^{i q U} d\phi,$$  \hspace{1cm} (34)
where, to the second order in $\tau$ and $\epsilon$,

$$
U = \frac{1}{2} - \tau + \epsilon \cos 2\phi - U_2, \quad U_2 = \left(\frac{\tau}{2} + \epsilon \cos 2\phi\right)^2
$$

(35)

and, to the first order in $\tau$ and $\epsilon$,

$$
G = 1 + \frac{3}{4}\tau.
$$

(36)

We are interested in the short-wavelength limit $q \gg 1$ of Eq. (34). The $U_2$ term in Eqs. (34), (35) can be neglected if $q |U_2| \ll \pi$. This condition holds when

$$
|\tau| \ll \frac{2}{\sqrt{q}}, \quad \epsilon \ll \frac{1}{2\sqrt{q}}, \quad q \gg 1.
$$

(37)

The above conditions have already been encountered in Eqs. (23), (26).

Since $|\tau| \ll 2/\sqrt{q}$ and $q \gg 1$, the $\tau$ term in Eq. (36) can be dropped. Therefore, provided Eq. (37) holds, up to the constant phase factor

$$
F = \frac{1}{2\pi} \int_{0}^{2\pi} e^{i q(\epsilon \cos 2\phi - \tau)} d\phi = \frac{1}{2\pi} \int_{0}^{2\pi} e^{i q(\epsilon \cos 2\phi - \xi \cos \phi - \eta \sin \phi)} d\phi.
$$

(38)

Rewritten in terms of $X$ and $Y$, the condition imposed on $\tau$ in Eq. (37) is $|X \cos \phi + Y \sin \phi| \ll 2b/\sqrt{q}$, i.e.,

$$
R \ll R_v = \frac{2b}{\sqrt{q}} = \sqrt{\frac{2}{\pi} \sqrt{\lambda Z}}, \quad R := \sqrt{X^2 + Y^2},
$$

(39)

where $R$ is the distance between the observer and the $z$-axis. For $\lambda \sim 10^{-6}$ m (i.e., for $q \sim 10^{10}$--$10^{11}$), the radius $R_v$ is about 10 km, while the maximum possible radius of the caustic is about 300 m, so the above condition of validity of Eq. (38) clearly holds in the region of interest.

Therefore, to get the amplitude of the EM field one can integrate over the circle $\rho = 1$ in the lens plane provided that Eq. (39) holds. This happens due to the fact that the “optical path” $S$ is extremal on the ray trajectories (see Eq. (2)) and small deformations of the integration contour do not significantly change the contribution from the “monopole part” $S_0$ of $S = S_0 + \epsilon S_1$ when Eq. (39) holds. Since the contour deformations are of the order of $\epsilon$, the error in the quadrupole contribution $\epsilon S_1$ is of the order of $\epsilon^2$, which is also negligible. In other words, the width of the stationary phase integration region (the “thickness of the Fresnel zone”; see Fig. 3) significantly exceeds the deviation of the integration contour $\rho = \rho_{st}(\xi, \eta, \phi)$ from the unit circle when Eq. (39) is true.

It follows from Eqs. (38) and (30) that $\mu(X, Y; q, \epsilon)/\mu_0(q)$ is essentially a function of three variables,

$$
\frac{\mu}{\mu_0} = f\left(\epsilon q, \frac{qX}{b}, \frac{qY}{b}\right).
$$

---

18 The ratio of the radius $R_v$ to the caustic radius $4\epsilon b$ equals $1/(2\epsilon \sqrt{q})$, which leads to the condition $\epsilon \ll 1/\sqrt{q}$. Since for the sun $\epsilon$ is at most about $10^{-7}$, this condition clearly holds in our case.

19 The second tangential derivative of $S$ also vanishes on the critical line.
when Eqs. (23) and (26) hold. Therefore our results are scalable. For instance, \( \mu_\epsilon / \mu_0 \) evaluated for equatorial observations at \( Z = 550 \) AU for \( \lambda = 1 \) \( \mu \)m is the same as that for \( \lambda = 0.5 \) \( \mu \)m at \( Z = 1100 \) AU.

Apart from the situation when Eqs. (38), (39) overlap with the approximation (19), an analytical study of Eq. (38) can be performed for three asymptotic cases:

1. The “degenerate” case \( \chi = q \epsilon \ll 1 \), i.e., the case of observations in directions that are close to the sun’s polar axis \( |\beta| \ll \sqrt{\frac{\lambda}{\lambda_0} Z_{\min}} \) (see Eq. (33)).

   At \( Z \sim 1000 \) AU and \( \lambda \sim 10^{-6} \) m this corresponds to \( \beta \) that are smaller than a fraction of a degree. These directions cover less than 0.01% of the celestial sphere.

2. The “strongly non-degenerate” case \( \chi = q \epsilon \gg 1 \), or equivalently \( \epsilon \gg \lambda/r_g \). In this case the scale of the diffraction pattern is much smaller than the transverse caustic size, which takes place for \( |\beta| \gg \sqrt{\frac{\lambda}{\lambda_0} Z_{\min}} \).

   At \( Z \sim 1000 \) AU and \( \lambda \sim 10^{-6} \) m, this corresponds to directions with \( \beta \) bigger than few degrees.

3. “On-axis” magnification, i.e., a value of GPSF at \( X = Y = 0 \) and an arbitrary \( \chi = \epsilon q \).

1. We start with the first case, the spherical lens. When \( \epsilon = 0 \), Eq. (38) degenerates to the zero-order Bessel integral:

\[
F = \frac{1}{2\pi} \int_0^{2\pi} e^{-iq(\xi \cos \phi + \eta \sin \phi)} d\phi = J_0 \left( q\sqrt{\xi^2 + \eta^2} \right).
\]

Since \( \xi = X/b, \eta = Y/b, q = 2kr_g = 4\pi r_g/\lambda \), and \( b = \sqrt{2r_g Z} \) (see Eqs. (7), (21)), from Eq. (30) we get

\[
\mu = \pi qJ_0^2 \left( q\sqrt{\xi^2 + \eta^2} \right) = \frac{4\pi^2 r_g^2}{\lambda} J_0^2 \left( \frac{2\pi r_g}{\lambda} \sqrt{\frac{2r_g Z}{R}} \right), \quad R = \sqrt{X^2 + Y^2}.
\]

Thus the \( \epsilon = 0 \) limit, obtained alternatively in Sect. 3 by exact 2D integration of Eq. (20), is recovered. This gives the well known result [8–11,19] for the GPSF of the spherical lens. The GPSF is circularly symmetric and reaches maximum \( \mu_0 \) at \( R = 0 \). The radius of the Airy disk (i.e., “diffraction radius”) and the spatial period of the radial Airy pattern is of the order of \( \lambda/\alpha_E(Z) \).

2. Let us now pass to the strongly non-degenerate case \( q \epsilon \gg 1 \). In this case the size of the caustic \( \sim \epsilon b \) greatly exceeds the diffraction radius \( \sim \lambda/\alpha_E \). First we consider the asymptotic of the GPSF in the cusp neighborhoods, where it reaches its maxima. For convenience we choose the cusp at \( X = 4\epsilon b, Y = 0 \) (i.e., at \( \xi = 4\epsilon, \eta = 0 \)).

We now introduce the cusp-related coordinates \( \tilde{\xi}, \tilde{\eta} \), such that

\[
\tilde{\xi} = \epsilon \left( 4 + \frac{2\tilde{\xi}}{2qe} \right), \quad \eta = \frac{\sqrt{2} \epsilon \tilde{\eta}}{(2qe)^{3/4}},
\]

and rescale the integration angle \( \phi \rightarrow \varphi \):

\[
\phi = \frac{2^{1/4} \varphi}{(qe)^{1/4}}.
\]
In these coordinates integral (38) can be rewritten as

\[ F = \frac{1}{2\pi(q\epsilon/2)^{1/4}} \int_{-\pi(q\epsilon/2)^{1/4}}^{\pi(q\epsilon/2)^{1/4}} e^{i\tilde{V}(\phi)} d\phi, \]  

(41)

where, modulo \( \phi \)-independent terms, \( \tilde{V} \) has the following form:

\[ \tilde{V} = -\tilde{\eta} \phi + \tilde{\xi} \phi^2 + \phi^4 + \tilde{V}_1, \quad \tilde{V}_1 = \frac{\sqrt{2}}{\sqrt{q\epsilon}} \left[ \tilde{\xi} \phi^4 P + \tilde{\eta} \phi^3 Q + \phi^6 H \right]. \]  

(42)

Here, \( P, Q, H \) are bounded functions (max(|\( P \)|, |\( Q \)|, |\( H \)|) \( \leq 1/6 \)) of the single variable \( \phi = \frac{\varphi}{(q\epsilon/2)^{1/4}} \), \( \varphi = -\pi \cdots \pi \).

Since \( q\epsilon \gg 1 \), the \( \tilde{V}_1 \) term in Eq. (42) can be neglected when

\[ |\tilde{\xi}| \ll \sqrt{q\epsilon}, \quad |\tilde{\eta}| \ll \sqrt{q\epsilon}. \]  

(43)

Then, provided that the above conditions hold, from Eqs. (41), (42) it follows that

\[ F = \frac{1}{2\pi(q\epsilon/2)^{1/4}} \text{Pe}(\tilde{\xi}, \tilde{\eta}), \]

where \( \text{Pe}(x, y) \) is the Pearcey integral:

\[ \text{Pe}(x, y) = \int_{-\infty}^{\infty} e^{i(-xy^2 + xy^4)} d\varphi. \]  

(44)

It follows from Eq. (30) that, in terms of the unscaled deviation \( \tilde{X} = X - 4\epsilon b, \tilde{Y} = Y \) from the cusp, the near-cusp GPSF equals

\[ \mu = \frac{1}{4\pi} \sqrt{\frac{2q}{\epsilon}} \left| \text{Pe} \left( \frac{\tilde{X}}{2\epsilon b} (2\epsilon q)^{1/2}, \frac{\tilde{Y}}{\epsilon b} (2\epsilon^3 q^3)^{1/4} \right) \right|^2. \]  

(45)

The domain of validity (43) of the above asymptotics can be rewritten in terms of the unscaled deviations from the cusp as follows:

\[ |\tilde{X}| = |X - 4\epsilon b| \ll \epsilon b, \quad |\tilde{Y}| = |Y| \ll \frac{\epsilon b}{(q\epsilon)^{1/4}}. \]  

(46)

We recall that the above conditions are in agreement with Eq. (39), since, as has been mentioned before, the “radius of validity” of Eq. (38) greatly exceeds the maximal possible size of the caustic when \( \epsilon \ll 1/\sqrt{q} \).

The absolute value of the Pearcey integral \( \left| \text{Pe}(\tilde{\xi}, \tilde{\eta}) \right| \) reaches a maximum at \( \tilde{\xi} \approx -2.02, \tilde{\eta} = 0 \), which is inside the domain of validity (43) of Eq. (45). The maximum of the GPSF in the \( \epsilon q \gg 1 \) limit then equals

\[ \mu_{\epsilon} = \frac{1}{4\pi} \sqrt{\frac{2q}{\epsilon}} \max |\text{Pe}|^2 = \sqrt{\frac{r_g}{2\pi\lambda\epsilon}} \max |\text{Pe}|^2, \quad \max |\text{Pe}|^2 \approx 7.02. \]
Fig. 5. Top: Normalized near-cusp GPSF obtained by numerical integration in Eqs. (30), (31) for $q\epsilon = 3700$. This corresponds, for instance, to $\beta = \pi/2$ and $\lambda = 10^{-6}$ m at 550 AU ($d_{\text{asteroid}} \approx d_{\text{max}} \approx 560$ m; the aspect ratio is preserved). Bottom: Square of the absolute value of the Pearcey integral.

Therefore the ratio of the maximum $\mu_{\epsilon}$ to that of the spherically symmetric case $\mu_0$ equals

$$\frac{\mu_{\epsilon}}{\mu_0} = \frac{\sqrt{2} \max |\text{Pe}|^2}{4\pi^2} \frac{1}{\sqrt{q\epsilon}} \approx 0.25 \frac{1}{\sqrt{q\epsilon}}, \quad q\epsilon \gg 1. \quad (47)$$

The distance between the point of maximum and the cusp equals $\approx 2.02\epsilon b/\sqrt{q\epsilon}/2$. This confirms the numerical results of Sect. 3.

In contrast to the symmetric case, where the circularly invariant diffraction pattern has radial oscillations, the near-cusp pattern in the $q\epsilon \gg 1$ case has a complicated 2D lattice-like structure (see Figs. 4, 5). The latter transforms towards a local 1D structure of smaller intensity (52) as one moves along the caustic away from the cusp.

We now evaluate the GPSF in the vicinity of caustic folds (regular points of the caustic) far from the cusps\textsuperscript{20}.

It is convenient to introduce the caustic-linked coordinates $\sigma, \theta$ (see Fig. 6):

$$\xi = 4\epsilon \cos^3 \theta - \sigma \sin \theta, \quad \eta = -4\epsilon \sin^3 \theta + \sigma \cos \theta, \quad (48)$$

with $\sigma$ being the dimensionless length of the perpendicular from the point $(\xi, \eta)$ to the caustic fold. According to Eq. (38), in these coordinates

$$F = \frac{1}{2\pi} \int_0^{2\pi} e^{iV} d\phi, \quad V = q\sigma \sin(\theta - \phi) + \epsilon q \left( \cos 2\phi + 4 \sin^3 \theta \sin \phi - 4 \cos^3 \theta \cos \phi \right). \quad (49)$$

\textsuperscript{20} A description of the diffraction pattern in a fold neighborhood is extensively presented in the literature on gravitational lensing. For reviews, see, e.g., Refs. [12,19].
Fig. 6. Correspondence between the Cartesian and caustic-linked coordinates in the observer plane.

Introducing a new integration variable \( \phi \) as well as making the change of \( \sigma \):

\[
\phi = \theta + \frac{\varphi}{(3\epsilon q \sin 2\theta)^{1/3}}, \quad \tilde{\sigma} = -\frac{\sigma q}{(3\epsilon q \sin 2\theta)^{1/3}},
\]

we rewrite Eq. (49) as

\[
F = \frac{1}{2\pi(3\epsilon q \sin 2\theta)^{1/3}} \int_{-(3\epsilon q \sin 2\theta)^{1/3}\pi}^{(3\epsilon q \sin 2\theta)^{1/3}\pi} e^{iV} d\varphi, \quad V = \tilde{\sigma} \varphi + \frac{1}{3} \varphi^3 + \tilde{W}, \quad (50)
\]

where

\[
\tilde{W} = \frac{\varphi^3}{(3\epsilon q \sin 2\theta)^{1/3}} \left[ \tilde{\sigma} \tilde{W}_1 + \varphi^2 \tilde{W}_2 + \varphi \tilde{W}_3 \cot 2\theta \right].
\]

In the above equation \( \tilde{W}_1, \tilde{W}_2, \tilde{W}_3 \) are bounded functions (max(|\( \tilde{W}_1 \)|, |\( \tilde{W}_2 \)|, |\( \tilde{W}_3 \)|) \( \leq 21/20 \)) of the single variable \( \phi = \frac{\varphi}{(3\epsilon q \sin 2\theta)^{1/3}}, \phi = -\pi \cdots \pi \):

\[
\tilde{W}_1 = \frac{\sin \phi - \phi}{\phi^3},
\]

\[
\tilde{W}_2 = 2 \frac{\sin \phi - \phi + \frac{\phi^3}{6} - 2 \sin 2\phi + 4\phi - \frac{(2\phi)^3}{3}}{\phi^5},
\]

\[
\tilde{W}_3 = \frac{\cos 2\phi - 4 \cos \phi + 3}{\phi^4}.
\]

When the conditions

\[
|\tilde{\sigma}| \ll (\epsilon q \sin 2\theta)^{2/3}, \quad |\sin 2\theta| \gg \frac{1}{(\epsilon q)^{1/4}}, \quad q\epsilon \gg 1 \quad (51)
\]
hold, the $\tilde{W}$ term can be neglected in Eq. (50), since $\epsilon q \gg 1$. Therefore

$$F = \frac{1}{2\pi (3q\epsilon \sin 2\theta)^{1/3}} \int_{-\infty}^{\infty} \exp \left[ i \tilde{\sigma} \varphi + \frac{\varphi^3}{3} \right] d\varphi = \frac{\text{Ai}(\tilde{\sigma})}{(3q\epsilon \sin 2\theta)^{1/3}},$$

where $\text{Ai}(\tilde{\sigma})$ is the Airy function. Finally we get

$$\frac{\mu}{\mu_0} = \frac{1}{(3q\epsilon \sin 2\theta)^{2/3}} \text{Ai}^2 \left( \frac{-\sigma q}{(3q\epsilon \sin 2\theta)^{1/3}} \right), \quad |\sigma| \ll \epsilon |\sin 2\theta|, \quad |\sin 2\theta| \gg \frac{1}{(\epsilon q)^{1/4}}.$$

In terms of the unscaled distance from the caustic fold $D = \sigma b$, $b = \sqrt{2rg}$ (see Fig. 6),

$$\mu = \frac{4\pi^2 rg}{\lambda K^2} \text{Ai}^2 \left( \frac{-2\pi D}{K\lambda} \sqrt{\frac{2rg}{Z}} \right), \quad K = \left( \frac{12\pi \epsilon rg}{\lambda \sin 2\theta} \right)^{1/3}. \quad (52)$$

The validity domain of Eq. (52),

$$|D| \ll b |\sin 2\theta|, \quad |\sin 2\theta| \gg \frac{1}{(\epsilon q)^{1/4}}, \quad (53)$$

is obviously contained in Eq. (39). The maximum of $\text{Ai}(\tilde{\sigma})$ is reached at $\tilde{\sigma} = -1.02$, which satisfies Eq. (51). The diffraction pattern in the fold neighborhood, far from the cusps, is locally 1D with the oscillation scale $\sim \lambda |3q\epsilon \sin 2\theta|^{1/3}/\alpha_E$ depending on the fold coordinate $\theta$. Also, for maximal magnification at fixed $\theta$, we have

$$\frac{\max_D \mu(D, \theta)}{\mu_0} = \frac{\max \text{Ai}^2}{(3q\epsilon \sin 2\theta)^{2/3}} \approx \frac{0.29}{(3q\epsilon \sin 2\theta)^{2/3}}, \quad \frac{\max_D \mu(D, \theta)}{\mu_e} \approx \frac{0.56}{(\epsilon q)^{1/6} \sin^{2/3} 2\theta}.$$

Taking into account that Eq. (52) is valid for $|\theta| \gg (\epsilon q)^{-1/4}$ and $\epsilon q \gg 1$, we conclude that

$$\max_{\theta, D} \mu(D, \theta) < \mu_e.$$

This confirms an obvious fact that in the neighborhood of folds the intensity is smaller than the maximum $\mu_e$ in the cusp region.

According to Eqs. (45), (52) the GPSF oscillates and the amplitude of oscillations falls as one moves away from the caustic and the “far-field” GPSF can be approximated by Eq. (19).

(3) Concluding this section, we mention the case in which the observer is on the $z$-axis, i.e., when $X = Y = 0$: Here, the integral (38) is expressed in terms of the zero-order Bessel function and

$$\mu_{\text{axis}} = \pi q J_0^2(\epsilon q), \quad \frac{\mu_{\text{axis}}}{\mu_0} = J_0^2(\epsilon q), \quad (54)$$

which is in agreement with the numerical results (see Fig. 4).

When $\epsilon q \gg 1$, i.e., when the size of the caustic exceeds greatly the diffraction radius$^{21}$,

$$\frac{\mu_{\text{axis}}}{\mu_0} = \frac{2}{\epsilon q} \cos^2(\epsilon q - \pi/4), \quad \epsilon q \gg 1,$$

which is in agreement with Eq. (19).

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$^{21}$That is, when an on-axis observer sees the perfect “Einstein cross” image in the lens plane.
5. PSF in the focal plane of a telescope

The GPSF (30, 31) is, in fact, a point spread function for a “zero-aperture” telescope that can be used only for the intensity scan in the observer plane. One should use the PSF of a compound system of a gravitational lens and a telescope (for more details see, e.g., Ref. [21]) if the diffraction resolution of the telescope is finer than the angular radius of the Einstein ring $\alpha_E$.

In the Fraunhofer approximation for a telescope lens of focal length $F$, the PSF in the telescope focal plane is expressed through the Fourier transform of the complex field amplitude at aperture

$$w(\gamma; \omega) = \frac{k}{2\pi i F} \int_{R\in\text{Aperture}} u(\tilde{R} - \tilde{\omega}Z) e^{ik(\tilde{\omega}+\gamma)\tilde{R}} d^2R,$$

(55)

where $\gamma$ and $\omega$ are the observation and point source angles respectively (see Fig. 7), and $\tilde{R} = (X, Y)$ are coordinates in the aperture (observer) plane. The complex amplitude of the EM field at the aperture plane $u(\tilde{R} - \tilde{\omega}Z) e^{ik\tilde{\omega}R}$ is expressed through $u(\tilde{R})$ given by Eq. (18). This expression is obtained by application of the small rotation\(^\text{22}\)

$$\tilde{R} \rightarrow \tilde{R} - \tilde{\omega}Z, \quad Z \rightarrow Z + \left(\tilde{\omega} \cdot \tilde{R}\right)$$

to the spatial arguments of the complex field amplitude $e^{ikZ} u(\tilde{R})$. We recall that the phase factor $e^{ikZ}$ was dropped in Eq. (18), so we restore it to get the complex amplitude at aperture. After getting the amplitude by application of the above rotation, this factor can be dropped again in Eq. (55). For simplicity, we will perform our computations modulo common phase factors.

After substituting Eq. (18) into Eq. (55) one gets (up to a common phase factor)

$$w = \frac{k^2}{4\pi^2 F Z} \int e^{\frac{ik}{2} \left[ \frac{(\tilde{R}_\perp)^2}{Z} - \Psi(\tilde{R}_\perp) \right]} \left[ e^{ik\tilde{\omega}\tilde{Z}} \tilde{R} + ik \tilde{R}_\perp \hat{Z} \right] A(\tilde{R}) d^2R d^2\tilde{r}_\perp,$$

(56)

where $\tilde{r}_\perp = (x, y)$ and $A$ is the aperture function, i.e.,

$$A(\tilde{R}) = \begin{cases} 1, & \tilde{R} \in \text{aperture} \\ 0, & \tilde{R} \notin \text{aperture} \end{cases}.$$

\(^{22}\) Here we have neglected $\mathcal{O}(\omega^2)$ terms, since the variation of the optical path across the aperture due to these terms is of the order of $a\omega^2$, where $a$ is the radius of the aperture. Even for $\omega \sim \alpha_E(Z_{\text{min}})$ this variation is $\sim 10^{-10}$ m, not to mention $\omega \sim$ apparent sizes of objects to be observed.
The last term $ik \frac{\alpha^2}{2Z}$ in the last exponential of Eq. (56) can be dropped when $a \ll \sqrt{\lambda Z}$, where $a$ is the radius of aperture. Therefore

$$w = \frac{k}{2\pi Z} \int e^{ik\left[\frac{(\alpha Z + \alpha)^2}{2Z} - \psi(\vec{r}_\perp)\right]} A_k(\vec{\gamma} - \vec{r}_\perp / Z) d^2r_\perp,$$  

(57)

where the function $A_k$ is proportional to the Fourier transform of the aperture function

$$A_k(\vec{\theta}) = \frac{k}{2\pi F} \int A(\vec{R}) e^{ik\vec{R} \cdot \vec{\theta}} d^2R.$$ 

For a circular aperture of radius $a$, $A_k$ can be expressed through the Bessel function of the first order:

$$A_k(\vec{\theta}) = a |\vec{\theta}| F J_1(k a |\vec{\theta}|).$$  

(58)

As in the case of the GPSF, the stationary phase integration in $\rho$ (recall that $r_\perp = b \rho$; see Eq. (8)) can be performed in Eq. (57). Indeed, a variation of the argument of the Bessel function in Eqs. (57), (58) across the stationary phase region is $\sim k a \delta r_\perp / Z$, where $\delta r_\perp \sim \sqrt{\lambda Z}$ is the width of this region. Therefore, such a variation can be neglected when $a \ll \sqrt{\lambda Z}$. The last condition holds for any realistic aperture.

When condition (39) holds, one can integrate over the unit circle $\rho = 1$ in the observer plane and

$$M = \mu_0 \left(\frac{ka^2}{2F}\right)^2 |F|^2, \quad F = \frac{1}{\pi} \int_0^{2\pi} e^{ik a_\alpha E (\epsilon b \cos 2\phi - X \cos \phi - Y \sin \phi)} h(k a_\alpha E |e^{i\phi} - \Gamma e^{i\theta}|) d\phi,$$ 

(59)

where the function of one variable $h(\cdot)$ is defined as

$$h(x) = J_1(x)/x$$

and $\Gamma, \theta$ are the dimensionless polar coordinates in the focal plane

$$\vec{\gamma} = (a_\alpha E \Gamma \cos \theta, a_\alpha E \Gamma \sin \theta).$$

In Eq. (59), $X, Y$ denote the deviation of the observer from the heliocentric projection of the point source ($(X, Y)$ equals $-\vec{R}'$ in Fig. 7):

$$(X, Y) = -\vec{R}' = -\vec{\alpha} Z.$$ 

If an aperture is much smaller than the diffraction radius $\lambda / a_\alpha E$, the PSF (59) is independent of $\theta$ and it is proportional to the GPSF (the focal plane image is the Airy spot rather than the limbs for such apertures). On the other hand, when an aperture is big enough, so that the telescope diffraction resolution is much finer than $a_\alpha E$ and when the object apparent size $\Delta \omega_{\text{max}}$ is much smaller than the diffraction resolution of the telescope (i.e., for $a_\alpha E \gg \lambda$ and $d|\Delta \omega_{\text{max}}| \ll \lambda$) this function is “concentrated” (within the telescope diffraction limit) in the neighborhood of the “Einstein circle” $\Gamma = 1$ (as in, e.g., Figs. 8 and 11).

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23 For $Z \sim 1000$ AU and $\lambda \sim 10^{-6}$ m, $\sqrt{\lambda Z} \sim 10^4$ m, so this term can be dropped for any realistic aperture.

24 As in the case of the GPSF, we drop the $cT(Z)$ term in Eq. (9), since the variation of the optical path over the aperture due to this term is at most $2r_{\text{atm}} / Z \approx a_\alpha E \omega$. Indeed, even for $\omega \sim a_\alpha E$ this variation is $\sim a \times 10^{-15}$, not to mention $\omega \sim$ apparent sizes of object to be observed.
Fig. 8. Normalized focal plane images for \( p := k a a_{E} = 60 \), which would correspond, e.g., to a 2 m aperture \((a = 1 \text{ m})\) at \( \lambda \approx 1 \mu \text{m} \), and \( Z \approx 550 \text{ AU} \). Images A and B show the \( \epsilon = 0 \) case. A: Telescope is centered at the caustic line. B. Maximum of the symmetric case \( M_{m} := \max_{\vec{\gamma}, \tilde{R}} M|_{\epsilon = 0} \) is observed at \( \theta = 0 \) and \( \theta = \pi \). The distance between the observer and the caustic line equals \( R_{m} \). C: Non-symmetric case \( \epsilon_{q} = 3700 \), which would correspond to the equatorial PSF at, e.g., the above-mentioned conditions. The observer coordinates are \( X \approx 0.479 \times d_{\text{astroid}} \), \( Y = 0 \). The focal plane intensity at \( /Gamma_{1} = 1, \theta = 0 \) exceeds the maximum \( M_{m} \) of the degenerate case by the factor \( \approx 1.1 \). The plots of \( M / M_{0} \) at \( /Gamma_{1} = 1 \) in the vicinity of point \( \theta = 0 \) are shown on the right. The intensities of images A and B are plotted by thin black solid lines, while the thick gray solid curve corresponds to image C. Note that in case C there are four geometrical optics images. The maxima corresponding to three of the four images (one global and two local) are visible in C. The local maximum at \( /Gamma_{1} = 1, \theta = \pi \) (corresponding to the fourth geometrical optics image; see Fig. 2) approximately equals \( 0.04 M_{0} \). It is too weak in comparison with the global maximum \( M|_{/Gamma_{1} = 1, \theta = 0} \approx 1.51 M_{0} \) to be noticeable on the normalized grayscale panel C. This local maximum is shown in the separate sub-panel in the lower-left corner of C; for better visibility the brightness inside the selected square is increased by \( \approx 33 \) times.

Note that the ratio \( M / M_{0} \), where \( M_{0} = \max_{\vec{\gamma}, \tilde{R}} M|_{\epsilon = 0} \) and \( \tilde{R} = (X, Y) \), is essentially a function of six arguments:

\[
\frac{M}{M_{0}} = g \left( \frac{\vec{\gamma}}{a_{E}}, \frac{p \tilde{R}}{a}, \epsilon_{q}, p \right), \quad p := k a a_{E}.
\]

Thus, our results are scalable. For example, \( \max_{\vec{\gamma}, \tilde{R}} M / M_{0} \) for \( \lambda = 10^{-6} \text{ m}, a = 1 \text{ m}, \) and \( Z = 550 \text{ AU} \) is the same as that for \( \lambda = 0.5 \times 10^{-6} \text{ m}, a \approx 0.71 \text{ m}, \) and \( Z = 1100 \text{ AU} \).

It is also important to note that, in contrast to the GPSF \( \mu \), the maximum of the focal plane PSF \( M \) is not necessarily smaller for equatorial observations in comparison with polar observations, when the aperture is big enough (see Fig. 8).

To examine this in more detail, consider first the \( \epsilon = 0 \) case.

1) The numerical integration in Eq. (59) shows (see Fig. 9) that the focal plane PSF, as a function of \( \vec{\gamma} \) and \( \tilde{R} \) (at fixed \( a, \lambda, Z, \) and \( \mathcal{F} \)), reaches its global maximum

\[
M_{m} := \max_{\vec{\gamma}, \tilde{R}} M|_{\epsilon = 0}
\]

when the telescope is placed at some non-zero distance \( R = R_{m} \) from the caustic line. The maximum is reached at \( \vec{\gamma} \) corresponding to positions of the geometrical optics images. Let us now find \( M_{m} \) and \( R_{m} \) analytically.
Fig. 9. The ratio of the maximal focal plane intensity for $R \neq 0$ to that for $R = 0$ in the $\epsilon = 0$ case is shown by the gray dotted plot. It is obtained by the numerical integration in Eq. (59) for $a \alpha E / \lambda \approx 10$. The ratio (vertical axis) is plotted against $Q = ka^2 \alpha E / R$ (horizontal axis). The ratio given by Eq. (61) is plotted by the black line. Note that, at fixed $R$, the number of focal plane points where the maximum is reached is either two or four. The two-point maxima (i.e., maxima at $\theta = 0$ and $\theta = \pi$) correspond to values of $Q$ at which the above two plots (approximately) coincide. The maximum $\max_{\hat{r}} \mathcal{M}\epsilon = 0 | R_M | \approx 1.35 M_0$ is a two-point maximum. It is seen when $Q = Q_m \approx 5.7$ (see Fig. 8B).

Without loss of generality we can set $X = R \geq 0, Y = 0$, so that one of the geometrical optics images would be at $\Gamma = 1, \theta = 0$. Considering the case when both the aperture and $R$ are much bigger than the diffraction radius (i.e., $a \gg \lambda / \alpha E$ and $R \gg \lambda / \alpha E$) we can approximate the value of $F$ in Eq. (59) at $\Gamma = 1, \theta = 0$ as follows:

$$F|_{\epsilon = 0, \Gamma = 1, \theta = 0, Y = 0} \approx \frac{1}{\pi} \int_{-\infty}^{\infty} e^{ik \alpha E R \phi^2 / 2} h(ka \alpha E \phi) d\phi = \frac{1}{\pi k a \alpha E} \int_{-\infty}^{\infty} e^{i \phi^2 / 2} h(\phi) d\phi, \quad (60)$$

where

$$Q = \frac{ka^2 \alpha E}{R} = \frac{pa}{R}. \quad (60)$$

We would like to compare the intensity at $R \neq 0, \theta = 0$ with the maximal intensity of the circularly symmetric ring seen by the on-axis observer. From Eq. (59) it follows that for such a ring (up to a common, $\theta$-dependent phase factor)

$$F|_{\epsilon = 0, \Gamma = 1, R = 0} = \frac{1}{\pi} \int_{-\pi}^{\pi} h(2ka \alpha E \sin \frac{\phi}{2}) d\phi \approx \frac{2}{\pi k a \alpha E}, \quad a \gg \lambda / \alpha E. \quad (60)$$

Therefore (see Eq. (59)), the maximal intensity of the circularly symmetric ring seen by the on-axis observer equals

$$\mathcal{M}_0 := \max_{\hat{r}} \mathcal{M}|_{\epsilon = 0, R = 0} = \mu_0 \left(\frac{a}{\pi F \alpha E}\right)^2 = \frac{q}{\pi} \left(\frac{a}{F \alpha E}\right)^2, \quad a \gg \lambda / \alpha E. \quad (60)$$

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25 When $\epsilon = 0, R \ll \sqrt{\lambda Z}, a \ll \sqrt{\lambda Z}$, and $\lambda \ll r_g$, the PSF is symmetric with respect to the central inversion $\theta \rightarrow \theta + \pi$. 

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25
Note that for small apertures $F|_{\epsilon=0,\Gamma=1, R=0} \approx 1$, and therefore in the asymptotic cases we have
\[
\mathcal{M}_0 = \begin{cases} 
\mu_0 \left( \frac{a}{\pi a_E} \right)^2, & a \gg \lambda / \alpha_E \\
\mu_0 \left( \frac{ka^2}{\pi} \right), & a \ll \lambda / \alpha_E.
\end{cases}
\]
For an intermediate range of apertures one has to evaluate $\mathcal{M}_0$ numerically. We will take $\mathcal{M}_0$ as a reference value in what follows.

The integral in Eq. (60) can be expressed in terms of the Bessel functions and we get
\[
\mathcal{M} \approx \frac{\pi}{8} (J_0(Q/4) + iJ_1(Q/4)), \quad a \gg \lambda / \alpha_E.
\] (61)
This ratio can exceed unity and takes the biggest value $\approx 1.35$ at $Q = Q_m \approx 5.7$ (see Figs. 8B and 9). In other words, the maximum $\max_{\vec{r}, R} \mathcal{M} = \mathcal{M}_0 \approx 1.35 \mathcal{M}_0$ is seen when the telescope is placed at the distance $R = R_m$ from the caustic line, where
\[
R_m = k \alpha E a^2 / Q_m \approx 1.1 \alpha E a^2 / \lambda, \quad a \gg \lambda / \alpha_E. \quad (62)
\]
Note that once an aperture is much bigger than the diffraction radius, a similar condition for the distance $R_m \gg \lambda / \alpha_E$ is satisfied automatically.

The domain of validity of Eq. (62) is also restricted by the condition (39), i.e., $R_m \ll \sqrt{\lambda Z}$ or $\lambda \gg (a^4 \alpha_E^2 / Z)^{1/3} = (2rg a^4 / Z^2)^{1/3}$.

For $a \sim 1$ m and $Z \sim 1000$ AU this restriction reads as $\lambda \gg 10^{-8}$ m.

The fact that $\max_{\vec{r}, R} \mathcal{M}|_{\epsilon=0}$ is not seen when the telescope is centered exactly at the caustic line is explained as follows: Although the telescope at $R = R_m$ takes a smaller total energy flux, the size of the bright limb/spot in the focal plane is also smaller (compared to the full ring for a telescope at $R = 0$), so the maximal flux density through the focal plane happens to be a bit bigger at $R = R_m$ than at $R = 0$. At big distances from the caustic line, the characteristic size of the focal plane images is defined only by the diffraction limit of the telescope and the ratio $\max_{\vec{r}} \mathcal{M}(\epsilon = 0; R) / \mathcal{M}_0$ decays proportionally to $1 / R$ (i.e., proportionally to $Q$) when $R \gg R_m$ (i.e., $Q \ll Q_m$) (see Fig. 9).

Now, let us turn to the non-degenerate $\epsilon \neq 0$ case.

2) Here, analysis of the behavior of the PSF is more involved, so we present only some general comments and preliminary (mostly numerical) results.

The behavior of the PSF can be easily described in both small and big aperture limits. When the aperture is much smaller than the scale of the diffraction pattern (i.e., when $\alpha E$ is much smaller than the diffraction limit of the telescope) the focal plane PSF is essentially a product of the GPSF and the PSF of the telescope lens. Therefore
\[
\max \mathcal{M} / \max \mathcal{M}(\epsilon = 0) \rightarrow \mu_\epsilon / \mu_0, \quad a \ll \lambda / \alpha_E.
\]
On the other hand, when the aperture is much bigger than the diameter of the astroid the maxima of the PSF should be approximately the same in the degenerate and non-degenerate case.

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26 Here $\int_{-\infty}^{\infty} e^{iQ\varphi} J_1(\varphi) d\varphi = \frac{\sqrt{\pi} \mu_0}{2} (J_0(Q/4) + iJ_1(Q/4))$. 

24/32
Fig. 10. $M/M_0$ is evaluated at the point of the focal plane corresponding to the geometric optics image (at $\Gamma = 1$, $\theta = 0$) for different observer positions. The observer moves along the $X$-axis in the vicinity of the cusp at $X = d_{\text{astroid}}/2$, $Y = 0$. Numerical evaluation of Eq. (59) is performed for $\epsilon q \approx 3700$. Values of $p := ka\alpha E$ for the curves A, B, C, D are 7.5, 30, 60, 120 respectively. For equatorial observations at $Z = Z_{\min}$ and $\lambda = 10^{-6}$ m, the curves A, B, C, and D would correspond to the apertures of 0.25, 1, 2, and 4 m. In the case of curves C and D the maximum of the PSF exceeds that $M_m$ of the symmetric case. For instance, for curve C it is $\max_{\Gamma=1,\theta=0,Y=0} M(\gamma) \approx 1.1 \times M_m \approx 1.5 \times M_0$. The focal plane image corresponding to the maximum of curve C is shown in Fig. 8C. As the aperture becomes larger, $M$ behaves similarly to the $\epsilon = 0$ case: $\max M/M_0$ decreases as $a$ increases and tends to that of the symmetric case $M_m/M_0$ (cf. curves C and D). Note the oscillatory behavior of curves in the interior of the astroid (i.e., for $2X/d_{\text{astroid}} < 1$) due to the diffraction pattern (see Fig. 5).

Cases. Numerical results as well as the analysis below show that the maxima can already be approximately the same at apertures that are much smaller than $d_{\text{astroid}}$ (see Fig. 10). To get an analytic estimate for the corresponding range of apertures and absolute maxima of $M/M_0$ (or that of $M_m/M_0$) we apply an approach similar to that of the symmetric case.

In more detail, numerical computations show that when the aperture is big enough, $\max_{\gamma,R} M$ is reached at four symmetrically situated points (two at the $X$-axis and two at the $Y$-axis) and at $\gamma$ corresponding to the position of the brightest (at given $X$ or $Y$) geometrical optical image. Therefore, we place an observer at the $X$-axis at the point $X = d_{\text{astroid}}/2 + \tilde{R}$, $Y = 0$, so that the distance from the cusp is much smaller than the diameter of the astroid $|\tilde{R}| \ll d_{\text{astroid}}$. The value of $M$ at the point $\Gamma = 1$, $\theta = 0$ of the focal plane (i.e., at the position of the brightest geometrical optical image) can be obtained in a manner similar to that of the symmetric case (see Eq. (60)). Then, up to a common phase factor, we get

$$F|_{\Gamma=1,\theta=0,Y=0} \approx \frac{1}{\pi ka\alpha E} \int_{-\infty}^{\infty} \exp \left[ i \left( \frac{d_{\text{astroid}}}{16 k^3 \alpha^3 \alpha E^3} \phi^4 + \frac{\phi^2}{2 \tilde{Q}} \right) \right] h(\phi) d\phi, \quad \tilde{Q} = \frac{ka^2 \alpha E}{\tilde{R}}. \quad (63)$$

When $\tilde{Q} \sim Q_m$, the $\phi^4$ term in the exponent can be neglected if

$$a \gg a_0 = \left( \frac{d_{\text{astroid}}}{\alpha E^3} \right)^{1/4}. \quad (63)$$
In this situation one observes a maximum that approximately equals $M_m$, i.e.,

$$\max M / \max M(\epsilon = 0) \to 1, \quad \text{when } a \gg a_0.$$ 

Like the degenerate case, the maximum is observed when $\bar{R} \approx \pm R_m$, i.e., at $X \approx d_{\text{astroid}}/2 \pm R_m$, $Y = 0$, which is confirmed by the numerical computations. Note that, due to approximation where the $\varphi^4$ term is neglected in Eq. (63), we got eight equal maxima (four inside and four outside the astroid) instead of four: Although for $a \gg a_0$ the difference between the maxima inside and outside the astroid is negligible (see, e.g., curve D of Fig. 10), only four or them are global ones.

In the intermediate range of apertures

$$a \sim a_0, \quad a \gg \lambda/\alpha_E$$

the analysis of the PSF becomes non-trivial. However, it follows from Eq. (63) that for $a \gg \lambda/\alpha_E$, the ratio $\max \bar{R} M|_{\Gamma=1,\delta=0}/M_0$ is approximately a function of $a/a_0$ only. Therefore, it is sufficient to perform a set of computations for a fixed $\epsilon q \gg 1$ and different $p$ (i.e., different apertures) to approximately get the absolute maximum of $M/M_0$ (i.e., that for all values of parameters) in the strongly non-degenerate case.

Numerical computations performed for the strongly non-degenerate case show that, once the aperture exceeds some value $\sim a_0$, the maximum of the “non-symmetric” PSF becomes even bigger than that of the degenerate case, i.e., the maximum of $M$ exceeds $M_m$ when the aperture is big enough. However, the difference in these maxima cannot exceed a few ($\sim 10$) %. The results of the numerical computations are shown in Fig. 10. Examples of the corresponding focal plane images are shown in Fig. 8.

Note that $a_0 \sim 1$ m for equatorial observations at $Z \sim 1000$ AU and $\lambda \sim 10^{-6}$ m. Therefore, for a telescope of a modest aperture ($\sim 1$ m), the maximum of the PSF (PSF as a function of $\tilde{\gamma}$ and $\bar{R}$) changes only by at most $\approx 10\%$ when the source is moved from the equatorial plane to the polar axis.

Finally, let us estimate the PSF when a telescope whose diffraction resolution is much finer than $\alpha_E$ is placed far away from the caustic. Here our computations are facilitated due to the fact that one can perform the 2D stationary phase integration in Eq. (57). Then, similarly to the case of the GPSF (cf. Eq. (19)), one gets

$$w(\tilde{\gamma}, \bar{R}) = \sum_j \sqrt{|\mu_j|} e^{i(k S_j - \pi n_j/2)} A_k \left( \tilde{\gamma} - \frac{\tilde{r}_j}{Z} \right),$$

where $\tilde{r}_j = (x_j, y_j)$ are coordinates of the $j$th image in the lens plane and $S_j = S(\bar{R}; r_j)$, $\mu_j, n_j$ are same as in Eq. (19). Computation of the PSF then reduces to the summation

$$M = \sum_j |\mu_j| P \left( \tilde{\gamma} - \tilde{r}_j/Z \right) + \sum_{j \neq l} \sqrt{|\mu_j\mu_l|} e^{i(k (S_j - S_l) - \pi (n_j - n_l)/2)} A_k \left( \tilde{\gamma} - \tilde{r}_j/Z \right) A^*_k \left( \tilde{\gamma} - \tilde{r}_l/Z \right),$$

where $P(\tilde{\varnothing}) = |A_k(\tilde{\varnothing})|^2$ is the PSF of the telescope lens.

When the angular separation between the geometrical optics images of a point source is much bigger than the diffraction limit of a telescope, the double sum in the last equation can be dropped.
Fig. 11. Normalized focal plane images of the equatorial point source viewed through a 1 m telescope ($a = 0.5 \text{ m}$) obtained by numerical integration in Eq. (59). The parameters $\lambda$, $\beta$, and $Z$ are set as in Fig. 5. The corresponding positions of the telescope with respect to the caustic are shown on the right. In contrast to images A and B, image C is taken at the caustic (at the cusp of the astroid): The brightest part of image C has an elongated (limb-like) form, while those of images A and B are round Airy spots of the telescope lens only. The maximum of the PSF corresponding to image A (“Einstein cross”) is about $2 \times 10^{-2} \mathcal{M}_0$. For image B it is $\approx 1.5 \times 10^{-2} \mathcal{M}_0$. For image C the global maximum exceeds $\mathcal{M}_0$ and is $\approx 1.2 \times \mathcal{M}_0$. Note that the value of the local maximum at $\Gamma = 1, \theta = \pi$, corresponding to the “weak image” of Fig. 2D, is too small in comparison with the global maximum at $\Gamma = 1, \theta = 0$ to be noticeable in panel C of the present normalized grayscale figure. This local maximum is shown in the separate sub-panel in the lower-left corner of panel C: For better visibility the brightness inside the selected square is increased by $\approx 127$ times.

Then the focal plane PSF equals the following convolution:

$$
\mathcal{M}(\gamma; R) = \int \tilde{\mu}(\gamma') P(\gamma - \gamma')d^2\gamma', \quad \tilde{\mu}(\gamma') = \sum_j |\mu_j| \delta(\gamma' - \vec{r}_j/Z).
$$

(64)

In other words, for a telescope of reasonably big aperture ($a \alpha_E \gg \lambda$) placed away from the caustic, the gravitational lensing can be described by the geometrical optics, while the wave effects due to finite aperture have to be taken into account. Related examples of the focal plane images are shown in Fig. 11.

6. Discussion and conclusions

In this section we would like to discuss the implications of the effects related to the quadrupole moment of the sun on prospective observations. For this we first summarize the main results of the present work.

The transverse size of the astroid caustic (due to the quadrupole moment of the sun) could reach several hundred meters at distances ranging from that of the closest observation (550 AU) up to several thousand AU. This size is comparable with the sizes of heliocentric projections of possible objects of observation, which are about several kilometers across. At such distances, the diffraction pattern of a monochromatic point source transforms significantly (in the region of interest, at, e.g., sub-micrometer wavelengths) when the direction of observation is changed from that along the sun’s polar axis to that in the sun equatorial plane. The maximum of the gravitational point spread function (GPSF) can differ by up to about two to three orders of magnitude. In the strongly non-degenerate case the maximum of the GPSF is reached in the neighborhood of the cusp of a caustic. The GPSF can be expressed in terms of the Pearcey integral in this neighborhood.

On the other hand, the behavior of the PSF of a compound system of the gravitational lens and a telescope depends on the telescope’s aperture. If the aperture is small, the focal plane PSF and GPSF are essentially the same. For big apertures (e.g., a 2 m aperture) the absolute maximum of the focal
plane PSF can be even bigger in the non-symmetric case. Although these maxima do not differ very much, the formation of images can be significantly different in the symmetric and non-symmetric cases. For instance, in contrast to the symmetric case, in the strongly non-degenerate case an image of a point source never forms a “bright” ring, but rather consists of small limbs/spots. The focal plane image is not generally centrally symmetric (which can be an advantage (S. Turyshev, private communication)); the number of images of a point source can be different etc.

We recall that the GPSF/focal plane PSF are magnifications of a monochromatic point source and not those of a realistic extended object. Therefore, in prospective missions, one does not expect to directly observe diffraction patterns. However, one needs the PSF for deconvolution of realistic images.

In more detail, the energy flux at the point \( \vec{R} = (X, Y) \) of the observer plane radiated by an extended, totally (spatially and temporally) incoherent source equals the convolution of the GPSF and the surface brightness of the source:

\[
I(\vec{R}) = \int I_s(\vec{R}'; q) \mu(\vec{R} - \vec{R}'; q) d^2 R' dq. \tag{65}
\]

Here \( I_s(\vec{R}'; q) dq \) is a non-magnified energy flux (times the filtering function) in the interval \( q, q + dq \) of the spectrum radiated by a surface element on the source, and \( \vec{R}' = (X', Y') \) stands for the coordinates of the heliocentric projection of this element to the observer plane (see Fig. 7).

The role of the caustic in the deconvolution process can already be seen in the geometrical optics: Suppose that one got the intensity (65) in the non-symmetric case and then performed deconvolution of \( I(\vec{R}) \) as if \( \mu \) were a magnification of the monopole. The finest spatial resolution of such a deconvolution will be of the order of the astroid diameter \( d_{\text{astroid}} \), since both non-symmetric and monopole \( \mu(\vec{R}) \) have the same asymptotic behavior at \( R \gg d_{\text{astroid}} \) and they start to differ significantly at \( R \sim d_{\text{astroid}} \). As has been mentioned before, the typical size of the heliocentric projection of an exo-planet to the observer plane is about several kilometers at 1000 AU and \( d_{\text{astroid}} \) is about 10% of that size. Therefore, the area of a minimal pixel will be of the order of 1% of that of the whole deconvoluted image, i.e., changes of \( \mu \) play an important role starting from about the hecto-pixel level of imaging, not to mention the mega-pixel imaging currently discussed in the literature [10,11]. At the latter level of resolution one has to take the diffraction pattern of the PSF into account.

Similar arguments can be applied to the deconvolution of the focal plane images. It is most likely that this type of deconvolution, rather than that of an intensity scan (65) will be used in a possible mission where a sequence of images of the Einstein rings along the path of a spacecraft across the “high-intensity” region will be taken. The intensity of a ring point in this set in a sense encodes “projection data” taken along a “section” of the source surface. Therefore, a kind of “tomographic” reconstruction algorithm should be developed for the deconvolution related to the focal plane PSF.

One might consider the hypothetical possibility of obtaining a relatively high-resolution image during a single arbitrary passage in the neighborhood and/or through the heliocentric projection of an exo-planet. Indeed, at \( Z \sim 1000–2000 \) AU, a modest \( \sim 1 \) m telescope equipped with coronagraphs resolves the Einstein ring with about \( \sim 20 \) circumferential elements. Therefore, taking about \( \sim 10^2 \) samples of rings along, e.g., a straight path crossing a high-intensity region could, in principle, result in a kilo-pixel image. Note that at this level of resolution the size and structure of the caustic plays an important role.

Completing this section, we would like to mention the influence of the higher multipole moments of the sun on the PSF. Computation of the PSF accounting for higher moments is a straightforward
generalization of the case considered in this work: one should add the higher-order harmonic terms to the potential $\psi$ in Eq. (9):

$$\psi = \text{Re} \left[ \log(\zeta) - \sum_{i=2}^{\infty} \epsilon_n \zeta^{-i} \right], \quad \zeta = \rho e^{i\phi},$$

where $\epsilon_n$ are complex harmonic moments. Then one could perform the geometric optics analysis and the stationary phase integration taking account of the new potential.

Note that the corrections to the PSF due to the fluctuations of plasma density in the solar atmosphere might be even more important than corrections accounting for higher harmonic moments.

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Appendix A. Algorithms of Theory of Diffraction Applied to Strongly Non-Degenerate Case

Below we perform a double-check of our main results for the strongly non-degenerate case $q \epsilon \gg 1$ ($\epsilon \gg \lambda/r_s$) using algorithms from the geometrical theory of diffraction.

We start with the computation of the intensity near a regular point of the caustic. According to the theory of uniform (caustic) expansions, the intensity magnification in the vicinity of a regular point $Q$ of the caustic surface is equal to (see, e.g., Refs. [13,17])

$$\mu = 2\pi U \left( \frac{2k^2}{R_s} \right)^{1/6} A^2 \left( -D \frac{2k^2}{R_s} \right), \quad (A.1)$$

where $D$ is the distance to the caustic from its convex side. Here, $R_s$ stands for the radius of curvature of the section of the caustic surface by the plane $P$ containing a light ray that is tangent to the caustic at $Q$. The plane $P$ also contains a vector normal to the caustic at $Q$ (see Fig. A.1). The pre-factor $U = U(Q)$ in Eq. (A.1) is determined by matching the geometrical optics value of magnification (12) in the vicinity of the caustic (taking into account the multiplicity of images) with the following asymptotics of Eq. (A.1) at $D \gg (R_s/k^2)^{1/3}$:

$$\mu \to \frac{U}{\sqrt{D}}, \quad (A.2)$$

To find all the above values, we use the expansion in the proximity of the critical line (13):

$$\rho = \rho_c(\theta) + \Delta \rho. \quad (A.3)$$

It is now convenient to introduce another set of caustic-linked coordinates $\Delta \rho, \theta$ (see also Eq. (14)):

$$(X, Y) = (b\xi_c(\theta), b\eta_c(\theta)) + \Delta \vec{r}, \quad \Delta \vec{r} = (2b\Delta \rho \cos \theta, 2b\Delta \rho \sin \theta). \quad (A.4)$$

The vector $\Delta \vec{r}$ is tangent to the caustic at $Q$. Therefore, for small $\Delta \vec{r}$, the distance from the caustic to the point $(X, Y, Z)$ equals

$$D = \frac{\Delta \vec{r}^2}{2R_a} = \frac{2b^2}{R_a} \Delta \rho^2,$$
where \( R_a \) is the radius of curvature of the astroid (14):

\[
R_a = 6be \sin(2\theta).
\]

The point \((X, Y)\) has two (“strong”) pre-images in the close vicinity of the critical line \(^{27}\) with

\[
\Delta \rho = \pm \frac{1}{2b} \sqrt{2R_a D}.
\]

On the other hand, it follows from Eqs. (A.3) and (12) that away from the caustic \( \mu = \frac{1}{4\Delta \rho} \) and, taking account of image multiplicity and signs, we get

\[
\mu = \frac{1}{2|\Delta \rho|} = \frac{b}{\sqrt{2R_a D}}.
\]

Comparing the above equation with Eq. (A.2) we get

\[
U = b/\sqrt{2R_a} = b/\sqrt{12\epsilon b \sin 2\theta}.
\]

Also,

\[
R_s = R_a/\alpha_E^2 = \frac{6\epsilon b \sin(2\theta)}{\alpha_E^2},
\]

since the plane \( P \) intersects the \( z \)-axis under the angle \( \alpha_E + \mathcal{O}(\alpha_E \epsilon) \) (see Fig. A.1).

Plugging the above values for \( U \) and \( R_s \) into Eq. (A.1), with the help of Eqs. (7), (16), we get the expected final expression (52) for the near-fold GPSF.

Let us now consider the pattern in regions near the turning points \( \theta = 0, \pi/2, \pi, 3\pi/2 \), where the intensity reaches its maximum. Without loss of generality we take the cusp at \( \theta = 0 \).

According to the theory of the uniform caustic expansions (see, e.g., Refs. [17,20]), in the local coordinates of the section plane \( \tilde{x}, \tilde{y} \) (see Fig. A.1), where the equation of the caustic has the approximate form

\[
\tilde{x}^3 = -\frac{9}{8} a \tilde{y}^2,
\]

\(^{27}\) It has up to four pre-images in total, depending on the observer’s position relative to the caustic.
the magnification equals
\[ \mu(\tilde{x}, \tilde{y}) = W \left| \text{Pe} \left( \tilde{x} \left( \frac{6k}{a} \right)^{1/2}, \tilde{y} \left( \frac{24k^3}{a} \right)^{1/4} \right) \right|^2. \] (A.6)

Here, the pre-factor \( W \) is determined by matching the geometrical optics value of magnification (12) in the cusp neighborhood with the corresponding asymptotics of Eq. (A.6). It is convenient to set \( \tilde{y} = 0 \) and use the asymptotics of Eq. (A.6) for \( \tilde{x} \gg \sqrt{\frac{2\pi}{6k}} \):
\[ \mu(\tilde{x}, 0) \rightarrow \frac{\pi W}{\tilde{x}} \sqrt{\frac{a}{6k}}. \] (A.7)

The section plane is parallel to the \( y \)-axis and intersects the \( z \)-axis under the angle \( \alpha_E + O(\alpha_E \epsilon) \) (see Fig. A.1). Therefore,
\[ \tilde{X} = \alpha_E \tilde{x}, \quad \tilde{Y} = \tilde{y}, \] (A.8)

where \( (\tilde{X}, \tilde{Y}) \) is the deviation from the cusp in the observer \( z = Z \) plane. Taking Eq. (A.8) into account, from Eqs. (A.5) and (14) we get
\[ a = 12 \epsilon b / \alpha_E^3. \]

From Eq. (12) it follows that for \( \tilde{Y} = 0 \) and \( \tilde{X} > 0 \), near the cusp \( \mu \rightarrow \frac{b}{2\tilde{x}} \). Then with the help of Eq. (A.7, A.8) we obtain
\[ W = \frac{1}{4\pi} \sqrt{\frac{2kb\alpha_E}{\epsilon}}. \]

Substituting the above values into Eq. (A.6) and taking Eq. (A.8) into account we get the expected expression (45) for the near-cusp GPSF.

Appendix B. Refraction in Solar Atmosphere

The solar atmosphere introduces corrections to the gravitational deflection picture. In the first approximation, the correction \( \alpha_{pl} \) to the total deflection angle \( \alpha \rightarrow \alpha + \alpha_{pl} \) due to refraction in the solar plasma is the cylindrically symmetric vector field (for a review see, e.g., Ref. [4] and references therein):
\[ \alpha_{pl} = \left( \frac{\lambda}{\Lambda} \right)^2 \left[ A \left( \frac{R_0}{r_\perp} \right)^2 + B \left( \frac{R_0}{r_\perp} \right)^6 + C \left( \frac{R_0}{r_\perp} \right)^{16} \right], \]

where \( \Lambda \approx 50 \text{ m}, A \approx 1.1, B \approx 2.28 \times 10^2, C \approx 2.952 \times 10^3 \). The atmosphere bends the rays outwards while gravity bends the rays inwards. The above correction corresponds to the circularly symmetric contribution \( \psi_{pl} = \psi_{pl}(\rho) \) to the dimensionless potential \( \psi \rightarrow \psi + \psi_{pl} \) (for definitions of \( \psi \) and \( \rho \) see Eqs. (9), (8)):
\[ \psi_{pl} = \epsilon_{pl} \left[ A \frac{R_0}{b} \frac{1}{\rho} + B \frac{R_0}{b} \frac{1}{\rho^5} + C \frac{R_0}{b} \frac{1}{\rho^{15}} \right]. \]
where
\[ \epsilon_{pl} = \left( \frac{\lambda}{\Lambda} \right)^2 \frac{R_0}{r_g} \]
and \( b = b(Z) \) is given by Eq. (7).

The effect of the solar plasma is small for sub-micrometer wavelengths. Indeed, for \( \lambda = 10^{-6} \) m, \( \epsilon_{pl} \approx 10^{-10} \).

It is easy to see that due to the circular symmetry of \( \psi_{pl} \) and the smallness of \( \epsilon_{pl} \) the atmospheric refraction can be taken into account in this approximation by the formal renormalization of the gravitational radius \( r_g \rightarrow \tilde{r}_g(Z) \) in all our previous results:
\[ \tilde{r}_g(Z) \approx r_g \left( 1 - \frac{\alpha_{pl}(Z)}{\alpha_E(Z)} \right), \]
where \( \alpha_{pl}(Z) \) is the \( \alpha_{pl} \) evaluated at \( r_\perp = b(Z) \). Note that the correction factor \( \alpha_{pl}/\alpha_E \approx 5 \times 10^{-9} \) for \( \lambda = 10^{-6} \) m and \( Z \approx 1000 \) AU, i.e., the effect of refraction in sub-micrometer range of the EM spectrum is extremely small in this approximation.

Higher-order (non-symmetric) corrections to \( \psi_{pl} \) are proportional to the product of \( \epsilon_{pl} \) and small deformation parameters, such as the oblateness coefficient of the columnar density of the solar atmosphere etc. In principle, the contribution to \( \psi_{pl} \) corresponding to this oblateness can be accounted for through an effective (Z-dependent) correction of the quadrupole moment of the sun. This and higher-moment corrections can also be discarded in the context of the present work due to their smallness for wavelengths of interest. However, the question of the deformations of the caustic/PSF due to fluctuations in the solar atmosphere is worth studying in the context of high-resolution and/or longer-wavelength imaging.

References