

# Mach Number Correction of Rectangular Duct Criticals

**Dave Bartran**

dbartran@primary.net

ASME Member

**Gregory Banyay**

gab5631@psu.edu

ASME Member

## **ABSTRACT**

*The theory involved in accounting for the velocity dependence of acoustic cut-off frequencies in rectangular ducts is briefly summarized. This is followed by a comparison with reported duct critical frequencies in a flow test involving isolated cylinders in cross-flow. The agreement with the theory illustrates the importance of addressing the Mach number effects in moderate to high velocity services.*

## **INTRODUCTION**

The role of fluid velocity in formulating duct mode resonances, often referred to as the cut-off frequencies in isentropic uniform flow is well known, and readily developed from the convected wave equations [1-5]. While it is of little importance for

23 fluid velocities less than 40 m/s as in air ducting and many heat exchanger applications,  
24 published flow test in [6] covering a range of Mach numbers of 0.1 - 0.4, illustrate its  
25 potential significance. Operating experience in power plants likewise suggests the  
26 importance of understanding acoustic resonance conditions in steam lines [7], yet is not  
27 directly addressed in current industry guidance [8].

28 The flow tests presently under consideration [6], rely on carefully arranged,  
29 isolated cylinders in uniform, low turbulence cross-flow, to excite the acoustic duct  
30 modes sequentially as the fluid velocity is increased. This allows us to carefully examine  
31 the velocity dependence of the duct modes excited, with identifiable near constant  
32 Strouhal numbers of the cylinders of the providing the acoustic excitation.

33

#### 34 DUCT CUT-OFF FREQUENCIES

35

36 The convected wave equations [1-5] involving the acoustic velocity potential or  
37 the linearized Euler equations, are used to quantify duct cut-off frequencies or  
38 resonances. In rectangular ducts, the velocity potential  $\phi$ , as in [2] is described by

39

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = \frac{1}{c_0^2} \left( \frac{\partial}{\partial t} + v_x \frac{\partial}{\partial x} \right)^2 \phi \quad (1)$$

40 The right hand side of (1) is based on the total derivative of the velocity potential in the  
41 presence of idealized uniform flow, with duct coordinates  $\{x,y,z\}$ , mean fluid  
42 velocity  $V_x$ , and speed of sound,  $c_0$ .

43 Expanding the total derivative in (1), we now have

$$44 \quad \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = \frac{1}{c_0^2} \left( \frac{\partial^2 \phi}{\partial t^2} + 2V_x \frac{\partial^2 \phi}{\partial t \partial x} + V_x^2 \frac{\partial^2 \phi}{\partial x^2} \right) \quad (2)$$

45 This equation is combined with the trial solutions in (3) that are periodic in both time  
46 and duct coordinates, with frequency  $\omega$ , and wave numbers  $\{k_x, k_y, k_z\}$ .

$$47 \quad \phi = A e^{j(k_x x - \omega t)} \cos(k_y y) \cos(k_z z) \quad (3)$$

48 With the duct length considerably greater than the transverse dimensions, we shall  
49 focus on the transverse modes. Applying hard-wall boundary conditions given by

$$50 \quad \left. \frac{\partial \phi(x, y, z)}{\partial y} \right|_{y=\pm H/2} = 0 \quad \text{and} \quad \left. \frac{\partial \phi(x, y, z)}{\partial y} \right|_{z=\pm W/2} = 0, \quad (4)$$

51 to the trial solutions in (3) requires  $k_y = m\pi/H$  and  $k_z = n\pi/W$ , with integer mode  
52 numbers  $\{m, n\}$ .

53 Substituting the trial solutions into the convected wave equation (2) and  
54 simplifying, results in the characteristic equation

$$55 \quad (k_x^2 + k_y^2 + k_z^2) + [(\omega - V_x k_x)^2 / c_0^2 - k_x^2] = 0 \quad (5)$$

56 Upon defining  $\beta_{m,n}^2 \equiv k_y^2 + k_z^2$ , and rearranging allows us to write

$$57 \quad (1 - M^2)k_x^2 + 2k_x M(\omega/c_0) - [(\omega/c_0)^2 + \beta_{m,n}^2] = 0, \quad (6)$$

58 which is readily solved, yielding the longitudinal wave number as

$$59 \quad k_x(m,n) = -(M\omega/c_0) / \sqrt{1 - M^2} \pm \sqrt{(\omega/c_0)^2 - \beta_{m,n}^2 / (1 - M^2)} \quad (7)$$

60 When  $k_x$  is real, longitudinal propagation is allowed, and cut-off (or attenuated)

61 otherwise.

62 In the absence of flow, the wave number  $k_x(m,n)$  is zero for  $\omega/c_0 = \beta_{m,n}$ , with

63 propagation cut-off at the traditional duct resonance condition

$$64 \quad \omega(m,n) = c_0 \pi \sqrt{(m/H)^2 + (n/W)^2} \quad (8)$$

65 In the presence of uniform flow, the cut-off frequency is now

$$66 \quad \omega(m,n) = c_0 \pi \sqrt{(n/H)^2 + (m/W)^2 / (1 - M^2)^{0.5}}, \quad (9)$$

67 with wave propagation allowed for  $\omega > \omega(m,n)$ .

68 Before comparing these predictions with those observed in [6], one can consider a

69 finite element solution of the acoustic wave equation in terms of the acoustic pressure

70 in the absence of flow as in [6]. The Helmholtz pressure wave equation is

71 
$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = \frac{1}{c_0^2} \frac{\partial^2 p}{\partial t^2} \quad (10)$$

72 With solutions of the form  $p = \varphi e^{-j\omega t}$ , now (10) reduces to

73 
$$\nabla^2 \varphi + k^2 \varphi = 0 \quad (11)$$

74 where  $k = \omega/c_0$ , with duct cut-off frequencies  $\omega$ , and speed of sound  $c_0$ . Using (11)

75 together with the hardwall boundary conditions  $\varphi(x, y)|_{y=\pm H/2} = 0$  and  $\varphi(x, y)|_{z=\pm W/2} = 0$ , one

76 can compute the cut-off frequencies  $\omega$ , and mode shapes  $\varphi$ , using finite element

77 methods described in [9,10].

78

## 79 FLOW TEST COMPARISON

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81 The duct dimensions used in the flow test [6] are shown in Fig. 1, with transverse

82 dimensions of the duct width  $W=127$  mm and height  $H=254$  mm, with cylinders having

83 diameter  $D=12.7$  mm. The reported Strouhal number as  $St(m, 0) = D f(m, 0) / V$ , is taken

84 as the mean of those reported in Table 1 of [6] as  $St=0.189$  for Reynolds number of

85 31,000 to 137,000. This Strouhal estimate agrees with the empirical correlations in [11]

86 commonly used with smooth cylinders in cross flow. The fluid temperature is not

87 stated, so we have assumed it to equal 20 degrees C, which corresponds to  $c_0=343$  m/s.

88 The reported duct resonances presented in Table 1, represent single and

89 multiple cylinders, which are carefully positioned to coincide with the acoustic particle

90 velocity antinodes, so as to engage the first three duct modes in sequential fashion with  
91 increasing fluid velocity. The frequencies reported for a given mode number are  
92 averaged to provide mean estimates of each frequency of each mode. The root mean  
93 square (RMS) of the fluctuations about the mean are less than 0.5% for each mode. In  
94 the case of the single cylinder located midway at  $y/H=0$  (a pressure node), the  $m=2$   
95 mode is not excited, and is omitted from the averaging process. The changes in the  
96 absolute temperature associated with the reported pressure drop are less than 0.22% at  
97 100 m/s and 0.58% at 150 m/s, and is omitted from consideration.

98 The mean duct criticals for the half-dozen cylinder arrangements tested, are  
99 plotted in Fig. 2, versus the transverse mode number  $m$ , with and without the Mach  
100 number correction. The finite element calculations, shown as a dashed line, coincide  
101 with the analytical solutions in (8). The fluid velocity associated with the modes excited  
102 by vortex shedding acoustics is shown adjacent to the mode number to facilitate  
103 association of the fluid velocity and this cylinder shedding rate associated with the  
104 modes excited.

105 In Table 1, the predicted duct criticals (9) versus mode number together with the  
106 mean fluid velocity at each excited duct resonance are summarized together with the  
107 frequency deviation from the mean for each mode. Clearly the transverse duct  
108 resonances, based on the convected wave equation agree with the flow test data. The  
109 criticals without the Mach number correction, over-estimate those observed by 0.4% for

110 velocities of 45 m/s ( $M=0.13$ ), increasing to 7.5% at 127 m/s ( $M=0.37$ ). This suggests  
111 that the Mach number correction is minimal in most industrial applications.

112

## 113 **CONCLUSIONS**

114

115 This rather brief comparison of the duct criticals intends to increase awareness  
116 of the convected wave equation where high velocity fluids are encountered. We began  
117 with the convective wave equation in (1) and the transverse resonance conditions in (9),  
118 with and without the Mach number correction to exemplify the importance of the Mach  
119 number in moderate to high velocity, duct acoustics.

120 While the flow test considered herein specifically to an atmospheric wind tunnel  
121 test, it provides definitive evidence of Mach number effects with increasing velocity,  
122 thus serving to validate the analysis. The convected wave equation may also be  
123 expressed in terms of acoustic pressure and particle velocity, but requires careful  
124 attention to the boundary conditions when less than ideal flow conditions are present.  
125 Applications involving high velocity fluids in cylindrical ducts are covered in [1-5], while  
126 those involving cross-flow heat exchangers as in [12,13] are beyond the scope of the  
127 present article and subject to ongoing study.

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129

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### Figure Captions List

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Fig. 1 Duct Coordinates showing side and end view and representative cylinder placements in [6]

Fig. 2 Comparison of duct criticals with and without uniform flow Mach No. correction

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### Table Caption List

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Table 1      Frequency of transverse duct modes for single and multiple cylinders [6]

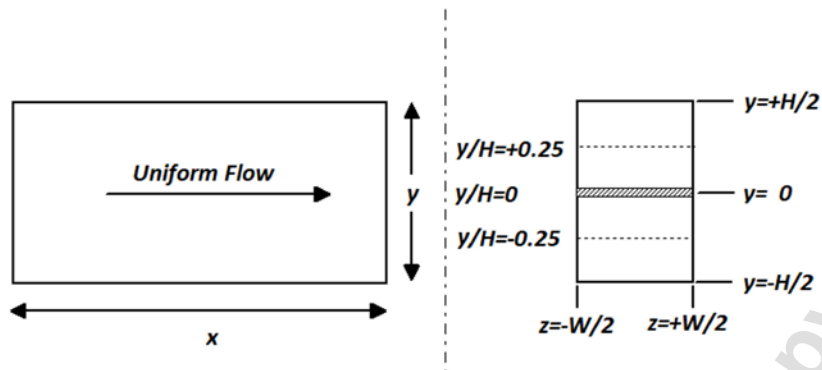
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Fig. 1



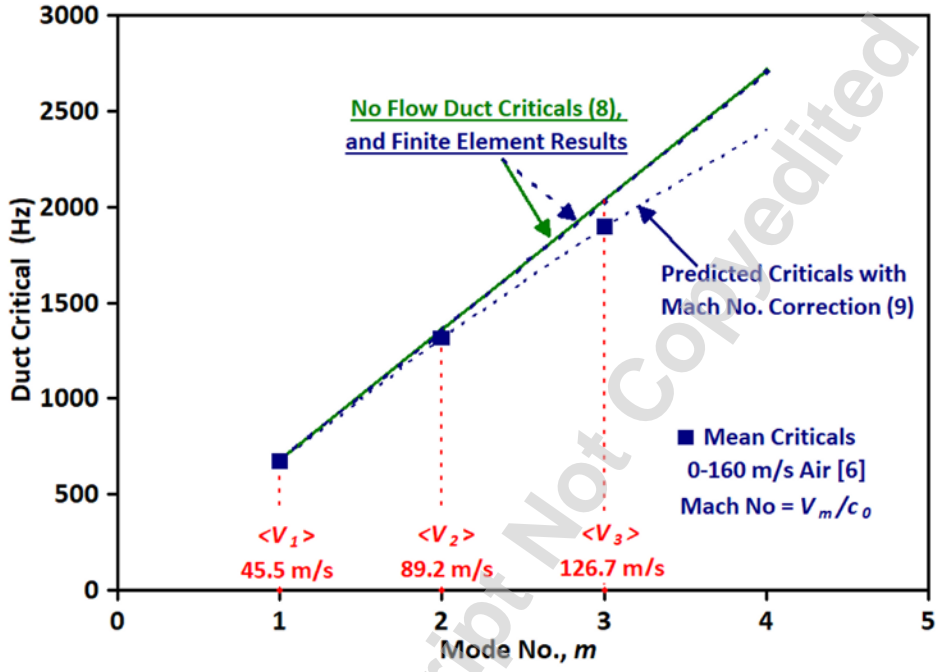
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Fig. 2



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Table 1

<b><u>Y/H Cylinder</u></b> <b><u>Location</u></b>	<b><u>First Transverse</u></b> <b><u>Mode;</u></b> <b><u>Mach=0.131</u></b>  <b><u>m=1</u></b>	<b><u>Second Transverse</u></b> <b><u>Mode;</u></b> <b><u>Mach=0.259</u></b>  <b><u>m=2</u></b>	<b><u>Third Transverse</u></b> <b><u>Mode;</u></b> <b><u>Mach=0.370</u></b>  <b><u>m=3</u></b>	<b><u>St Nos.</u></b> <b><u>Reported</u></b> <b><u>m=1; n=0</u></b>
-	(Hz)	(Hz)	(Hz)	-
0	678	<u>Not excited</u>	1884	0.190
0.25	681	1326	1886	0.189
0, 0.25	679	1332	1889	0.193
0.25, -0.25	678	1324	1880	0.194
0, 0.25 & -0.25	679	1332	1879	0.187
0, 0.167 & -0.167	677	1326	1926	0.184
<b>RMS Scatter</b>	<b><u>0.1%</u></b>	<b><u>0.1%</u></b>	<b><u>0.4%</u></b>	<b><u>0.7%</u></b>
$\langle f(m,n)=0 \rangle$	<b><u>679</u></b>	<b><u>1330</u></b>	<b><u>1891</u></b>	<b><u>0.189</u></b>

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