INCORPORATING MINIMUM SUBSISTENCE CONSUMPTION INTO INTERNATIONAL COMPARISONS OF REAL INCOME

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Abstract—Cross-country demand data are often consistent with the existence of a representative consumer with homothetic preferences. While homotheticity allows the construction of tight bounds to quantity indexes and their variance, it contradicts the biological reality that humans require minimum consumption of food, clothing, and shelter. This paper presents an approach for nonparametrically estimating bounds to utility from above-subsistence consumption. OECD data are used to show that homotheticity markedly compresses the real income distribution relative to what is found under the more general class of affine-homothetic preferences, and this has major consequences for estimates of convergence.

I. Introduction

THE considerable body of research on aggregation methods in the context of international comparisons of real income has been driven by the availability of comparable and disaggregated price and quantity data for countries, and also significant demand for the comparative data that these methods produce. Rankings of countries in terms of real GDP per capita are used as an indicator of comparative economic performance, and the measurement of convergence (or lack thereof) of real GDP is central to distinguishing between competing models of economic growth.

Those working on international comparisons of income are faced with the "index number problem," which refers to the ambiguity associated with comparing consumption bundles valued at different sets of relative prices. One aspect of this problem is the issue of nonuniqueness—real income comparisons between two countries can validly be made using either country's price vector in the calculation; international comparisons thus focus on multilateral indexes which, by construction, satisfy base-country invariance (measurement is independent of which country is being used as the base in the comparison).[1] The second aspect of the index number problem is the indeterminacy resulting from a lack of complete information on consumer preferences—we do not directly observe utility.

A major approach to multilateral comparisons has developed from the nonparametric demand analysis of Afriat (1972, 1981), Diewert (1973), and Varian (1982, 1983).[2] Of particular relevance here is the work of Dowrick and Quiggin (1997), who provide new results on the construction of multilateral true quantity indexes (and propose a new index, the Ideal Afriat Index) and the bounds to the variance of these indexes. However, multilateral true comparisons require the data to be consistent with a representative consumer exhibiting homothetic preferences, implying that in consumption space, the (linear) income expansion paths originate from a single point (the minimum subsistence consumption bundle), which is constrained to the origin. Consequently, for any given relative price vector, budget shares are constant across income levels (income elasticities are constrained to 1). Representative agents in rich and poor countries are therefore restricted to devote the same share of their budget to food, for example, and this contradicts within-country empirical evidence that food is a necessity.

Despite the restrictiveness of homotheticity, many international comparison data sets are found to be compatible with this assumption, and Dowrick and Quiggin (1997) used their approach to study convergence of real GDP between 1980 and 1990. The main contribution of the present paper is to show that the assumption of homotheticity, while empirically supported by real-world data sets, is not inconsequential. In particular, homotheticity leads to a marked compression of the real income distribution relative to what is found under the more general class of affine-homothetic preferences, and this has implications for real income comparisons and the measurement of convergence. With affine homotheticity, the linear expansion paths originate from a subsistence bundle that is not necessarily the origin and hence while income elasticities for *marginal consumption* (consumption in excess of a minimum subsistence bundle) are constrained to 1, the income elasticities for *consumption* may differ from 1 (and thus goods can be necessities or luxuries).

This paper presents a method for constructing bounds to true marginal indexes using affine-homothetic Afriat envelope functions and tests the implications for the measurement of real GDP and convergence using 1980 and 1990 OECD data. The approach also enables the nonparametric estimation of demand elasticities, and it is shown that while the bounds to these elasticities are relatively wide, the midpoints conform with economic intuition. It is possible to impose constraints on the bounds to the subsistence bundle (and hence income elasticities), and this leads to significantly tighter bounds on true marginal indexes and their variance.

The structure of the paper is as follows. Section II presents results on the existence and construction of a multilateral true quantity index. In section III, an approach for constructing bounds to a multilateral true marginal index is outlined. Section IV provides an empirical investigation

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[1] A multilateral index is an index that satisfies circularity: the real income of observation i relative to observation j is the same whether the two are compared directly or via an arbitrary intermediate third observation k.

[2] The other main approaches to multilateral comparisons are the fixed-weight Geary-Khamis (GK) index (Geary, 1958; Khamis, 1970, 1972), recently extended by Neary (2004) to incorporate substitution in consumption, and the "superlative" EKS and CCD indexes (Diewert, 1976, 1978).

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of marginal real income indexes using 1980 and 1990 OECD data. Section V presents conclusions and avenues for future research.

II. Homothetic Preferences and the Multilateral True Index

In this section, it is shown that the bounds to the true index can be calculated by adapting results of Chavas and Cox (1997) on the construction of Afriat envelope functions. While the Afriat homothetic envelope function presented here is an interesting theoretical insight in its own right, it is empirically redundant in that it produces bounds to true indexes that can be more conveniently calculated via a minimum path combinatorial algorithm. However, in the next section, an affine-homothetic version of the Afriat envelope function is introduced, and this function is empirically important since it can be used to construct tight bounds to marginal true indexes (and these bounds cannot be calculated via alternative means).

A. A Quantity Index

Let $\mathcal{N} = \{1, \ldots, N\}$ denote the set of observations and let $\mathbf{p}^i$ and $\mathbf{q}^i$ denote $K \times 1$ vectors of prices and quantities consumed, respectively, for observation i. The Allen Quantity Index (Allen, 1949) comparing observation i and j at reference prices $\mathbf{p}^r$ is defined by

$$Q(\mathbf{q}^i, \mathbf{q}^j; \mathbf{p}^r) = \frac{e(U^i, \mathbf{p}^r)}{e(U^j, \mathbf{p}^r)},$$

where $U^i = u(\mathbf{q}^i)$, $u(\ )$ is a utility function, and $e(U, \mathbf{p})$ is the expenditure function. The Allen index gives the fraction of the cost of attaining observation j's utility level required to attain observation i's utility level at the reference prices. In general, different reference prices will result in different values of the Allen index; however, as shown by Samuelson and Swamy (1974) and others, the unique true index, defined as

$$Q(\mathbf{q}^i, \mathbf{q}^j; \mathbf{p}^r) \equiv Q(\mathbf{q}^i, \mathbf{q}^j) = \frac{u(\mathbf{q}^i)}{u(\mathbf{q}^j)},$$

exists when preferences are homothetic.

B. Testing the Homothetic Representative Consumer Hypothesis

Afriat (1972, 1981) and Diewert (1973) showed that a necessary and sufficient condition for a given set of demand data to be rationalized by a homothetic utility function is the existence of a set of Afriat numbers.

Definition 1. Afriat numbers: *A set of positive numbers* $\mathbf{A} = (A^1, A^2, \ldots, A^N)$ *such that*

$$A^i/A^j \le \frac{\mathbf{p}^j \cdot \mathbf{q}^i}{\mathbf{p}^j \cdot \mathbf{q}^j}, \quad i, j \in \mathcal{N}, \tag{1}$$

is a true quantity index.

Varian (1983) showed that the existence of a locally nonsatiated homothetic function that rationalizes the data is equivalent to the condition that the data satisfy the Homothetic Axiom of Revealed Preference (HARP), namely that for all distinct choices of indexes $(i, j, \ldots, m)$ we have $\left(\frac{\mathbf{p}^i \cdot \mathbf{q}^j}{\mathbf{p}^i \cdot \mathbf{q}^i}\right)\left(\frac{\mathbf{p}^j \cdot \mathbf{q}^k}{\mathbf{p}^j \cdot \mathbf{q}^j}\right) \cdots \left(\frac{\mathbf{p}^m \cdot \mathbf{q}^i}{\mathbf{p}^m \cdot \mathbf{q}^m}\right) \ge 1$. If we define $\mathbf{L}$ as the matrix of logarithms of Laspeyres indexes with typical element: $\{L_{ij}\} = \log(\mathbf{p}^j \cdot \mathbf{q}^i/\mathbf{p}^j \cdot \mathbf{q}^j)$, then a test of HARP involves using Warshall's algorithm to construct the minimum path matrix $\mathbf{M}$: $M_{ij} = \min_{k,\ldots,m} \{L_{ij}, (L_{ik} + L_{kl} + \cdots + L_{mj})\}$. If any of the diagonal elements of $\mathbf{M}$ are negative, then HARP is violated. Dowrick and Quiggin (1997) define $\mathcal{A}$ to be the set of true indexes that can be constructed from the elements of $\mathbf{M}$, and show that the bounds to any true index $\mathbf{a} \in \mathcal{A}$ can also be calculated from $\mathbf{M}$.

C. The Homothetic Afriat Envelope Function

The bounds to true indexes provided by $\mathbf{M}$ can equivalently be constructed by adapting the results of Chavas and Cox (1997) who construct Afriat envelope functions under the assumption of general preferences.[3] The homothetic outer envelope function is a monotonic concave polyhedral function and is defined for a given set of Afriat numbers $\mathbf{A}$ as

$$a_O(\mathbf{q}, \mathbf{A}) = \min_i [A^i \mathbf{v}^i \cdot \mathbf{q} : i \in \mathcal{N}], \tag{2}$$

where $\mathbf{v}^i = \mathbf{p}^i/\mathbf{p}^i \cdot \mathbf{q}^i$. Define the Marshallian demand correspondence for a particular homothetic utility function $a(\mathbf{q})$ as

$$\mathbf{q}(\mathbf{v}) = \text{argmax}_{\mathbf{q}} [a(\mathbf{q}) : \mathbf{v} \cdot \mathbf{q} \le 1, \mathbf{q} \ge 0].$$

The homothetic utility function $a(\mathbf{q})$ *rationalizes* (or could have generated) the demand data $\{(\mathbf{q}^i, \mathbf{v}^i) : i \in \mathcal{N}\}$ if $\mathbf{q}^i \in \mathbf{q}(\mathbf{v}^i)$, $i \in \mathcal{N}$.

Proposition 1. $a_O(\mathbf{q}, \mathbf{A})$ *is a representation of consumer preferences that rationalizes the demand data* $(\mathbf{q}^i, \mathbf{v}^i)$, $i \in \mathcal{N}$.

Proof: See appendix.

Proposition 2. *For a given set* $\mathbf{A}$ *satisfying equation (1), the function* $a_O(\mathbf{q}, \mathbf{A})$ *provides the homothetic outer-bound representation of consumer preferences:*

$$a(\mathbf{q}) \le a_O(\mathbf{q}, \mathbf{A}),$$

[3] Chavas and Cox (1997) draw from Sidney Afriat's pioneering work in this area.

where $a(\mathbf{q})$ is any concave, monotonic, continuous, nonsatiated homothetic utility function that rationalizes the data $(\mathbf{q}^i, \mathbf{v}^i)$, $i \in \mathcal{N}$, and satisfies $a(\mathbf{q}^i) = A^i$ for all $i \in \mathcal{N}$.

PROOF: See appendix.

Proposition 2 shows that $a_O(\mathbf{q}, \mathbf{A})$ provides the widest upper bound to all possible concave and nonsatiated homothetic utility representations of the (unobserved) underlying utility function $u(\mathbf{q})$. However, this bound is not unique and in fact is *conditional* on a particular set of Afriat numbers, $\mathbf{A}$. Adapting results in Chavas and Cox (1997), we now calculate an *unconditional* upper bound for $a(\mathbf{q})$.

The concave function $a(\mathbf{q})$ is only defined up to a positive linear transformation and hence, without loss of generality, the Afriat inequalities (1) can be rewritten as

$$
\begin{aligned}
&A^i/A^j \le \mathbf{v}^j \cdot \mathbf{q}^i, \quad i, j \in \mathcal{N},\\
&A^i > 0, \quad i \in \mathcal{N}, \qquad\qquad (3)\\
&A^b = 1,
\end{aligned}
$$

where A^b is the utility level for some base observation $b \in \mathcal{N}$.

We are now able to define the *unconditional* homothetic outer bound at point $\mathbf{q}$:

$$A_O(\mathbf{q}) = \max_{\mathbf{A}} [a_O(\mathbf{q}, \mathbf{A}) : \text{Eq. (3)}]. \qquad (4)$$

Result 1. *Given equation (3), the function $A_O(\mathbf{q})$ in equation (4) provides the unconditional homothetic outer-bound representation of consumer preferences at point $\mathbf{q}$:*

$$a(\mathbf{q}) \le A_O(\mathbf{q}).$$

Result 1 shows that the function $A_O(\mathbf{q})$ provides the widest possible upper bound on all possible concave and nonsatiated homothetic utility representations of $u(\mathbf{q})$. While $A_O(\mathbf{q})$ can be calculated using the linear program in equation (4), the task of constructing this bound is simplified by the following proposition.

Proposition 3. *The unconditional homothetic outer-bound envelope function can be calculated as*

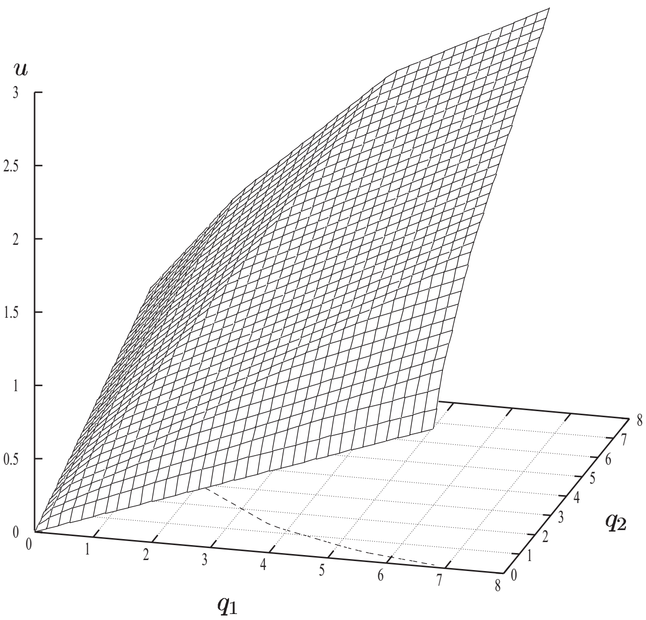
$$A_O(\mathbf{q}) = \min_h [\exp(M_{bi})\mathbf{v}^i \cdot \mathbf{q} : i \in \mathcal{N}], \qquad (5)$$

where M_{ij} is the ijth element of the minimum path matrix, $\mathbf{M}$, $\mathcal{N}$ is the set of countries that share common homothetic preferences, and $A^b = 1$.

PROOF: By construction, $\exp(M_{bi})$ gives the upper bound to A^i consistent with the data being rationalized by a homothetic utility function and $A^b = 1$. ■

It is apparent that since $\mathbf{v}^i \cdot \mathbf{q}^i = 1$, $A_O(\mathbf{q}^i) = \exp(M_{bi})$, for all $i \in \mathcal{N}$. Thus, $A_O(\mathbf{q})$ is redundant for utility comparisons involving observations in $\mathcal{N}$. However, equation (5) can be used to impute utility bounds for another country r where $r \notin \mathcal{N}$. That is, one can calculate the utility of the representative consumer, whose preferences are homothetic and consistent with observations in the set $\mathcal{N}$, given they consumed the consumption bundle $\mathbf{q}^r$, even though the homothetic consumer would not have in fact chosen bundle $\mathbf{q}^r$ at prices $\mathbf{p}^r$. Equation (5) was first proposed by Varian (1983) [equation (19)]; a contribution of the present paper is to show how this equation can be derived from the Afriat homothetic envelope approach.

FIGURE 1.—UNCONDITIONAL HOMOTHETIC OUTER ENVELOPE FUNCTION



In figure 1, the unconditional homothetic outer envelope function (with observation 3 as the base) is shown for a five-observation, two-good data set where the price vectors are $\mathbf{p}^1 = \{3, 1\}$, $\mathbf{p}^2 = \{0.5, 1\}$, $\mathbf{p}^3 = \{1, 1\}$, $\mathbf{p}^4 = \{5, 1\}$, and $\mathbf{p}^5 = \{0.25, 1\}$, and the commodity bundles are $\mathbf{q}^1 = \{1.25, 2.25\}$, $\mathbf{q}^2 = \{7, 2.5\}$, $\mathbf{q}^3 = \{2.5, 2.5\}$, $\mathbf{q}^4 = \{1.5, 7.5\}$, and $\mathbf{q}^5 = \{6, 0.5\}$.

III. Affine-Homothetic Preferences and the Multilateral True Marginal Index

This section presents results on the existence and bounds of a multilateral true marginal quantity index, which measures the utility gained from marginal or supernumerary consumption (consumption in excess of a minimum subsistence bundle).

A. A Marginal Quantity Index

The Allen Marginal Quantity Index comparing observation i and j at reference prices $\mathbf{p}^r$ is defined by

$$\tilde{Q}(\mathbf{q}^i, \mathbf{q}^j; \mathbf{p}^r) = \frac{e(U^i, \mathbf{p}^r) - e(U^s, \mathbf{p}^r)}{e(U^j, \mathbf{p}^r) - e(U^s, \mathbf{p}^r)},$$

where U^s is an arbitrary level of subsistence utility. The Allen marginal index gives the fraction of the cost of attaining observation j's marginal utility level required to attain observation i's marginal utility level at the reference prices. In general, different reference prices will result in different values of the Allen marginal index.

Result 2. EXISTENCE OF A UNIQUE TRUE MARGINAL INDEX:[4] *If, and only if, preferences are affine-homothetic, with* $\tilde{u}(\mathbf{q}) \equiv u(\mathbf{q} - \boldsymbol{\gamma})$ *representing the affine-homothetic utility function*[5] *measuring utility gained from consumption in excess of the subsistence level of consumption* $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_K)$ *will, for all* $\mathbf{p}^r$,

$$Q(\mathbf{q}^i, \mathbf{q}^j; \mathbf{p}^r) \equiv \tilde{Q}(\mathbf{q}^i, \mathbf{q}^j) = \frac{u(\mathbf{q}^i - \boldsymbol{\gamma})}{u(\mathbf{q}^j - \boldsymbol{\gamma})}.$$

B. *Testing the Affine-Homothetic Representative Consumer Hypothesis*

Given that the data do not reject the null hypothesis of common affine-homothetic preferences, then we cannot reject the hypothesis of the existence of a multilateral true marginal index. However, we are now confronted by an even worse problem of indeterminateness than that encountered in the construction of multilateral true indexes. In particular, we do not know the quantities in the "true" subsistence bundle $\boldsymbol{\gamma}$ and even if we did know $\boldsymbol{\gamma}$, then there is a whole family of affine-homothetic utility functions that would be consistent with this subsistence bundle.

In the face of this indeterminateness, it is necessary to construct bounds to all the possible affine-homothetic utility functions that are consistent with the data. First assume that a given subsistence bundle denoted $\mathbf{g}^s = (g_1^s, \ldots, g_K^s)$ is the true subsistence bundle, that is, $\boldsymbol{\gamma} = \mathbf{g}^s$. It is possible to construct bounds to the true marginal index consistent with $\mathbf{g}^s$ using the Laspeyres and Paasche marginal quantity indexes, defined respectively:

$$\frac{\mathbf{p}^j \cdot (\mathbf{q}^i - \mathbf{g}^s)}{\mathbf{p}^j \cdot (\mathbf{q}^j - \mathbf{g}^s)},$$

$$\frac{\mathbf{p}^i \cdot (\mathbf{q}^i - \mathbf{g}^s)}{\mathbf{p}^i \cdot (\mathbf{q}^j - \mathbf{g}^s)}.$$

The bundle $(\mathbf{q}^i - \mathbf{g}^s)$ is one way of achieving $U^i = u(\mathbf{q}^i - \mathbf{g}^s)$, but not necessarily the cheapest when prices are $\mathbf{p}^j$; hence $\mathbf{p}^j \cdot (\mathbf{q}^i - \mathbf{g}^s) \geq e(U^i, \mathbf{p}^j) - a(\mathbf{p}^j)$. By definition, $\mathbf{p}^j \cdot (\mathbf{q}^j - \mathbf{g}^s) = e(U^j, \mathbf{p}^j) - a(\mathbf{p}^j)$. Hence:

$$\frac{\mathbf{p}^j \cdot (\mathbf{q}^i - \mathbf{g}^s)}{\mathbf{p}^j \cdot (\mathbf{q}^j - \mathbf{g}^s)} \geq \frac{e(U^i, \mathbf{p}^j) - a(\mathbf{p}^j)}{e(U^j, \mathbf{p}^j) - a(\mathbf{p}^j)} = \tilde{Q}(\mathbf{q}^i, \mathbf{q}^j; \mathbf{p}^j),$$

that is, the Laspeyres marginal quantity index is the upper bound to the base-weighted Allen marginal index. Using similar reasoning, the Paasche marginal quantity index is the lower bound to the current-weighted Allen marginal index:

$$\frac{\mathbf{p}^i \cdot (\mathbf{q}^i - \mathbf{g}^s)}{\mathbf{p}^i \cdot (\mathbf{q}^j - \mathbf{g}^s)} \leq \frac{e(U^i, \mathbf{p}^i) - a(\mathbf{p}^i)}{e(U^j, \mathbf{p}^i) - a(\mathbf{p}^i)} = \tilde{Q}(\mathbf{q}^i, \mathbf{q}^j; \mathbf{p}^i).$$

This leads to the following empirical definition of a multilateral true marginal quantity index.

Definition 2. AFRIAT MARGINAL NUMBERS: *If two sets of numbers* $\tilde{\mathbf{A}} = (\tilde{A}^1, \tilde{A}^2, \ldots, \tilde{A}^N)$ *and* $\mathbf{g} = (g_1, \ldots, g_K)$ *such that*[6]

$$\tilde{A}^i > 0, \quad i \in \mathcal{N},$$
$$0 \leq g_l \leq q_l^i, \quad i \in \mathcal{N}, \quad l \in \mathcal{K}, \tag{6}$$
$$\tilde{A}^i/\tilde{A}^j \leq \frac{\mathbf{p}^j \cdot (\mathbf{q}^i - \mathbf{g})}{\mathbf{p}^j \cdot (\mathbf{q}^j - \mathbf{g})}, \quad i, j \in \mathcal{N},$$

exist, then $\tilde{\mathbf{A}}$ *is a multilateral true marginal index.*

C. *The Affine-Homothetic Afriat Outer Envelope Function*

For a given data set satisfying affine homotheticity, bounds to Afriat marginal numbers can be constructed using an affine-homothetic version of the unconditional Afriat homothetic envelope function (4).

Definition 3. AFFINE-HOMOTHETIC OUTER ENVELOPE FUNCTION: *The affine-homothetic outer envelope function is a monotonic concave polyhedral function and is defined for a given* $\tilde{\mathbf{A}}$ *and* $\mathbf{g}$ *satisfying the Afriat marginal inequalities (6) as*

$$\tilde{a}_O(\mathbf{q}, \tilde{\mathbf{A}}, \mathbf{g}) = \min_i \left[\tilde{A}^i \frac{\mathbf{p}^i \cdot (\mathbf{q} - \mathbf{g})}{\mathbf{p}^i \cdot (\mathbf{q}^i - \mathbf{g})} : i \in \mathcal{N} \right]. \tag{7}$$

Note that equation (7) can alternatively be expressed as the following primal nonlinear programming problem:

$$\tilde{a}_O(\mathbf{q}, \tilde{\mathbf{A}}, \mathbf{g}) = \max_{\eta} \left[\eta : \eta \leq \tilde{A}^i \frac{\mathbf{p}^i \cdot (\mathbf{q} - \mathbf{g})}{\mathbf{p}^i \cdot (\mathbf{q}^i - \mathbf{g})}, i \in \mathcal{N} \right]. \tag{8}$$

[4] It has long been recognized that an affine-homothetic utility function is homothetic in the marginal quantities $(\mathbf{q} - \boldsymbol{\gamma})$ and Lloyd (1979, p. 684) mentions that the existence theorem of Samuelson and Swamy (1974) for homothetic functions "carries over" to affine-homothetic functions. In the interest of space, the proof is therefore omitted here.

[5] Note that an affine-homothetic utility function is formally defined as $f[u(\mathbf{q} - \boldsymbol{\gamma})]$, where $f[\ ]$ is a monotonic increasing transformation and $u(\ )$ is homogeneous of degree 1. Since consumer behavior is invariant with respect to monotonic transformations of the utility function, we chose $u(\ )$ itself as the representation of preferences.

[6] The constraints on $\mathbf{g}$ are required to keep the problem bounded.

Proposition 4. *$\tilde{a}_O(\mathbf{q}, \mathbf{A}, \mathbf{g})$ is a representation of consumer preferences that rationalizes the demand data $(\mathbf{q}^i, \mathbf{p}^i)$, $i \in \mathcal{N}$.*

PROOF: Analogous to proof of proposition 1.

Proposition 5. *For a given set of $\tilde{\mathbf{A}}$ and $\mathbf{g}$ satisfying equation (6), the function $\tilde{a}_O(\mathbf{q}, \tilde{\mathbf{A}}, \mathbf{g})$ provides the outer-bound affine-homothetic representation of consumer preferences:*

$$\tilde{a}(\mathbf{q} - \mathbf{g}) \leq \tilde{a}_O(\mathbf{q}, \tilde{\mathbf{A}}, \mathbf{g}),$$

where $\tilde{a}(\mathbf{q} - \mathbf{g})$ is any concave, monotonic, continuous, nonsatiated affine-homothetic utility function that rationalizes the data $(\mathbf{q}^i, \mathbf{p}^i)$, $i \in \mathcal{N}$, and satisfies $\tilde{a}(\mathbf{q} - \mathbf{g}) = \tilde{A}^i$ for all $i \in \mathcal{N}$.

PROOF: Analogous to proof of proposition 2.

Proposition 5 shows that $\tilde{a}_O(\mathbf{q}, \tilde{\mathbf{A}}, \mathbf{g})$ provides the widest upper bound to all possible concave and nonsatiated affine-homothetic utility representations of $u(\mathbf{q})$ (conditional on a given subsistence bundle, $\mathbf{g}$). However, $\mathbf{g}$ is just one of a potentially infinite number of subsistence bundles that are consistent with the data, and $\tilde{\mathbf{A}}$ is just one of a potentially infinite number of sets of Afriat marginal numbers consistent with $\mathbf{g}$. We need to calculate an *unconditional* upper bound for $\tilde{a}(\mathbf{q} - \boldsymbol{\gamma})$, that is, a bound that is not conditional on a particular $\tilde{\mathbf{A}}$ and $\mathbf{g}$. The concave function $\tilde{a}(\mathbf{q} - \mathbf{g})$ is only defined up to a positive linear transformation and hence, without loss of generality, the Afriat marginal inequalities (6) can be rewritten as

$$\begin{aligned} &\tilde{A}^i > 0, \quad i \in \mathcal{N}, \\ &\tilde{A}^b = 1, \\ &0 \leq g_l \leq q_l^i, \quad i \in \mathcal{N}, \quad l \in \mathcal{K}, \\ &\tilde{A}^i/\tilde{A}^j \leq \frac{\mathbf{p}^j \cdot (\mathbf{q}^i - \mathbf{g})}{\mathbf{p}^j \cdot (\mathbf{q}^j - \mathbf{g})}, \quad i, j \in \mathcal{N}, \end{aligned} \tag{9}$$

where $\tilde{A}^b$ is the marginal utility level for some base observation $b \in \mathcal{N}$.

We are now able to define the *unconditional* affine-homothetic outer bound at observation $\mathbf{q}$ (with base b):

$$\tilde{A}_{O,b}(\mathbf{q} - \mathbf{g}) = \max_{\tilde{\mathbf{A}}, \mathbf{g}} [a_O(\mathbf{q}, \tilde{\mathbf{A}}, \mathbf{g}) : \text{Eq. (9)}]. \tag{10}$$

Result 3. *Given equation (9), the function $\tilde{A}_{O,b}(\mathbf{q} - \mathbf{g})$ provides the unconditional outer-bound affine-homothetic representation of consumer preferences at point $\mathbf{q}$:*

$$\tilde{a}(\mathbf{q} - \boldsymbol{\gamma}) \leq \tilde{A}_{O,b}(\mathbf{q} - \mathbf{g}).$$

Result 3 shows that the function $\tilde{A}_{O,b}(\mathbf{q} - \mathbf{g})$ provides the widest possible upper bound on all possible concave and nonsatiated affine-homothetic utility representations of $u(\mathbf{q})$. Equation (10) is a nonlinear programming problem consisting of $N + K + 1$ variables and $(N + K + 1)(N + 1)$ constraints. For the test data introduced earlier, an affine-homothetic outer envelope function will resemble that depicted in figure 1 but with the envelope no longer constrained to be touching the origin. The function $\tilde{A}_{O,b}(\mathbf{q} - \mathbf{g})$ can be used in money-metric welfare comparisons.

Definition 4. BOUNDS TO TRUE BILATERAL COMPARISONS: *Let $(\mathbf{q}, \mathbf{p})$ be demand data that satisfy affine homotheticity. The upper bound to the marginal utility comparison between observation v and base observation b is given by the unconditional affine-homothetic outer bound estimated at $\mathbf{q}^v$, $\tilde{A}_{O,b}(\mathbf{q}^v - \mathbf{g}^{vb})$, where $\mathbf{g}^{vb}$ is the subsistence bundle that maximizes $\tilde{A}_{O,b}(\mathbf{q}^v - \mathbf{g}^{vb})$.*

Note, however, that once $\mathbf{g}^{vb}$ has been found using equation (10), the upper bound to the true marginal index can be alternatively calculated by applying the minimum path algorithm to the relevant marginal demand data.

Result 4. *Let $\tilde{\mathbf{M}}^{vb}$ represent the minimum path matrix constructed using the marginal demand data $(\mathbf{q} - \mathbf{g}^{vb}, \mathbf{p})$, where $\mathbf{g}^{vb}$ is calculated for observation v using equation (10). It is apparent that*

$$\tilde{M}_{ji}^{vb} = \log \tilde{A}_{O,b}(\mathbf{q}^i - \mathbf{g}^{vb}) \text{ for } i = v \text{ and } j = b.$$

The elements of $\tilde{\mathbf{M}}^{vb}$ therefore provide upper bounds to the marginal utility ratios between observations, conditional on the subsistence bundle $\mathbf{g}^{vb}$. However, given that $\mathbf{g}^{vb}$ has been calculated via equation (10) so as to maximize the marginal utility ratio between observations v and b, it will be the case that $\tilde{M}_{ji}^{vb} = \log \tilde{A}_O(\mathbf{q}^i - \mathbf{g}^{vb})$ when $i = v$ and $j = b$. There will be N^2 minimum path matrices than can be constructed from marginal demand data, one for each subsistence bundle estimated using equation (10).

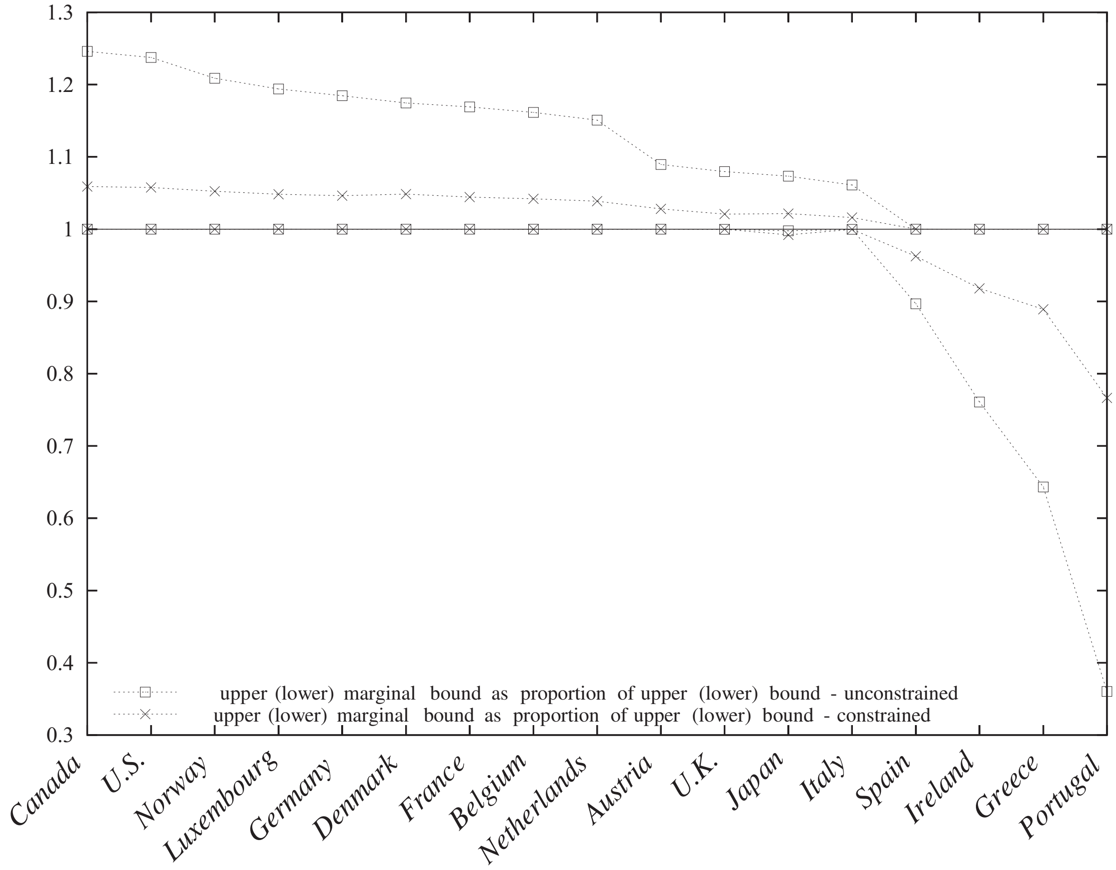
IV. Measures of True Marginal GDP and Convergence for OECD Countries

Dowrick and Quiggin (1997) calculated true measures of GDP and convergence using 1980 and 1990 International Comparison Program data on around forty components of GDP expenditure for seventeen OECD countries. In this section, the same data are used to show the impact of relaxing homotheticity on calculated bounds to GDP and convergence.

The construction of a complete set of $N^2 = 289$ bilateral bounds to marginal GDP involved solving 289 nonlinear programs (10) in 56 variables and 1,008 constraints.[7] The

[7] The nonlinear programs were solved using the AMPL-MINOS optimization software.

FIGURE 2.—TRUE MARGINAL BOUNDS AND TRUE BOUNDS, 1980



estimated subsistence bundles were then used to calculate bounds to true marginal GDP using the 1980 and 1990 OECD data. Dowrick and Quiggin (1997) show that the upper bound to the true index is the vector of column means from the minimum path matrix, while the lower bound is the vector of negative row means. The calculation of the bounds to true marginal GDP is analogous: the vector of upper marginal bounds is calculated as the maximum over each of the vector of column means of the 289 minimum path matrices constructed using the estimated subsistence bundles, while the vector of lower marginal bounds is the minimum over the 289 vectors of negative row means. Thus, while the true upper and lower bounds themselves form a true index, this is not necessarily the case with the true marginal bounds (in other words, the upper marginal bound for the United States might be calculated from one of the 289 marginal minimum path matrices, while the upper marginal bound for Canada may be from another).

In figure 2, normalized bounds to true marginal GDP are presented for the seventeen countries for 1980, with the countries ranked in descending order according to the Ideal Afriat Index of Dowrick and Quiggin (1997). To aid visual comparison, each marginal true bound is presented as a proportion of the respective true bound (which itself is expressed as a proportion of the sample mean). These numbers can be hard to interpret since they reflect proportions of proportions, so an example is warranted. For the unconstrained case (the constrained case is explained below), the richest country (Canada) has an upper marginal bound that is 25% greater than the upper bound, while its lower marginal bound is equal to the lower bound. Thus, while Canada's true income is estimated to be up to 44% above the sample mean (this number is not shown in figure 2), its true marginal income is up to $1.25 \times 44 = 55\%$ above the sample mean. In contrast, the upper marginal bound and upper bound for the poorest country (Portugal) are equal, while the lower marginal bound is only 36% of the lower true bound. Thus, while Portugal's true income is estimated to be as low as 42% below the sample mean (again, this number is not shown in figure 2), its marginal income is estimated to be as low as only $0.36 \times 42 = 15\%$ of the sample mean.

From figure 2, it is clear that the assumption of homotheticity leads to a marked compression of the real income distribution relative to what is found under the more general assumption of affine-homotheticity. In general, one can show that for demand data that are consistent with homotheticity the variance of any true marginal index must be greater than or equal to the variance of its counterpart true index. Let $\mathcal{Q}_b \equiv \{Q(\mathbf{q}^1, \mathbf{q}^b), \ldots, Q(\mathbf{q}^N, \mathbf{q}^b)\}$ and $\tilde{\mathcal{Q}}_b \equiv$

TABLE 1.—ESTIMATES OF VARIANCE OF REAL LOG GDP AND σ CONVERGENCE

	(1)	(2)	(3)	(4)	(5)
		True Marginal Indexes			
			Constrained		
	True Index	Unconstr.	Necess./Lux.	Homot. ± 1%	Homot. ± 10%
Variance—1980					
Upper extreme bound	0.117	0.413	0.176	0.120	0.161
Upper observed bound	0.109	0.396	0.164	0.112	0.150
Lower observed bound	0.094	0.094	0.094	0.094	0.094
Lower extreme bound	0.087	0.087	0.087	0.087	0.087
Variance—1990					
Upper extreme bound	0.086	0.300	0.133	0.089	0.118
Upper observed bound	0.083	0.289	0.125	0.085	0.112
Lower observed bound	0.075	0.075	0.075	0.075	0.075
Lower extreme bound	0.071	0.071	0.071	0.071	0.071
σ convergence					
Upper extreme bound	−0.030	−0.113	−0.043	−0.031	−0.043
Upper observed bound	−0.026	−0.106	−0.039	−0.027	−0.038
Lower observed bound	−0.019	−0.019	−0.019	−0.019	−0.019
Lower extreme bound	−0.016	−0.016	−0.016	−0.016	−0.016

Note: The last two entries in column 1 differ slightly from table 5 in Dowrick and Quiggin (1997) because of rounding errors. In column 3, income elasticities for certain goods have been constrained to be either necessities or luxuries. In columns 4 and 5, the assumption of homothetic preferences has been relaxed by 1% and 10%, respectively.

$\{\tilde{Q}(\mathbf{q}^1, \mathbf{q}^b), \ldots, \tilde{Q}(\mathbf{q}^N, \mathbf{q}^b)\}$ be counterpart true and true marginal indexes, respectively, where b is some arbitrarily chosen base observation. It is apparent that

$$\text{var}\ (\mathcal{Q}_b) = \text{var}\left[e(U^1, \mathbf{p}^r), \ldots, e(U^N, \mathbf{p}^r)\right] \times e(U^b, \mathbf{p}^r)^{-2}.$$

Affine-homothetic preferences are characterized by a specific version of the Gorman Polar Form (GPF) expenditure function, $e(U, \mathbf{p}) = a(\mathbf{p}) + b(\mathbf{p})U$, where U is the utility level in excess of the subsistence level of utility (normalized to be zero), $a(\mathbf{p}) = \sum_l \gamma_l p_l$ is the fixed cost of attaining subsistence utility and thus is the cost of purchasing a subsistence bundle of goods, and $b(\mathbf{p})$ is the marginal price of attaining utility U above the subsistence level. $a(\mathbf{p})$ and $b(\mathbf{p})$ are both positive and homogeneous of degree 1 functions of prices. Thus,

$$\text{var}\ (\tilde{\mathcal{Q}}_b) = \text{var}\left[e(U^1, \mathbf{p}^r), \ldots, e(U^N, \mathbf{p}^r)\right] \times \{e(U^b, \mathbf{p}^r) - a(\mathbf{p}^r)\}^{-2}.$$

It will be the case that var $(\mathcal{Q}_b) \leq$ var $(\tilde{\mathcal{Q}}_b)$ if $e(U^b, \mathbf{p}^r)^{-2} \leq \{e(U^b, \mathbf{p}^r) - a(\mathbf{p}^r)\}^{-2}$. This implies that $a(\mathbf{p}^r) \geq 0$, which is true by the fact that $a(\mathbf{p}^r)$ is a positive and homogeneous degree 1 function.

Column 1 of table 1 presents bounds to the variance of the true index, replicated from table 5 in Dowrick and Quiggin (1997), while column 2 presents the bounds to the variance of the true marginal index for the "unconstrained" affine-homothetic case (the remaining columns of table 1 are discussed below). The upper observed bound for variance of log marginal GDP in 1980 (0.396) was nearly four times that found for log GDP (0.109).[8]

An unexpected finding in figure 2 was that for twelve of the seventeen countries the lower marginal true bound coincides with the lower true bound, while for four countries (Spain, Ireland, Greece, Portugal) the upper marginal bound coincides with the upper true bound.[9] Thus, for the first set of "rich" countries, the marginal approach indicates that they are unambiguously better off (relative to the sample mean) as soon as we acknowledge the potential existence of a minimum subsistence bundle (which is the same for all countries) and factor this into welfare comparisons. The second set of "poor" countries are unambiguously worse off under the marginal true comparisons, compared with the true comparisons. When applied to a data set covering the entire development spectrum (not just OECD countries), the marginal true index approach presented here can therefore be used to construct a poverty line for international comparisons of poverty (possibly set to be equal to the per capita income of the richest country that is unambiguously worse off under the marginal true comparisons, compared with true comparisons). An alternative approach for constructing a minimum subsistence poverty line would be to value the subsistence bundles obtained from equation (10) in local currencies—this could be compared with the World Bank's \$1-per-person-per-day International Poverty Line.

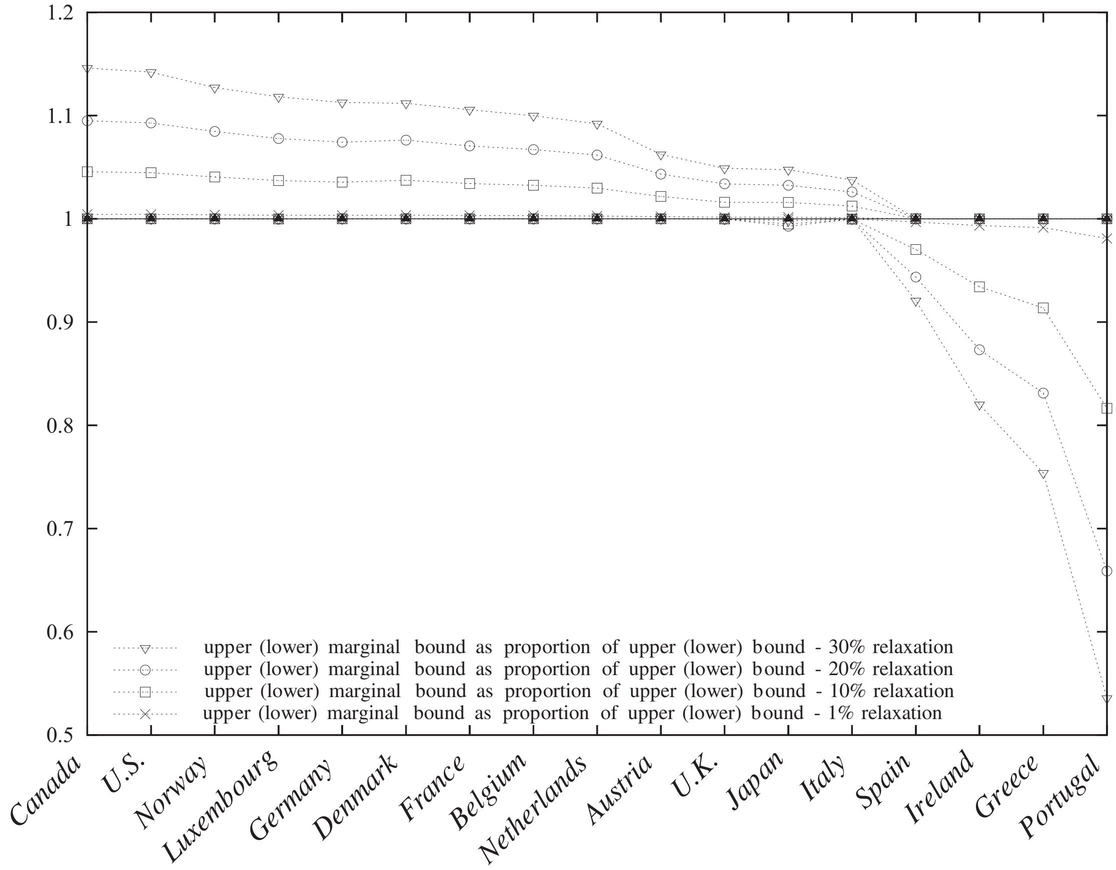
A. *Bounds to Income Elasticities*

With GPF preferences, the demand function for good l is linear in supernumerary income ($\tilde{x} = \mathbf{p} \cdot (\mathbf{q} - \boldsymbol{\gamma})$): $q_l = \gamma_l + \beta_l \tilde{x}$. The marginal propensity to consume therefore equals the ratio of supernumerary consumption of good l to

[8] Table 1 also shows extreme bounds to the variance of the various indexes, defined in equation (7) of Dowrick and Quiggin (1997).

[9] Japan was an unusual case in that the lower marginal true bound is 0.2% below the lower true bound.

FIGURE 3.—TRUE MARGINAL BOUNDS AND TRUE BOUNDS: RELAXATION OF HOMOTHETICITY, 1980



total supernumerary expenditure: $\beta_l = \tilde{q}_l/\tilde{x}$, where $\tilde{q}_l = q_l - \gamma_l$. The income elasticity of consumption is therefore

$$\epsilon_l \equiv \frac{x}{q_l}\frac{\delta q_l}{\delta x} = \frac{x}{q_l}\beta_l = \frac{x}{q_l}\frac{\tilde{q}_l}{\tilde{x}}.$$

The above formula was used to estimate nonparametric bounds to income elasticities for the average OECD country. The calculation of the complete set of 289 bilateral bounds to true marginal GDP for the 1980 data resulted in 289 nonparametric estimates of income elasticity for each good. The bounds to the elasticity for each good tend to be very wide (the income elasticity of consumption for bread and cereals for 1980, for example, ranges between 0.33 and 1.31). However, the midpoints of these ranges suggest that, for example, bread and cereals, fruits and vegetables, and clothing are necessities (income elasticity less than 1), while beverages and transport equipment/services are luxuries (income elasticity greater than 1). Thus, the nonparametric approach produces income elasticities that make economic sense.

The wideness of the nonparametric bounds to the income elasticities parallels the finding, displayed in figure 2, that bounds to the true marginal index are wide. It is possible to tighten the bilateral bounds to the true marginal index by using a priori information on the signs of the income elasticities.[10] Based on the midpoints of the estimated elasticities and a priori expectations, eight (thirteen) goods were constrained to be necessities (luxuries) and the nonlinear programs (10) were rerun with these additional constraints on the derived income elasticities (and thus constraints on the calculated subsistence bundles).[11] As shown in figure 2, the use of constraints on the income elasticity of demand results in much tighter bounds to the true marginal index; the elasticity-constrained lower bound to the true marginal GDP of Portugal is $0.64 \times 42 = 28\%$ of the sample mean. The application of constraints on income elasticities compresses the real income distribution—from column 3 of table 1, the upper observed bound of log marginal real GDP variance in 1980 was 0.164, which is less than half what was found for the unconstrained case.

B. *Measuring* σ *Convergence*

Dowrick and Quiggin (1997) provide estimates of the rate of true quantity convergence, defined as the measured change in the variance (σ^2) of log GDP, between 1980 and 1990 for the seventeen OECD countries—these results are

[10] See Chavas and Cox (1997) for a related example of using a priori information.

[11] See the data annex for a complete listing of the estimated income elasticities and constraints.

replicated in column 1 of table 1. The upper extreme bound to variance in 1980 (1990) is 0.117 (0.086), which implies an upper extreme bound to σ convergence between 1980 and 1990 of $0.086 - 0.117 = -.030$. The lower extreme bound to σ convergence is similarly the difference between the lower extreme bounds to variance in the two years, and hence does not indicate the theoretically lowest rate of convergence (or highest rate of divergence) that could have occurred. However, as noted in Dowrick and Quiggin (1997, footnote 7), even the least favorable calculation of σ convergence, involving the extreme upper bound to variance in 1990 (0.086) and the extreme lower bound for 1980 (0.087), still supports the conclusion that true GDP converged rather than diverged between the two years. The upper observed bound to variance for 1980 (1990) is 0.109 (0.083), implying an upper observed bound to convergence of $0.083 - 0.109 = -0.026$ (the lower observed bound to true convergence is calculated in a similar manner).

From column 2 of table 1, the upper extreme bound to σ convergence for real marginal GDP is −0.113, which indicates a rate of convergence nearly four times that found for real GDP. The lower observed and extreme bounds to σ convergence for real marginal GDP are equal to these bounds for real GDP. However, while the least favorable calculation of σ convergence indicates that true GDP converged between the two years, the same cannot be said for true marginal GDP. The extreme upper bound to variance of real marginal GDP in 1990 (0.3) is much greater than the extreme lower bound for 1980 (0.087), indicating potentially marked divergence of real marginal GDP between the two years. Even with the constraints on income elasticity (column 3 of table 1), we find that the least favorable comparison of variances between the two years indicates possible divergence in real marginal GDP.

C. The Impact of Relaxing Homotheticity

Finally, in figure 3, the normalized true marginal bounds are presented for various deviations from the assumption of homotheticity. A 1% relaxation of homotheticity implies that the estimated income elasticities for the average country were bounded within the range of 0.99 and 1.01. Of interest is the fact that a 1% relaxation of homotheticity has a larger than 1% impact on the measured real income of the poorest country, Portugal: relaxing homotheticity by 1% leads to the lower marginal bound for Portugal being 2% below the lower true bound, and a 10% relaxation results in the lower true marginal bound being only 82% of the true counterpart bound. The last two columns of table 1 show the impact of relaxing the assumption of homotheticity on estimates of variance and σ convergence. With only a 1% relaxation in homotheticity, the upper extreme bound to log variance in 1990 (0.089) lies above the lower extreme bound to log variance in 1980 (0.087). Dowrick and Quiggin's finding that real income unambiguously converged over the time period is therefore altered with a marginal relaxation of homotheticity.

V. Conclusions

Due to our inability to directly observe utility, international comparison of real income will always involve indeterminacy. For a given set of demand data satisfying homotheticity, the indeterminacy arises because there exists a family of empirically indistinguishable homothetic utility functions that could have generated the data. Dowrick and Quiggin (1997) show that for data sets consistent with homothetic preferences (as many real-world data sets are found to be), it is possible to construct tight bounds to real income and, relatedly, bounds to convergence between given years. However, by assuming homothetic preferences, a second source of indeterminacy arising from incomplete knowledge as to composition of the minimum subsistence bundle is ignored.

The present paper has focused on this second source of indeterminacy in international comparisons. A new approach to international comparisons involving the construction of bounds to multilateral true marginal indexes via the affine-homothetic Afriat envelope function was presented. Using data on OECD countries for 1980 and 1990, it was shown that homotheticity leads to a marked compression of the real income distribution relative to what is found under the more general class of affine-homothetic preferences, and this impacts on the measurement of convergence. For demand data that are consistent with homotheticity, the variance of any true marginal index must be greater than or equal to the variance of its counterpart true index (intuitively, this is because the same minimum subsistence bundle applies to all countries, both rich and poor). The finding that the affine-homothetic approach generates greater dispersion in per capita incomes compared with the homothetic approach will therefore be robust across different data sets. Thus, while the assumption of homotheticity may not be contradicted by the data, this assumption is not inconsequential.

There are other interesting areas for further research on multilateral true marginal indexes. The impact of income elasticity constraints on estimates of the rate of real marginal GDP convergence could be more fully investigated. For example, it is possible to constrain income elasticities within tighter ranges (based on empirical evidence from other studies) and also to apply constraints to the maximum allowable change in income elasticities between the two years being studied. Also, the methods outlined in this paper are also directly applicable to cross-country leisure-inclusive comparisons of real GDP. In the absence of direct cross-country measures of leisure time consumed, the construction of a leisure-inclusive real GDP index requires an estimation of minimum subsistence leisure (that is, time spent eating, sleeping, and performing other necessary biological functions). While previous attempts at constructing leisure-inclusive welfare measures have required an essentially arbitrary assumption about subsistence leisure, the results in the present paper can be used to nonparametrically estimate the bounds to this parameter.

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APPENDIX A

The proofs of propositions 1 and 2 presented here are an adaptation to the homothetic case of proofs relating to general preferences found in Chavas and Cox (1997) (which in turn are based on proofs in Afriat, 1987).

PROOF OF PROPOSITION 1: We first need to show that $a_O(\mathbf{q}, \mathbf{A})$ is a representation of consumer preferences, that is, $a_O(\mathbf{q}^i, \mathbf{A}) = A^i$ for all $i \in \mathcal{N}$. If we evaluate equation (2) at $\mathbf{q} = \mathbf{q}^j$, it is apparent that under nonsatiation,

$$a_O(\mathbf{q}^j, \mathbf{A}) \leq A^j. \tag{A1}$$

Note that equation (2) can alternatively be expressed as the following primal nonlinear programming problem:

$$a_O(\mathbf{q}, \mathbf{A}) = \max_{\eta} [\eta : \eta \leq A^i \mathbf{v}^i \cdot \mathbf{q}, i \in \mathcal{N}]. \tag{A2}$$

It is clear from equation (1) that when $\mathbf{q} = \mathbf{q}^j$, $\eta = A^j$ is a feasible choice in equation (A2). It therefore follows from the maximization in equation (A2) that

$$a_O(\mathbf{q}^j, \mathbf{A}) \geq A^j. \tag{A3}$$

Therefore, from equations (A1) and (A3) it follows that $a_O(\mathbf{q}^j, \mathbf{A})$ is both an upper bound and a lower bound to A^j, implying that $a_O(\mathbf{q}^j, \mathbf{A}) = A^j$, for all $j \in \mathcal{N}$.

It has been shown that $a_O(\mathbf{q}, \mathbf{A})$ is a utility function—we now need to show that it is a utility function that rationalizes the data. To do this, we must show that $a_O(\mathbf{q}, \mathbf{A})$ attains its maximum at the chosen bundles. From equation (2) it is apparent that

$$\max_{\mathbf{q}} [a_O(\mathbf{q}, \mathbf{A})) : \mathbf{v}^j \cdot \mathbf{q} \leq 1, \mathbf{q} \geq 0] \leq \max_{\mathbf{q}} [A^j \mathbf{v}^j \cdot \mathbf{q} : \mathbf{v}^j \cdot \mathbf{q} \leq 1, \mathbf{q} \geq 0]. \tag{A4}$$

However, when $\mathbf{q} = \mathbf{q}^j$ (under nonsatiation), the right-hand side of equation (A4) is equal to A^j. Hence, $\mathbf{q}^j$ is a solution to the utility maximization problem, $\mathbf{q}^j = \text{argmax}_{\mathbf{q}} [a_O(\mathbf{q}, \mathbf{A}) : \mathbf{v} \cdot \mathbf{q} \leq 1, \mathbf{q} \geq 0]$, with $a_O(\mathbf{q}, \mathbf{A})$ being a representation of consumer preferences satisfying $a_O(\mathbf{q}^j, \mathbf{A}) = A^j$, $j \in \mathcal{N}$. ■

PROOF OF PROPOSITION 2: Let $a(\mathbf{q})$ be any concave, monotonic, continuous, nonsatiated homothetic utility function that rationalizes the data $(\mathbf{q}^i, \mathbf{v}^i)$, $i \in \mathcal{N}$, and satisfies $a(\mathbf{q}^i) = A^i$ for all $i \in \mathcal{N}$. From the maximization problem (A2), it is clear that $a_O(\mathbf{q}, \mathbf{A})$ provides the tightest upper-bound representation of utility levels consistent with the Afriat inequalities (1). ■

APPENDIX B

Data Annex

1. Data Source

The data source is the International Comparison Program (ICP) data for 1980 and 1990, used by Dowrick and Quiggin (1997). For reasons of data availability, Dowrick and Quiggin (1997) restricted their sample to seventeen OECD member countries. The 1980 ICP data cover 38 components of GDP expenditure and are from United Nations and Commission of the European Communities (1987), while the 1990 ICP data cover forty components of GDP expenditure and are from World Bank (1993). The original data files constructed by Dowrick and Quiggin (1997) were retrieved from ftp://coombs.anu.edu.au/coombspapers/coombsarchives/economics-rsss/dowrick/.

2. Perl Programs for Data Analysis

The Perl code and data files used to generate the results in this paper are available at http://hdl.handle.net/1885/46914.

3. Estimated Income Elasticities

The nonparametric bounds to income elasticities for the average OECD country are presented in Tables B1 and B2 (see section IV A for more details). The tables also show the income elasticity constraints that were placed on each good (goods were constrained to be either necessities or luxuries, or else were unconstrained)—see column 3 of table 1.

TABLE B1.—ESTIMATED INCOME ELASTICITIES FOR 17 OECD COUNTRIES, 1980

	Minimum	Midpoint	Maximum	Constraint
Bread and cereals	0.33	0.82	1.31	n
Meat	0.89	1.07	1.25	n
Fish	0.74	0.98	1.21	n
Milk, cheese, and eggs	0.68	0.95	1.21	n
Oils and fats	0.89	1.11	1.32	n
Fruits and vegetables	0.57	0.86	1.14	n
Other food	0.74	0.98	1.21	u
Nonalcoholic beverages	0.61	1.05	1.48	l
Alcoholic beverages	0.74	1.00	1.26	l
Tobacco	0.95	1.17	1.39	u
Clothing including repairs	0.46	0.89	1.31	n
Footwear including repairs	0.55	0.94	1.32	n
Gross rent and water charges	0.65	1.05	1.45	u
Fuel and power	0.78	1.00	1.21	u
Furniture, floor coverings, and repairs	0.87	1.06	1.24	u
Household textiles and repairs	0.83	1.15	1.46	u
Household appliances and repairs	0.64	0.98	1.31	u
Other household goods	0.58	0.95	1.32	u
Medical products, drugs	0.55	1.00	1.44	l
Medical and health services	0.62	1.03	1.43	l
Personal transport equipment	0.91	1.14	1.37	l
Operation of transport equipment	0.87	1.04	1.21	l
Transport services	0.78	1.12	1.46	l
Communication	0.80	1.12	1.43	l
Recreational equipment and repairs	0.87	1.10	1.32	l
Recreational and cultural services	0.66	0.96	1.26	l
Books, magazines, and newspapers	0.72	1.10	1.47	l
Education	0.65	1.04	1.43	l
Restaurants, cafés, and hotels	0.79	1.07	1.34	l
Nonelectrical equipment	0.76	0.99	1.21	u
Electrical equipment	0.87	1.10	1.32	u
Transport equipment	0.75	1.04	1.32	u
Residential buildings	0.67	1.05	1.43	u
Nonresidential buildings	0.99	1.15	1.31	u
Civil engineering works	0.91	1.12	1.32	u
Change in stocks	0.93	1.13	1.32	u
Balance of exports and imports	1.00	1.81	2.61	u
Government	0.67	1.02	1.37	u

Note: The last column indicates whether the item was constrained to be a necessity (n), luxury (l), or was unconstrained (u).

TABLE B2.—ESTIMATED INCOME ELASTICITIES FOR 17 OECD COUNTRIES, 1990

	Minimum	Midpoint	Maximum	Constraint
Bread and cereals	0.64	0.95	1.26	n
Meat	0.78	1.01	1.23	n
Fish	0.74	1.04	1.34	n
Milk, cheese, and eggs	0.63	0.88	1.12	n
Oils and fats	0.88	1.17	1.46	n
Fruit and vegetables	0.44	0.90	1.35	n
Other food	0.85	1.01	1.17	u
Nonalcoholic beverages	0.82	1.02	1.22	l
Alcoholic beverages	0.67	1.05	1.43	l
Tobacco	0.63	1.04	1.45	u
Clothing including repairs	0.52	0.89	1.26	n
Footwear including repairs	0.68	1.03	1.37	n
Gross rent and water charges	1.00	1.23	1.46	u
Fuel and power	0.71	0.95	1.18	u
Furniture, floor coverings, and repairs	0.83	1.04	1.25	u
Household textiles and repairs	0.66	1.05	1.44	u
Household appliances and repairs	0.56	0.91	1.26	u
Other household goods	0.51	0.85	1.18	u
Medical products, drugs	0.64	1.05	1.45	l
Medical and health services	0.88	1.17	1.45	l
Personal transport equipment	0.83	1.03	1.23	l
Operation of transport equipment	0.70	0.94	1.18	l
Transport services	0.75	1.05	1.35	l
Communication	0.65	1.05	1.45	l
Recreational equipment and repairs	0.80	1.03	1.26	l
Recreational and cultural services	0.86	1.10	1.34	l
Books, magazines, and newspapers	0.78	1.02	1.26	l
Education	1.00	1.25	1.49	l
Restaurants, cafés, and hotels	0.66	0.92	1.18	l
Other goods and services	0.76	1.07	1.37	u
Government collective services	0.46	0.90	1.34	u
Government individual services	0.72	1.06	1.40	u
Residential buildings	0.66	1.05	1.44	u
Nonresidential buildings	0.81	1.02	1.22	u
Civil engineering works	0.71	0.98	1.25	u
Transport equipment	0.84	1.05	1.26	u
Nonelectrical equipment	0.66	0.89	1.12	u
Electrical equipment	0.84	1.15	1.45	u
Increase in stocks	0.85	1.03	1.21	u
Balance of exports and imports	0.99	1.23	1.47	u

Note: The last column indicates whether the item was constrained to be a necessity (n), luxury (l), or was unconstrained (u).

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