Intertemporal Substitution and Equity Premium*

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Abstract
This article presents a model that incorporates habit formation and long-run risks into the Epstein–Zin preferences, and reveals intertemporal substitution as a distinctive channel, separate from risk aversion, in generating key asset market phenomena. With habit formation, both the risk aversion and intertemporal substitution channels enhance the market price of short-run consumption risk. With long-run risks, intertemporal substitution reduces the market prices of long-run consumption risks, working against risk aversion. The contrasting effects of the intertemporal substitution channel drive key differences in the model implications of habit formation and long-run risks.

JEL classification: E21, G12

1. Introduction
Risk aversion has long been the focus of the consumption-based asset pricing literature on the equity premium puzzle. Since Mehra and Prescott (1985), many studies have recognized that large risk aversion is necessary in order to reconcile the high levels of average stock returns and return volatilities with smooth consumption growth.¹

In contrast, the role of intertemporal substitution has been largely under-researched as most studies cannot effectively differentiate its effect from that of risk aversion. Often this is because models are based on the power utility. With the constraint that the elasticity parameter is the inverse of the risk aversion parameter, the two parameters cannot vary independently. Moreover, even in models using the Epstein and Zin (1989) utility that differentiates the two parameters, an i.i.d. random walk consumption process renders the Epstein–Zin utility equivalent to the power utility (Kocherlakota, 1990).

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This article presents a model that reveals intertemporal substitution as a distinctive and important channel, separate from risk aversion, in generating the equity premium, the return volatility, and their variations over time. The model is able to distinguish the two channels as a result of two features. First, the model is based on the Epstein–Zin utility that differentiates the two parameters. Second, the model incorporates external habit of Campbell and Cochrane (1999) and long-run risks as in Bansal and Yaron (2004). With habit, the agent cares about surplus consumption (i.e., consumption in excess of habit), and both the expected growth of surplus consumption and the volatility of the shocks to surplus consumption are time-varying. With long-run risks, both the expected consumption growth and the volatility of consumption growth shocks are time-varying. Hence, with either habit or long-run risks, or both, surplus consumption and/or consumption is not an i.i.d. random walk, and the model does not reduce to the power utility.

Intuitively, the intertemporal substitution channel arises because the agent prefers a smooth stream of surplus consumption, and demands compensation for the variations in surplus consumption across time. This is different from the risk aversion channel, in which the agent dislikes and requires compensation for the variations in surplus consumption across future states. A lower elasticity of intertemporal substitution parameter implies a stronger tendency to smooth over time, and thus a stronger effect of the intertemporal substitution channel. A higher risk aversion parameter implies a stronger preference to smooth across states, and thus a stronger effect of the risk aversion channel.

In identifying the separate channel of intertemporal substitution, the model reveals an interesting contrast between the habit and long-run risks mechanisms. With habit, the intertemporal substitution channel contributes positively to the market price of short-run consumption risk. With long-run risks, the intertemporal substitution channel contributes negatively to the market prices of long-run consumption risks. For both habit and long-run risks, the risk aversion channel contributes positively. Hence, with habit, the two channels work together; with long-run risks, the two channels oppose each other.

Consistent with the opposing roles of the two channels, the long-run risks mechanism hinges crucially on the agent’s preference for early resolution of uncertainty. As consumption growth is positively autocorrelated due to the persistent expected component, it is intuitive that the risk premium increases with the wedge between risk aversion and the inverse of the elasticity of intertemporal substitution. Thus, the agent would rather prefer late resolution of uncertainty in surplus consumption. While a low elasticity of intertemporal substitution may generate volatile risk-free rates, the habit sensitivity function counterbalances this effect.

I derive the approximate analytical solutions of the model to demonstrate the economic intuition underlying the two channels. To investigate the quantitative implications, I study

2 For habit formation, also see Sundaresan (1989), Abel (1990), Constantinides (1990), Ferson and Constantinides (1991), Heaton (1993, 1995), and Wachter (2006), among others. For long-run risks, also see Bansal, Dittmar, and Lundblad (2005), Hansen, Heaton, and Li (2008), and Bansal (2008) for a survey, among others.

3 Lustig, Van Nieuwerburgh, and Verdelhan (2013) also compare the habit formation and long-run risks models in terms of the implications on the wealth–consumption ratio. The habit formation model in their paper is based on the power utility.
three variants of the general model: Model I with habit only, Model II with long-run risks only, and Model III with both mechanisms. The sample period for consumption and asset prices is 1929–2012. I calibrate the models, compute numerical solutions, generate model-implied, simulated data, and then compare with the empirical data. Both analytical and numerical results confirm that risk aversion and intertemporal substitution drive different aspects of model implications. These results mitigate the concern that differentiating risk aversion and the elasticity of substitution merely results in unwarranted parameter proliferation and over-fitting.

In Model I with habit only, driven by the effect of the intertemporal substitution channel, both the equity premium and the return volatility increase with a decreasing elasticity of intertemporal substitution parameter. With the risk aversion and the intertemporal substitution channels working together, Model I requires a risk aversion of five and an intertemporal substitution of one to replicate the equity premium and the return volatility in the empirical data. For Model II with long-run risks only, the opposing effects from the two channels suggest that a strong risk aversion channel and a weak intertemporal substitution channel, or equivalently, a large wedge for the preference of early resolution of uncertainty, are required. This leads to the choice of a risk aversion of ten and an intertemporal substitution of 1.5 to replicate the empirical asset pricing moments.

The habit and long-run risks mechanisms also differ in their effects on the risk-free rate. Similar to Campbell and Cochrane (1999), Model I is devised to achieve a constant risk-free rate. In addition, both the risk aversion and the intertemporal substitution channels contribute positively to a strong precautionary savings effect that also generates a low risk-free rate. In Model II, with long-run risks only, both the level and the volatility of the risk-free rate increase with a decreasing intertemporal substitution parameter. To generate low and smooth risk-free rates, Model II requires a high intertemporal substitution parameter.

Model III incorporates both habit and long-run risks. Since the two channels operate through different consumption growth risks, their effects run parallel and are complementary to each other. For example, the equity premiums and the return volatilities in Model III are higher than those in Models I and II with the same parameters.

Given the model parameters and calibrations in the article, the habit mechanism appears to be a stronger driver of the quantitative results than long-run risks. A comparison of Models I and II indicates that habit generates quantitative results that are larger in magnitudes and also spread out over wider ranges. When both mechanisms are put together, the results of Model III are often closer in magnitudes to those of Model I, and also vary over the risk aversion and intertemporal substitution parameters mostly following the patterns in Model I. In the preferred calibration of Model III, habit contributes 80% of the equity premium, while long-run risks provide 20%.

This study brings together two prominent traditions in the literature. In response to the failure of the conventional power utility in resolving the equity premium puzzle, researchers have often introduced non-separabilities in the utility function. In one prominent tradition, habit formation introduces time non-separability. In another prominent tradition, the Epstein–Zin recursive utility introduces state non-separability. The model in this article

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4 For the recursive utility, also see Kreps and Porteus (1978), Epstein and Zin (1991), Weil (1989), Kandel and Stambaugh (1991), and Campbell (1996), among others.
embeds habit in the Epstein–Zin utility and investigates the interaction of the two non-separabilities.\(^5\)

The explicit decomposition of the intertemporal substitution and risk aversion channels in this article extends the insight in extant studies that explore their asset pricing implications. In a model based on the power utility, Constantinides (1990) points out that habit formation creates a wedge between risk aversion and intertemporal substitution. With the Epstein–Zin utility, my article is able to explicitly differentiate and quantitatively characterize the contributions from the two channels. Melino and Yang (2003) introduce the Epstein–Zin utility into the economy of Mehra and Prescott (1985) and show that exogenously imposed variations in the elasticity of substitution improve the performance of the model in explaining the equity premium puzzle. In my study, the effective elasticity of intertemporal substitution varies over time as a result of the slow moving habit, which is an economically interpretable process. Routledge and Zin (2010) show that generalized disappointment aversion endogenously generates asymmetric preference of consumption smoothing in bad times relative to good times, and provide an axiomatic foundation for their mechanism. My article does not provide an axiomatic foundation as in Routledge and Zin (2010). In my article, the surplus consumption ratio is low in bad times, and so is the effective elasticity of intertemporal substitution, which implies a stronger preference for consumption smoothing than in good times. Hence, my article suggests that habit provides an alternative mechanism for asymmetric preference of consumption smoothing, similar to generalized disappointment aversion.

In the rest of the article, I first present the general model in Section 2. Then I calibrate the model parameters and study the details of the three variants of the general model in Section 3. The concluding section discusses possible extensions of the article, and the appendices collect additional details of the model solution.

2. Model

This section presents the general model that integrates external habit and long-run risks into the Epstein–Zin preferences. The general model nests the habit model of Campbell and Cochrane (1999) and the long-run risks model of Bansal and Yaron (2004). I derive approximate analytical solutions to demonstrate the model intuition. In particular, I focus on how the model differentiates the contributions of intertemporal substitution and risk aversion to the pricing kernel, the market prices of risks, and the equity premium. More details of the model and the approximate analytical solutions are presented in the appendices.

2.1 Preferences

Utility \(U_t\) is defined recursively by embedding a certainty equivalent of a constant relative risk aversion parameter \(\gamma\),

\[
J(U_{t+1}) = \left( E_t \left[ U_{t+1}^{1-\gamma} \right] \right)^{\frac{1}{1-\gamma}},
\]

\(^5\) The utility function can also be non-separable across different goods or include labor. See, among others, Eichenbaum, Hansen, and Singleton (1988), Eichenbaum and Hansen (1990), Yogo (2006), Piazzesi, Schneider, and Tuzel (2007), and Uhlig (2006).
in an aggregator of a constant elasticity of intertemporal substitution parameter $\psi$,

$$U_t = \left(1 - \delta\right)(C_t - H_t)^{1 - \frac{1}{\psi}} + \delta f(U_{t+1})^{1 - \frac{1}{\psi}}.$$ (2)

Here, $C_t$ is consumption, $H_t$ is external habit, and $0 < \delta < 1$ is the time discount factor. If $\gamma = 1$ and/or $\psi = 1$, the power functions in the definition are replaced by log functions. More details for these special cases are in the appendices.

With this utility, the agent cares about surplus consumption rather than consumption per se. Hence, both risk aversion and intertemporal substitution are interpreted in terms of surplus consumption, the utility function is homogeneous in surplus consumption, and the utility is measured on the same scale as surplus consumption. In particular, if the agent consumes a deterministic stream of $C_t - H_t = C - H$, then $U_t = C - H$.

### 2.2 Consumption and Habit Processes

As in Bansal and Yaron (2004), consumption growth $\Delta C_{t+1}$ follows the dynamics:

$$\Delta C_{t+1} = \log \frac{C_{t+1}}{C_t} = \mu_c + x_t + \sigma_c \omega_{t+1} e_{c,t+1},$$ (3)

$$x_{t+1} = \phi_x x_t + \sigma_x \omega_{t} e_{x,t+1},$$ (4)

$$\omega_{t+1}^2 - 1 = \phi_{\omega} (\omega_t^2 - 1) + \sigma_{\omega} k_{\omega,t+1} e_{\omega,t+1} e_{x,t+1} e_{e,t+1} \sim \text{i.i.d. } N(0, 1).$$ (5)

Consumption growth has a mean of $\mu_c$ and contains two components. The expected component $x_t$ is an AR(1) process of the persistence parameter $0 < \phi_x < 1$, driven by shocks of the time-varying volatility $\sigma_x \omega_t$. The other component is random walk shocks with the time-varying volatility $\sigma_c \omega_t$. The underlying state variable for the time-varying volatilities is the variance $\omega_t^2$, which has a mean of 1 by construction, and follows an AR(1) process with the persistence $0 < \phi_{\omega} < 1$ and driven by shocks of the volatility $\sigma_{\omega}$.

In total, the growth dynamics contains three risks. The random walk shocks in consumption growth are short-run growth risk. Long-run growth risk comes from the shocks to the expected component $x_t$. The shocks to $\omega_t^2$ represent long-run volatility risk. As in Bansal and Yaron (2004), for parsimony, all shocks are mutually independent.

Following Campbell and Cochrane (1999), the evolution of external habit $H_t$ is specified through the surplus consumption ratio $S_t = (C_t - H_t) / C_t$. The log surplus ratio $s_t$ follows a mean-reverting process driven by consumption growth innovations,

$$s_{t+1} - \bar{s} = \phi_s (s_t - \bar{s}) + \lambda (s_t) \sigma_c \omega_{t} e_{c,t+1}.$$ (6)

Here, $\bar{s}$ is the mean, $0 < \phi_s < 1$ is the mean reversion parameter, and $\lambda (s_t)$ is the sensitivity function whose value varies with $s_t$. The process implies that

$$\Delta s_{t+1} = s_{t+1} - s_t = (\phi_s - 1)(s_t - \bar{s}) + \lambda (s_t) \sigma_c \omega_{t} e_{c,t+1},$$ (7)

$$E_t[\Delta s_{t+1}] = (\phi_s - 1)(s_t - \bar{s}).$$ (8)

A high surplus ratio today implies high consumption relative to habit. Going forward, habit will gradually catch up, and thus surplus consumption is expected to drop. In contrast, a low $s_t$ today implies that consumption is low and too close to habit. Habit will subsequently drop, and thus surplus consumption is expected to rise.
The log growth of surplus consumption is

$$\log \frac{C_{t+1} - H_{t+1}}{C_t - H_t} = \log \frac{C_{t+1} S_{t+1}}{C_t S_t} = \Delta C_{t+1} + \Delta S_{t+1}. \quad (9)$$

The expected surplus consumption growth

$$E_t[\Delta C_{t+1} + \Delta S_{t+1}] = \mu_t + x_t + (\phi_t - 1)(s_t - \bar{s}) \quad (10)$$

is time-varying. It varies positively with the persistent expected component $x_t$, and negatively with the mean-reverting $s_t$ process.

Following Campbell and Cochrane (1999), the sensitivity $\lambda(s_t)$ satisfies

$$\lambda(\bar{s}) = e^{-\gamma} - 1, \quad \lambda'(\bar{s}) = -e^{-\gamma}, \quad (11)$$

so that habit is pre-determined both at the steady state $s_t = \bar{s}$ and in a neighborhood of the steady state. These restrictions imply that $\lambda(s_t)$ is a decreasing function of $s_t$, and its level and slope cannot vary independently—a higher level of $\lambda(s_t)$ also implies a more negative slope. In addition to these two conditions, the sensitivity function is specified to achieve a risk-free rate that is constant with respect to $s_t$. More discussions of this requirement will come subsequently.

2.3 Pricing Kernel

Given wealth $W_t$, the agent maximizes

$$U_t = \left(1 - \delta \right) (C_t - H_t)^{1 - \frac{1}{\gamma}} + \delta \left( E_t \left[ U_{t+1}^{1 - \gamma} \right] \right)^{\frac{1}{1 - \gamma}} \frac{1}{1 - \gamma}. \quad (12)$$

The agent chooses consumption $C_t$ and invests the savings in a portfolio of the risk-free and risky assets to yield the next period wealth. The first-order condition suggests a pricing kernel of

$$M_{t+1} = \left( \frac{1}{1 - \delta} \right) \left( E_t \left[ U_{t+1}^{1 - \gamma} \right] \right)^{\frac{1}{1 - \gamma}} (C_t - H_t)^{\frac{1}{\gamma}} U_{t+1}^{1 - \gamma} \frac{\partial U_{t+1}}{\partial W_{t+1}}. \quad (13)$$

The envelope theorem implies that

$$\frac{\partial U_t}{\partial W_t} = (1 - \delta) \left( \frac{U_t}{C_t - H_t} \right)^{\frac{1}{\gamma}}. \quad (14)$$

Taken together, the pricing kernel, or the time $t$ price of one extra real dollar payoff at $t + 1$, is

$$M_{t+1} = \delta \left( E_t \left[ U_{t+1}^{1 - \gamma} \right] \right)^{\frac{1}{\gamma - 2}} (C_t - H_t)^{\frac{1}{\gamma}} U_{t+1}^{1 - \gamma} \left( \frac{U_{t+1}}{C_{t+1} - H_{t+1}} \right)^{\frac{1}{\gamma}} \propto U_{t+1}^{1 - \gamma} \left( \frac{U_{t+1}}{C_{t+1} - H_{t+1}} \right)^{\frac{1}{\gamma}}. \quad (15)$$

The second line above highlights how risk aversion and intertemporal substitution drive the dependence of the pricing kernel on time $t + 1$ variables. Risk aversion enters in the first term, which suggests that the present value of one extra dollar payoff in the future is low if utility is high. This is a result of the diminishing marginal utility. The effect of intertemporal substitution arises through the second term, equivalently the marginal value of wealth
\( \partial U_{t+1} / \partial W_{t+1} \). It suggests that the present value of one extra dollar payoff in the future is high if the marginal value of wealth is high, and the marginal value of wealth is high when utility is high relative to surplus consumption.

Hence, the future utility, operating through two different channels, yields almost opposite pricing implications. Through the risk aversion channel, high future utility diminishes the state price, and the diminishing effect is stronger with a higher \( \gamma \). Through the intertemporal substitution channel, high future utility relative to surplus consumption enhances the state price, and the enhancing effect is stronger with a lower \( \psi \).

Intuitively, the intertemporal substitution channel arises because the agent prefers a smooth stream of surplus consumption, and demands compensation for the variations in surplus consumption across time. This is different from the risk aversion channel, in which the agent dislikes and requires compensation for the variations in surplus consumption across future states. A lower elasticity of intertemporal substitution parameter implies a stronger tendency to smooth over time, and thus a stronger effect of the intertemporal substitution channel. A higher risk aversion parameter implies a stronger preference to smooth across states, and thus a stronger effect of the risk aversion channel.

Without habit formation, the persistent expected component, or time-varying volatility, consumption growth reduces to a process of a constant mean and i.i.d. shocks, which must imply a constant utility–consumption ratio (Kocherlakota, 1990, 1996). Consequently, the intertemporal substitution channel becomes silent and does not generate any variations in the pricing kernel. In addition, since the utility moves one-to-one with consumption, the risk aversion term reduces to the familiar marginal consumption.

With habit, the agent cares about surplus consumption. Both the expected growth of surplus consumption and the volatility of the shocks to surplus consumption are time-varying. With long-run risks, both the expected consumption growth and the volatility of consumption growth shocks are time-varying. As shown subsequently, habit and/or long-run risks give rise to a time-varying ratio of utility to surplus consumption, driving the pricing kernel through the intertemporal substitution channel.

Taken together, both the Epstein–Zin preferences and the time-varying utility-to-surplus consumption ratio are critical for the model’s capability in revealing the separate channel of intertemporal substitution. The Epstein–Zin utility decouples the elasticity parameter from the risk aversion parameter, while the time-varying utility-to-surplus consumption ratio allows variations to enter into the pricing kernel through the intertemporal substitution channel.

2.4 Utility–Surplus Consumption Ratio

To obtain the pricing kernel, I directly solve for the utility–surplus consumption ratio \( U_t / (C_t - H_t) \). This approach is complementary to the traditional method in Epstein and Zin (1989) and Cochrane (2008), both of which focus on the case without habit formation. This approach also extends the solution of the utility-to-consumption ratio in Hansen, Heaton, and Li (2008) to incorporate habit formation. Restoy and Weil (2011) develop an approximation to the pricing kernel that accommodates both homoskedastic and heteroskedastic consumption processes. With habit formation and long-run risks, consumption growth is not an i.i.d. process. Hence, the model and the solution in my study focus on a specific setting encompassed by the general platform in Restoy and Weil (2011).
I rewrite the recursive utility function as

$$
\left( \frac{U_t}{C_t - H_t} \right)^{1 - \frac{1}{\delta}} = 1 - \delta + \delta \left( E_t \left[ \left( \frac{U_{t+1}}{C_{t+1} - H_{t+1}} \right)^{1 - \gamma} - \right] \right)^{1 + \frac{1}{\psi}},
$$

and solve the utility–surplus consumption ratio as a function of the three state variables. Appendix A covers the variants of this equation when $\gamma = 1$ and/or $\psi = 1$.

To highlight the intuition, I obtain approximate analytical solutions as log-linear functions of the state variables. The log utility–surplus consumption ratio is

$$
\log \frac{U_t}{C_t - H_t} = \gamma_t = V_0 + V_1(s_t - \bar{s}) + V_2x_t + V_3(\sigma^2_t - 1),
$$

$$
V_1 \approx \frac{\kappa_{1y}(\phi_k - 1)}{1 - \kappa_{1y}\phi_k},
$$

$$
V_2 \approx \frac{\kappa_{1y}}{1 - \kappa_{1y}\phi_x},
$$

$$
V_3 \approx \kappa_{1y} \left\{ 1 - \gamma \left( \frac{V_1\lambda(\bar{s}) + 1 + \lambda(\bar{s})}{2} \sigma^2_x + V_2\sigma^2_x \right) \right\}.
$$

Here, $0 < \kappa_{1y} < 1$, $V_0$, $V_1$, $V_2$, and $V_3$ are constants.

The utility–surplus consumption ratio varies negatively with $s_t$, and this results from the negative relation between the expected surplus consumption growth and $s_t$. The magnitude of the negative slope $V_1$ is smaller than 1 since $\kappa_{1y} < 1$. The utility–surplus consumption ratio varies positively with $x_t$, and this results from the positive relation between the expected surplus consumption growth and $x_t$. Finally, the utility–surplus consumption ratio also varies with $\sigma^2_t$ because surplus consumption growth shocks exhibit time-varying volatility. In particular, the ratio increases (decreases) with $\sigma^2_t$ if $\gamma < (>) 1$. Altogether, the time-varying utility–surplus consumption ratio results from the time-varying expected growth of surplus consumption and the time-varying volatility of the shocks to surplus consumption growth.

### 2.5 Market Prices of Risks

The log pricing kernel innovation is

$$
m_{t+1} - E_t[m_{t+1}] 
\approx -\gamma(\lambda(\hat{s}_t + 1 + \lambda(\hat{s}_t))\sigma_x\omega_t\hat{e}_{s,t+1} + V_2\sigma_x\omega_t\hat{e}_{x,t+1} + V_3\sigma_\theta\bar{e}_{\theta,t+1}) 
+ \frac{1}{\psi} \left( V_1\lambda(\hat{s}_t)\sigma_x\omega_t\hat{e}_{s,t+1} + V_2\sigma_x\omega_t\hat{e}_{x,t+1} + V_3\sigma_\theta\bar{e}_{\theta,t+1} \right) 
= -\gamma(\lambda(\hat{s}_t + 1 + \lambda(\hat{s}_t))\sigma_x\omega_t\hat{e}_{s,t+1} 
- (\gamma V_2 - \frac{1}{\psi} V_2)\sigma_x\omega_t\hat{e}_{x,t+1} - (\gamma V_3 - \frac{1}{\psi} V_3)\sigma_\theta\bar{e}_{\theta,t+1} 
= -\xi_{s}(s_t)\sigma_x\omega_t\hat{e}_{s,t+1} - \xi_{x}\sigma_x\omega_t\hat{e}_{x,t+1} - \xi_{\theta}\sigma_\theta\bar{e}_{\theta,t+1}.
$$

The result demonstrates the market prices for the three risks, and for each, highlights the contributions from the risk aversion and the intertemporal substitution channels.
The market price for short-run growth risk $\xi_{c,t+1}$ is

$$\xi_c(s_t) = \gamma (V_1 \lambda(s_t) + 1 + \lambda(s_t)) + \frac{1}{\psi} (-V_1 \lambda(s_t)).$$  \hspace{1cm} (22)

Since $-1 < V_1 < 0$, both channels contribute positively to the market price of short-run risk. Thus, following $\lambda(s_t)$, $\xi_c(s_t)$ also varies negatively with $s_t$, and the higher the level, the more negative the slope. Further, following the almost counteracting roles of the future utility through the two channels in the pricing kernel, the negative dependence of the utility–surplus consumption ratio on $s_t$ also generates opposing effects on $\xi_c(s_t)$. Through the risk aversion effect, $V_1 < 0$ reduces $\xi_c(s_t)$ (the net contribution is still positive because $V_1 > -1$). Through the intertemporal substitution effect, $V_1 < 0$ contributes positively to $\xi_c(s_t)$. Clearly, even in the special case of $\gamma = 1/\psi$, both channels still arise in $\xi_c(s_t)$, even though the result reduces to that for the power utility.

The market price for long-run growth risk $\xi_{x,t+1}$ is

$$\xi_x = \gamma V_2 + \frac{1}{\psi} (-V_2).$$  \hspace{1cm} (23)

Because $V_2 > 0$, the risk aversion channel contributes positively, while the intertemporal substitution channel contributes negatively. The net contributions are positive if $\gamma > 1/\psi$, or when the agent prefers early resolution of uncertainty. This conclusion is the same as in Bansal and Yaron (2004).

Finally, the market price for long-run volatility risk $\xi_{o,t+1}$ is

$$\xi_o = \gamma V_3 + \frac{1}{\psi} (-V_3).$$  \hspace{1cm} (24)

As shown earlier, if $\gamma > 1$, then $V_3 < 0$ — an increase in the volatility reduces the utility–surplus consumption ratio. Further, if $\gamma > 1/\psi$, then $\xi_o$ is negative, and long-run volatility risk carries a negative market price of risk. This conclusion is the same as in Bansal and Yaron (2004).

Overall, the agent demands compensation for risks through both the risk aversion and the intertemporal substitution channels. Further, the results point to an interesting contrast between the habit and the long-run risks mechanisms. With habit, both the risk aversion and the intertemporal substitution channels contribute positively to the market price of short-run risk. With long-run growth and volatility risks, the risk aversion and the intertemporal substitution channels work against each other in the market prices of long-run risks. This contrast is subsequently shown to drive key differences in the model implications.

The wedge between risk aversion and the inverse of the elasticity of intertemporal substitution, $\gamma - 1/\psi$, determines the agent’s preference for early or late resolution of uncertainty. This provides another perspective to compare the habit and the long-run risks mechanisms. With long-run risks, consumption growth is positively autocorrelated due to the persistent expected component, and the risk premium increases with $\gamma - 1/\psi$, or when the agent prefers early resolution of uncertainty. In contrast, with mean-reverting habit, surplus consumption growth is negatively autocorrelated, and the risk premium decreases with increasing $\psi$. Thus, the agent would rather prefer late resolution of uncertainty in surplus consumption.
2.6 Risk-Free Rate

The log risk-free rate is

\[ r_{f,t} = -\log E_t[e^{m_{t+1}}] \approx -E_t[m_{t+1}] - \frac{1}{2} \text{var}_t[m_{t+1}], \]  

(25)

\[-E_t[m_{t+1}] \approx -\log \delta + \frac{1}{\psi} (\mu_c + x_t + (\phi - 1)(s_t - \bar{s})) \]

\[ -\left( \gamma - \frac{1}{\psi} \right) \frac{1-\gamma}{2} \left( (V_1 \lambda(s_t) + 1 + \lambda(s_t))^2 \sigma_c^2 \omega_t^2 + V_2^2 \sigma_c^2 \omega_t^2 + V_3^2 \sigma_c^2 \right), \]

(26)

\[-\frac{1}{2} \text{var}_t[m_{t+1}] \approx -\frac{1}{2} \left( \gamma (V_1 \lambda(s_t) + 1 + \lambda(s_t)) + \frac{1}{\psi} (-V_1 \lambda(s_t)) \right)^2 \sigma_c^2 \omega_t^2 \]

\[-\frac{1}{2} \left( \gamma V_2 - \frac{1}{\psi} V_2 \right)^2 \sigma_c^2 \omega_t^2 - \frac{1}{2} \left( \gamma V_3 - \frac{1}{\psi} V_3 \right)^2 \sigma_c^2 \]

(27)

\[ = -\frac{1}{2} \xi_c(s_t)^2 \sigma_c^2 \omega_t^2 - \frac{1}{2} \xi_a^2 \sigma_a^2 \omega_t^2 - \frac{1}{2} \xi_w^2 \sigma_w^2. \]

The first term in \(-E_t[m_{t+1}]\) is the time discount effect. If the agent has a large \(\delta\) and thus discounts the future less, then she wants to save, driving down the risk-free rate. The second term is the intertemporal substitution effect, and a small \(\psi\) implies a strong tendency to smooth surplus consumption over time. When the expected growth in surplus consumption is high, the agent wants to borrow from the future, driving up the risk-free rate. The third term is the Jensen’s term from the denominator of the pricing kernel. Finally, \(-\frac{1}{2}\text{var}_t[m_{t+1}]\) demonstrates the effects of three consumption risks. High market prices for future uncertainties generate strong precautionary savings, driving down the risk-free rate. As shown subsequently, with the habit mechanism, the precautionary savings effect is very strong and generates a low risk-free rate, thus resolving the risk-free rate puzzle (Weil, 1989).

For the period 1929–2012, Table VIII reports a volatility of 4% for the ex post real risk-free rate, which contains unexpected inflation. The volatility of the ex ante real risk-free rate is most likely much lower. Campbell and Cochrane (1999) require that the risk-free rate be constant with respect to the only state variable \(s_t\) in their study. My study subsequently investigates three variants of the general model, and two of them include the habit process. In Model I which contains habit only, the requirement on the risk-free rate is the same as in Campbell and Cochrane (1999). In Model III with both habit and long-run risks, the risk-free rate is a function of all three state variables. Thus, I require that it be constant with respect to \(s_t\) at the steady states of the other two variables, \(x_t = 0\) and \(\omega_t^2 = 1\).

The intertemporal substitution term, the Jensen’s term, and the precautionary savings term all potentially drive the risk-free rate to vary over time. The negative dependence of the expected growth of surplus consumption on \(s_t\), through the intertemporal substitution effect, drives the risk-free rate to vary negatively with \(s_t\). In contrast, the negative dependence of \(\lambda(s_t)\) on \(s_t\) results in a negative dependence of \(\xi_c(s_t)\) on \(s_t\) and through the precautionary savings effect, drives the risk-free rate to vary positively. The effect of the Jensen’s term depends on the values of \(\gamma\) and \(\psi\). Overall, a carefully chosen \(\lambda(s_t)\) can lead to a flat risk-free rate. To determine \(\lambda(s_t)\), I rely on a numerical approach which is described in Appendix G.
2.7 Dividend Claim

The log aggregate dividend growth nests the specifications in both Campbell and Cochrane (1999) and Bansal and Yaron (2004).

\[ \log \frac{D_{t+1}}{D_t} = \Delta d + \eta_d x_t + \sigma_d \omega_t e_{d,t+1}, \quad \text{corr}[e_{t+1}, e_{d,t+1}] = \chi, \]

(28)

Here, \( \mu_d \) is the mean, \( \eta_d \) is the loading on the expected growth component, \( \sigma_d \omega_t \) is the volatility of random walk shocks, and \( \chi \) is the correlation between dividend shocks and consumption shocks.

The pricing equation for the aggregate stock return is

\[ E_t[M_{t+1} R_{t+1}] = 1, \quad R_{t+1} = \left( 1 + \frac{P_{t+1}}{D_{t+1}} \right) \frac{D_{t+1}}{D_t} / \left( \frac{P_t}{P_{t+1}} \right). \]

(29)

Hence,

\[ \frac{P_t}{D_t} = E_t \left[ M_{t+1} \left( 1 + \frac{P_{t+1}}{D_{t+1}} \right) \frac{D_{t+1}}{D_t} \right]. \]

(30)

The approximate analytical solution for the log price-dividend ratio is

\[ \log \frac{P_t}{D_t} = z_t \approx A_0 + A_1 (s_t - \bar{s}) + A_2 x_t + A_3 (\omega_t^2 - 1), \]

(31)

\[ A_1 \approx -\xi'(s) \frac{\sigma_d \sigma_d}{1 - \kappa_1 \phi_x}, \]

(32)

\[ A_2 \approx \eta_d \frac{1}{1 - \kappa_1 \phi_x}, \]

(33)

\[ A_3 \approx \frac{-F_3 + \frac{1}{2} \left( \kappa_1 A_1^2 \sigma_x^2 \xi'(\bar{s})^2 + \kappa_1^2 A_2^2 \sigma_x^2 + 2 \chi \kappa_1 A_1 \sigma_d \xi'(\bar{s}) \right) \left( 1 - \kappa_1 A_1 \sigma_d \xi'(\bar{s}) - \xi' \kappa_1 A_2 \sigma_x^2 \right)}{1 - \kappa_1 \phi_x}. \]

(34)

Here, \( 0 < \kappa_1 < 1 \), \( F_3, A_0, A_1, A_2, \) and \( A_3 \) are constants.

The market price of short-run consumption risk \( \xi_c(s_t) \) has a negative slope. With a positive covariance between dividend and consumption growth shocks, the price–dividend ratio varies positively with \( s_t \). The dependence of the price–dividend ratio on \( x_t \) arises due to the loading of dividend growth on the expected growth component, and the relation is positive if \( \eta_d > 1/\psi \). Finally, the slope on \( \omega_t^2 \) is determined by various terms of time-varying volatility in the pricing equation.
The log return innovation is

\[ r_{t+1} - E_t[r_{t+1}] \]

\[ \approx \kappa_1 A_1 \lambda(s_t) \sigma_c \omega_t e_{c,t+1} + \kappa_1 A_2 \sigma_x \omega_t e_{x,t+1} + \kappa_1 A_3 \sigma_o \omega_o e_{o,t+1} + \sigma_d \omega_t e_{d,t+1} \]

\[ = (\kappa_1 A_1 \lambda(s_t) \sigma_c + \chi \sigma_d) \omega_t e_{c,t+1} + \kappa_1 A_2 \sigma_x \omega_t e_{x,t+1} + \kappa_1 A_3 \sigma_o \omega_o e_{o,t+1} \]

\[ + \sqrt{1 - \chi^2} \sigma_d \omega_t e_{d,t+1} \]

\[ = \beta_c(s_t) \sigma_c \omega_t e_{c,t+1} + \beta_x \sigma_x \omega_t e_{x,t+1} + \beta_o \sigma_o \omega_o e_{o,t+1} + \sqrt{1 - \chi^2} \sigma_d \omega_t e_{d,t+1}, \]

in which dividend growth shocks \( e_{d,t+1} \) are decomposed as

\[ e_{d,t+1} = \chi e_{c,t+1} + \sqrt{1 - \chi^2} e_{d,t+1}. \]

The exposure of the stock return to short-run consumption risk is

\[ \beta_c(s_t) = \kappa_1 A_1 \lambda(s_t) + \chi \frac{\sigma_d}{\sigma_c}. \]

Following \( \lambda(s_t) \), the return exposure \( \beta_c(s_t) \) varies negatively with \( s_t \). The conditional variance is

\[ \text{var}_t[r_{t+1}] \approx \beta_c(s_t) \sigma_c^2 \omega_t^2 + \beta_x^2 \sigma_x^2 \omega_t^2 + \beta_o^2 \sigma_o^2 + (1 - \chi^2) \sigma_d^2 \omega_t^2. \]

The conditional volatility thus follows \( \beta_c(s_t) \) and varies negatively with \( s_t \), and it also increases with \( \omega_t^2 \). The conditional expected excess return is

\[ E_t[r_{t+1}] - r_f,t + \frac{1}{2} \text{var}_t[r_{t+1}] \approx -\text{cov}_t[m_{t+1}, r_{t+1}] \]

\[ \approx \zeta_c(s_t) \beta_c(s_t) \sigma_c^2 \omega_t^2 + \xi_x \beta_x^2 \sigma_x^2 \omega_t^2 + \xi_o \beta_o^2 \sigma_o^2. \]

Since \( \zeta_c(s_t) \) and \( \beta_c(s_t) \) both vary negatively with respect to \( s_t \), so is the expected return. In addition, the expected return increases with \( \omega_t^2 \).

3. Solutions and Results

This section begins with a description of the empirical data and the calibration approach. Then I discuss the three variants of the model: Model I contains habit only, Model II is long-run risks only, and Model III has both. Finally, I examine the quantitative implications of the three models at their respective preferred calibrations.

3.1 Data and Model Calibration

The annual consumption data are from the Bureau of Economic Analysis. Following the literature, consumption is the sum of real personal consumption expenditures on nondurable goods and services, and divided by the population to obtain per capita values. The stock market returns are value-weighted annual returns for NYSE, and the risk-free rates are 3-month T-bill rates, both adjusted for inflation. All asset data series are obtained from the Center for Research in Security Prices. Year-end price–dividend ratios and annual real dividend growth rates are computed from the value-weighted annual returns for NYSE with and without distributions, and adjusted for inflation. The sample period is 1929–2012, and thus 1930–2012, or 83 years, for the growth rates.
The preceding section utilizes log-linear approximate analytical solutions to demonstrate model intuition. In this section, following the convention in the literature, I use the approach of calibration, numerical solution, and simulation to investigate the quantitative results of three variants of the model: Model I contains habit only, Model II is long-run risks only, and Model III has both.

Specifically, I calibrate the parameters for consumption and dividend growth rates at the monthly interval. Then I generate 10,000 simulated samples of growth rates. Each simulated sample is $83 \times 12$-month long, and is then time-aggregated to the annual frequency. The parameters are chosen so that the simulated growth rates replicate major properties of the empirical growth data. I solve the models numerically and investigate the key properties of the model solutions. Then I simulate 10,000 monthly series of asset pricing data, each of $83 \times 12$ months, time-aggregate to the annual frequency, and compare with the annual empirical asset pricing data. The details of the numerical solution methodology are in Appendix G.

The calibrated parameters are listed in Table I. Across the three models, except for turning on or off various features, the calibrations are based on the same set of parameters. This preserves consistency and facilitates comparison between the three models. The mean parameters $\mu_c$ and $\mu_d$ are chosen to match the average annual consumption and dividend growth, and the volatility parameters $\sigma_c$ and $\sigma_d$ are chosen to match the standard deviations. These four parameters are the same for all three models. The correlation between the random walk shocks of consumption and dividend growth, $\chi$, is set to 0.45 in Model I (habit only) and Model III (both habit and long-run risks). This value is chosen to match the empirical correlation between consumption and dividend growth rates. In Model II with long-run risks only, to highlight the effect of long-run risks, $\chi$ is set to 0 as in Bansal and Yaron (2004).

For the habit process, the persistence parameter $\phi_h$ is 0.99, very close to that in Campbell and Cochrane (1999). The sensitivity function is determined numerically so as to generate a flat risk-free rate, and will be discussed subsequently.

The long-run risks parameters are similar to those in Bansal and Yaron (2004), with small differences mostly due to the longer sample period of the empirical data used in this article. The persistence of the expected component, $\phi_e$, is set to 0.98, and the persistence of time-varying volatility, $\phi_\sigma$, is 0.99. The volatility parameter $\sigma_e$ is chosen to match the autocorrelations of consumption growth. The volatility of the volatility process, $\sigma_\sigma$, is chosen to match the parameters of a GARCH(1,1) conditional volatility process estimated on consumption growth innovations, obtained as residuals from an AR(1) regression of annual consumption growth. The loading of dividend growth on the expected growth component, $g_d$, is set to 3.5. This value is chosen to match the autocorrelations of dividend growth. As noted above, in Model II (long-run risks only), the random walk shocks of consumption and dividend growth are not correlated, and thus the correlation between the two growth rates results solely from their loadings on the expected component.

Table II shows that the key properties of the empirical growth rates are well replicated in the samples simulated from all three models. In Model I with habit only, the standard deviations of the growth rates are smaller than those in the other two models that further...
Table II. Empirical and model-implied growth rates
This table compares the properties of the annual empirical growth rate data for the period 1929–2012 with those of the model-implied, simulated data. Here, AC(1) to AC(3) are the first- to third-order autocorrelations. The GARCH and ARCH parameters are for a GARCH(1,1) process estimated on the residuals from an AR(1) regression of annual consumption growth. The empirical estimates are obtained with GMM, and standard errors are Newey and West (1987) corrected with five lags. The model-implied results are the means and percentiles over 10,000 samples simulated from the models of the growth rates. Each simulated sample is 83 × 12-month long, and is then time-aggregated to the annual frequency.

<table>
<thead>
<tr>
<th>Data</th>
<th>Model I (habit only)</th>
<th>Model II (long-run risks only)</th>
<th>Model III (both)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Consumption growth (Δc)</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Mean (%)</td>
<td>1.82 (0.32)</td>
<td>1.81 1.36 2.23 1.81 0.81 2.82</td>
<td></td>
</tr>
<tr>
<td>Standard deviation (%)</td>
<td>2.16 (0.48)</td>
<td>1.97 1.71 2.24 2.54 2.09 3.05</td>
<td></td>
</tr>
<tr>
<td>AC(1)</td>
<td>0.48 (0.11)</td>
<td>0.47 0.28 0.63</td>
<td>Same as</td>
</tr>
<tr>
<td>AC(2)</td>
<td>0.18 (0.15)</td>
<td>0.23 −0.01 0.46</td>
<td>Model II</td>
</tr>
<tr>
<td>AC(3)</td>
<td>−0.06 (0.09)</td>
<td>0.17 −0.08 0.39</td>
<td>(long-run risks only)</td>
</tr>
<tr>
<td>GARCH</td>
<td>0.85 (0.07)</td>
<td>0.62 0.16 0.96</td>
<td></td>
</tr>
<tr>
<td>ARCH</td>
<td>0.08 (0.05)</td>
<td>0.11 0.01 0.29</td>
<td></td>
</tr>
<tr>
<td><strong>Dividend growth (Δd)</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Mean (%)</td>
<td>1.01 (1.1)</td>
<td>1.00 −1.88 3.83 1.01 −3.06 5.09</td>
<td>Same as</td>
</tr>
<tr>
<td>Standard deviation (%)</td>
<td>13.2 (1.1)</td>
<td>13.0 11.2 14.8 13.8 11.6 16.1</td>
<td>Model II</td>
</tr>
<tr>
<td>AC(1)</td>
<td>0.27 (0.09)</td>
<td>0.30 0.12 0.47</td>
<td>(long-run risks only)</td>
</tr>
<tr>
<td>AC(2)</td>
<td>−0.07 (0.11)</td>
<td>0.06 −0.14 0.27</td>
<td></td>
</tr>
<tr>
<td>corr(Δc, Δd)</td>
<td>0.47 (0.08)</td>
<td>0.45 0.29 0.59 0.22 0.01 0.42</td>
<td>0.54 0.39 0.68</td>
</tr>
</tbody>
</table>
incorporate long-run risks. In Model II with long-run risks only, absent correlated shocks, the correlation between consumption and dividend growth is smaller than in the other two models. Overall, the empirical estimates are well matched by the means and bounded within the 5th and 95th percentiles of the results obtained from the simulated samples.

For parameters in the Epstein–Zin preferences, I solve each model for $\gamma = (2/3), 1, 2, 5, 10$ and $\psi = 0.2, 0.5, 1, 1.5$. As shown subsequently, there are considerable variations in the model results as the parameters move within these ranges. The choice of the time discount $\delta$ is discussed separately for each model.

### 3.2 Model I (Habit Only)

In Model I, long-run risks are shut down, and the only state variable is the log surplus consumption ratio $s_t$. To solve the model, I also need to determine the sensitivity function $\lambda(s_t)$ so that the risk-free rate is flat with respect to $s_t$, and the time discount factor $\delta$ so that the risk-free rate matches the target, which is 0.76% per year as reported in Table VIII. For the power utility with $\gamma = 1/\psi$, Model I reduces to that in Campbell and Cochrane (1999), for which $\lambda(s_t)$ is known analytically, and $\delta$ is also easily imputed. For general cases, $\lambda(s_t)$ and $\delta$ are computed numerically as described in Appendix G.

Figure 1 plots key variables obtained from the model solution for the case of $\gamma = 5$ and $\psi = 1$, which is subsequently shown to provide a good match to the empirical asset pricing data. For this case, $\lambda(s_t)$ is not known analytically and is numerically computed.

The constraints on the level and the derivative of $\lambda(s_t)$ at $s_t = \overline{s}$ imply that a higher level is associated with a more negative slope. Hence, $\overline{s}$ well characterizes the shape of $\lambda(s_t)$. Panel A of Table III reports $\overline{s}$ for different $\gamma$ and $\psi$ values, and indicates that $\overline{s}$ increases with increasing $\gamma$ but stays largely flat with respect to $\psi$. Panel B of Table III reports the time discount $\delta$ and shows that it increases with increasing $\psi$ but stays largely flat with respect to $\gamma$.

When $\gamma$ is fixed, a smaller $\psi$ implies a stronger intertemporal substitution effect, which increases the level of the risk-free rate, and makes its slope on $s_t$ more negative. In the meantime, because the intertemporal substitution channel contributes positively to the market price of short-run risk $\xi_c(s_t)$, a smaller $\psi$ also implies a stronger precautionary savings effect, which reduces the risk-free rate and makes its slope on $s_t$ more positive. The almost flat $\overline{s}$ with respect to $\psi$ suggests that, for the slope of the risk-free rate, the two effects largely balance each other, and thus the slope of the sensitivity stays the same. In contrast, for the level of the risk-free rate, the latter effect dominates, and in order for the risk-free rate to remain at the target level, $\delta$ needs to decrease with decreasing $\psi$. At $\psi = 0.2$, the annualized $\delta$ drops to remarkably low values of about 0.8. This is opposite to the behavior of $\delta$ in conventional models without habit formation, where the precautionary savings effect is too small, and thus $\delta$ has to rise to counterbalance the effect of a small $\psi$ (Weil, 1989).

When $\psi$ is fixed, the level and the negative slope of the risk-free rate on $s_t$ due to the intertemporal substitution effect stay the same, regardless of $\gamma$ or $\delta$. Because the risk aversion channel contributes positively to the market price of short-run risk $\xi_c(s_t)$, a higher $\gamma$ would generate a stronger precautionary savings effect, reducing the risk-free rate and making its slope on $s_t$ more positive. To offset these effects, the level and the slope of $\lambda(s_t)$ need to decrease in magnitudes, resulting in an increasing $\overline{s}$. Indeed, as a confirmation that the effects of increasing $\gamma$ and decreasing $\lambda(s_t)$ on the risk-free rate level balance each other, $\delta$ stays largely flat as $\gamma$ varies.

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Figure 1 confirms that the utility–surplus consumption ratio is a negative function of \( s_t \), which results from the negative dependence of expected surplus consumption growth on \( s_t \).

The slope \( V_1 \) in the log-linear approximation can be computed from the numerical solution. This makes it possible to compare the respective contributions of the risk aversion and the intertemporal substitution channels to the market price of short-run risk. Specifically, the ratio between the two contributions is

\[
\frac{c \left( V_1 \lambda(s_t) + 1 + \lambda(s_t) \right)}{\psi (-V_1 \lambda(s_t))}.
\]  

(40)

A useful way to interpret the ratio is to define an effective risk aversion

\[
\gamma_{\text{eff}} = \gamma (V_1 \lambda(s_t) + 1 + \lambda(s_t))
\]  

(41)

and an effective elasticity of substitution

\[
\psi_{\text{eff}} = \frac{\psi}{(-V_1 \lambda(s_t))}.
\]  

(42)

The sensitivity \( \lambda(s_t) \) is a negative function of \( s_t \), and is typically larger than 1 at \( s_t = \bar{s} \). Thus, the effective risk aversion varies negatively with \( s_t \) and is larger than \( \gamma \) at \( s_t = \bar{s} \), while...
the effective elasticity of substitution varies positively with $s_t$, and is smaller than $\psi$ at $s_t = \bar{s}$. For the case of $\gamma = 5$ and $\psi = 1$, $\lambda(\bar{s}) \approx 5.4, V_1 \approx -0.57$. Hence, the effective risk aversion at the steady state is $\gamma(V_1\lambda(\bar{s}) + 1 + \lambda(\bar{s})) \approx 16.6$, while the effective elasticity of substitution is $\psi/(−V_1\lambda(\bar{s})) \approx 0.325$. The risk aversion contributes 84% of the market price of short-run risk, while intertemporal substitution makes up 16%.

Figure 1 also confirms that the price–dividend ratio is a positive function of $s_t$, and the expected excess return and the return volatility, following $\lambda(s_t)$, are both negative functions of $s_t$.

Table IV reports key asset pricing results obtained from the simulated data for different values of $\gamma$ and $\psi$. Panels A and B display the average values of the equity premium and the return volatility. Both quantities not only directly depend on $\gamma$ and $\psi$, but also indirectly through the dependence of $\lambda(s_t)$ on $\gamma$ and $\psi$, which is driven by the constant risk-free rate requirement discussed earlier. When $\psi$ decreases, the intertemporal substitution channel becomes stronger, while $\lambda(\bar{s})$ stays flat. Hence, both the equity premium and the return volatility increase with decreasing $\psi$. When $\gamma$ increases, the risk aversion channel becomes stronger, but $\lambda(\bar{s})$ decreases—the direct and indirect effects of increasing $\gamma$ oppose each other. For the return volatility, the indirect effect dominates, and the volatility decreases as $\gamma$ becomes larger. The equity premium, however, exhibits a U-shaped pattern, suggesting that the indirect effect dominates at the lower end of the $\gamma$ values while the direct effect dominates at the upper end of the $\gamma$ values.

Hence, for a considerable segment of Panel A in Table IV, given a fixed $\psi$, the equity premium decreases with increasing $\gamma$. As discussed above, this result arises due to the
Table IV. Model-implied asset pricing results of Model I (habit only)
This table reports key asset pricing results in the model-implied, simulated data for Model I (habit only). The results are reported for different parameters of risk aversion $\gamma$ and the elasticity of intertemporal substitution $\psi$. Here, $r$ is the log stock return, $r_f$ is the log risk-free rate, $p - d$ is the log price–dividend ratio, and slope(3 yr) and $R^2(3yr)$ are from regressing the cumulative excess log stock return from the end of year $t$ to the end of year $t + 3$ on the log price–dividend ratio at the end of year $t$. In addition, $\mu[\cdot]$ denotes the mean and $\sigma[\cdot]$ denotes the standard deviation. The results are the means over 10,000 samples simulated from the numerical solution of the model. Each simulated sample is $83 \times 12$-month long, and is then time-aggregated to the annual frequency.

<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>$\psi$</th>
<th>0.2</th>
<th>0.5</th>
<th>1</th>
<th>1.5</th>
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<tbody>
<tr>
<td>A. $\mu[r - r_f] (%)$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2/3</td>
<td>28.36</td>
<td>8.42</td>
<td>3.61</td>
<td>2.36</td>
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<tr>
<td>1</td>
<td>15.99</td>
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<td>3.49</td>
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<td></td>
</tr>
<tr>
<td>2</td>
<td>8.82</td>
<td>5.62</td>
<td>3.45</td>
<td>2.51</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>8.70</td>
<td>6.13</td>
<td>4.08</td>
<td>3.09</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>10.18</td>
<td>7.50</td>
<td>5.37</td>
<td>4.21</td>
<td></td>
</tr>
<tr>
<td>B. $\sigma[r]$</td>
<td></td>
<td></td>
<td></td>
<td></td>
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</tr>
<tr>
<td>2/3</td>
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<td>0.251</td>
<td>0.242</td>
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<td>0.175</td>
<td>0.173</td>
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<td>C. $\mu[p - d]$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
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<td>3.46</td>
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<td>2.95</td>
<td>3.47</td>
<td>3.81</td>
<td></td>
</tr>
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<td>2.84</td>
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<td>3.56</td>
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<tr>
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<td>2.63</td>
<td>2.98</td>
<td>3.24</td>
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<tr>
<td>D. $\sigma[p - d]$</td>
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</tr>
<tr>
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<td>0.263</td>
<td>0.239</td>
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<td>0.192</td>
<td></td>
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<td>0.164</td>
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<td>0.139</td>
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<tr>
<td>10</td>
<td>0.132</td>
<td>0.123</td>
<td>0.114</td>
<td>0.109</td>
<td></td>
</tr>
</tbody>
</table>

(continued)
dominating effect of the sensitivity function decreasing with $c$. I also study a variant of Model I in which both $k_{st}$ and $d$ are fixed at those for $c = 5$ and $w = 1$. The model-implied equity premium, reported in Table V, increases with increasing $c$. Note that the equity premiums reported in Table V are expected excess log returns, and thus can be negative because of the Jensen’s term $-\frac{1}{2} \text{var}_t[r_{t+1}]$.

Table IV. Continued

<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>$\Psi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>0.5</td>
</tr>
</tbody>
</table>

E. $\mu[d\text{lope}(3 \text{ yr})]$ |

| 2/3 | -1.431 | -0.708 | -0.555 | -0.536 |
| 1   | -0.868 | -0.611 | -0.526 | -0.513 |
| 2   | -0.597 | -0.517 | -0.466 | -0.453 |
| 5   | -0.495 | -0.433 | -0.371 | -0.342 |
| 10  | -0.423 | -0.356 | -0.289 | -0.254 |

F. $\mu[R^2(3 \text{ yr})]$ |

| 2/3 | 0.617 | 0.250 | 0.129 | 0.112 |
| 1   | 0.343 | 0.167 | 0.112 | 0.101 |
| 2   | 0.138 | 0.101 | 0.079 | 0.074 |
| 5   | 0.074 | 0.058 | 0.044 | 0.039 |
| 10  | 0.049 | 0.038 | 0.030 | 0.026 |

Table V. Model-implied equity premium of Model I (habit only) with fixed sensitivity function and time discount

This table reports the equity premium in the model-implied, simulated data for Model I (habit only) at different parameters of risk aversion $\gamma$ and the elasticity of intertemporal substitution $\psi$, but with the sensitivity function $\gamma(s_t)$ and the time discount $\delta$ fixed at those for the case of $\gamma = 5$ and $\psi = 1$. Here, $r$ is the log stock return, $r_f$ is the log risk-free rate, and $\mu[\cdot]$ denotes the mean. The results are the means over 10,000 samples simulated from the numerical solution of the model. Each simulated sample is $83 \times 12$-month long, and is then time-aggregated to the annual frequency.

<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>$\mu[r - r_f]$ (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>0.5</td>
</tr>
</tbody>
</table>

| 2/3 | 5.72 | 1.29 | -0.08 | -0.47 |
| 1   | 6.66 | 1.83 | 0.25  | -0.21 |
| 2   | 9.77 | 3.43 | 1.28  | 0.61 |
| 5   | 18.61 | 7.51 | 4.08  | 3.22 |
| 10  | 46.11 | 22.70 | 14.25 | 11.57 |
In Panel A of Table IV, when $\psi$ is allowed to change together with $c$ so as to stay in the groove of the power utility, the diagonal from the top right to the bottom left shows that the average return increases. The discussions above suggest that this results mainly from the increasingly stronger intertemporal substitution channel with decreasing $\psi$.

The average price–dividend ratio varies negatively with the expected stock return. Since the risk-free rate is the same for all values of $\gamma$ and $\psi$, the average price–dividend ratio varies negatively with the equity premium. As reported in Panel C of Table IV, the average price–dividend ratio increases with increasing $\psi$, but displays an inverted U-shaped pattern with respect to $\gamma$. Panel D of Table IV shows that the volatility of the price–dividend ratio varies with $\gamma$ and $\psi$ following the same pattern as that of the stock return volatility.

In Model I, the positive relation between the price–dividend ratio and $s_t$ and the negative relation between the equity premium and $s_t$ imply that excess returns are predictable by the price–dividend ratio with negative slopes. Panels E and F of Table IV report the model-implied results for the 3-year-ahead prediction of cumulative excess log stock returns by the log price–dividend ratio. The magnitude of the negative slope and the $R^2$ depend on how much of the future excess return variation is due to conditional dependence on $s_t$ and how large is the volatility of the log price–dividend ratio as a function of $s_t$. The results show that the slope becomes more negative and the $R^2$ becomes larger with a decreasing $\psi$ that raises the effect of the intertemporal substitution channel, or a falling $c$ that is accompanied by the sensitivity function rising in the level and steepening in the slope.

In Table IV, across the $\gamma$ and $\psi$ values, the asset pricing results cover wide ranges of values. To match key asset pricing moments of the empirical data, in particular the equity premium of 4.78% and the return volatility of 19.4% for the aggregate US stock market, the preferred calibration appears to be the case of $\gamma = 5$ and $\psi = 1$. More simulation results for this case are presented in Tables VIII and IX, and detailed discussions of the results are deferred to Section 3.5.

As discussed earlier, the sensitivity $\lambda(s_t)$ is a negative function of $s_t$, and the higher the level, the more negative the slope. While the downward sloping $\lambda(s_t)$, $s_t$ is a heteroskedastic process. To investigate the effects of the level and the slope of the sensitivity function, I study a variant of the preferred case of $\gamma = 5$ and $\psi = 1$ for Model I, in which the sensitivity function is constant and thus $s_t$ is a homoskedastic process. When the sensitivity level is set to that of the original sensitivity function at $s_t = 3$, the solution indicates that the risk-free rate is a negative function of $s_t$, while the equity premium and the return volatility are flat with respect to $s_t$. In the simulation results, compared with those for the baseline version, the average levels of the equity premium, the return volatility, and the risk-free rate remain about the same, the risk-free rate volatility is higher, while the excess stock return predictability by the price–dividend ratio disappears. If I further lower the constant sensitivity level, the equity premium and the return volatility decrease, while the risk-free rate

---

9 This is different from the preferred calibration of $\gamma = 2$ for the power utility (and thus $\psi = 0.5$) in Campbell and Cochrane (1999). There are two potential reasons. First, my calibration is based on a longer empirical sample period than that in Campbell and Cochrane (1999), and thus the parameters are chosen to match different values of empirical data moments. Second, for stock returns, my study focuses on a match to the dividend claim, while Campbell and Cochrane (1999) largely focuses on a match to the consumption claim. In addition, as shown in Wachter (2005) and also discussed in Appendix G, in order to obtain accurate numerical solutions for the habit formation model, it is important to use a long and dense grid.
increases. Taken together, these results suggest that a constant sensitivity function is able to generate a sizable equity premium and return volatility, but a negative slope is necessary to generate the flat risk-free rate and the excess return predictability.

3.3 Model II (Long-Run Risks Only)

When habit is turned off, the model with long-run risks only is the same as that in Bansal and Yaron (2004). There are two state variables: $x_t$ is the expected growth, and its shocks are long-run growth risk; $\omega_t^2$ is the time-varying variance and its shocks are long-run volatility risk. In Model II there is no sensitivity function. With regard to the time discount $\delta$, as shown subsequently, for all $\gamma$ and $\psi$ values studied, Model II cannot attain the empirical annual risk-free rate of 0.76% even with a monthly calibrated $\delta = 0.999$. Hence, I set the same $\delta = 0.999$ for all cases of Model II. Figure 2 plots key asset pricing variables obtained from the numerical solution for the case of $\gamma = 10$ and $\psi = 1.5$, which is later selected as providing a good match to the empirical asset pricing data.

The utility–consumption ratio varies positively with $x_t$. If $\gamma < 1$, the utility–consumption ratio is a positive function of $\omega_t^2$. If $\gamma > 1$, such as in Figure 2, the dependence is negative. The price–dividend ratio is a positive function of $x_t$ when $\eta_d > 1/\psi$, such as shown in Figure 2. If $\eta_d < 1/\psi$, e.g., when $\psi = 0.2$, the relation is negative.

Panel A of Table VI reports the average annual risk-free rates in the simulated asset pricing data. These results are obtained with the same monthly $\delta = 0.999$, and all the values are higher than the empirical value of 0.76%.

With $\delta$ fixed, the risk-free rate is driven by the remaining three terms: the intertemporal substitution effect, the Jensen’s term, and the precautionary savings effect. The variation in the risk-free rate level with $\psi$ in Panel A of Table VI is mainly determined by the intertemporal substitution effect. A lower value of $\psi$ implies a stronger borrowing demand, which raises the risk-free rate. The intertemporal substitution effect also generates the positive dependence of the risk-free rate on $x_t$, as shown in Figure 2, since high expected consumption growth motivates more borrowing from the future and raises the risk-free rate. A lower $\psi$ makes the slope of the risk-free rate on $x_t$ more positive, which implies a higher risk-free rate volatility. Indeed, Panel B of Table VI shows that the risk-free rate volatility varies almost solely with $\psi$.

The Jensen’s term and the precautionary savings effect determine the variation of the risk-free rate with $\gamma$. As $\gamma$ increases, the Jensen’s term largely raises the risk-free rate, while the precautionary savings effect mainly drives down the risk-free rate. Panel A of Table VI suggests that for small $\psi$, the Jensen’s term dominates and the risk-free rate increases with increasing $\gamma$; for large $\psi$, the precautionary savings effect dominates and the risk-free rate falls with increasing $\gamma$. The Jensen’s term and the precautionary savings effect also determine the dependence of the risk-free rate on $\omega_t^2$. With the Jensen’s term, the risk-free rate increases with increasing volatility, while with the precautionary savings effect, the risk-free rate decreases with increasing volatility. For the values of $\gamma$ and $\psi$ studied, the latter dominates and thus the risk-free rate varies negatively with volatility, such as plotted in Figure 2 for $\gamma = 10$ and $\psi = 1.5$.

Panel C of Table VI reports the average equity premium in the simulated data. In the model, since short-run consumption and dividend shocks are uncorrelated, only long-run risks generate the equity premium. The contributions by long-run growth and volatility risks to the equity premium are $\xi_x \beta_x$ and $\xi_\omega \beta_\omega$, respectively. In this article, similar to Bansal and Yaron (2004), the equity premium is mainly due to long-run growth risk. For
example, with $\gamma = 10$ and $\psi = 1.5$, long-run growth risk generates almost 95% of the equity premium, while long-run volatility risk contributes the rest. Hence, the magnitude of the equity premium is almost solely determined by $c_x\beta_x$, the product of the market price and the return exposure to long-run growth risk, which varies with $\gamma$ and $\psi$ following

$$c_x\beta_x \propto \left(\frac{\gamma - 1}{\psi}\right)\left(\eta_d - \frac{1}{\psi}\right).$$  \hspace{1cm} (43)$$

For $\psi = 0.2$, this product decreases with increasing $\gamma$. For $\psi = 0.5, 1$ and $1.5$, this product increases with increasing $\gamma$. The equity premium follows the same patterns. Note that the
Table VI. Model-implied asset pricing results of Model II (long-run risks only)

This table reports key asset pricing results in the model-implied, simulated data for Model II (long-run risks only). The results are reported for different parameters of risk aversion $\gamma$ and the elasticity of intertemporal substitution $\psi$. Here, $r$ is the log stock return, $r_f$ is the log risk-free rate, $p - d$ is the log price–dividend ratio, and slope(3 yr) and $R^2(3yr)$ are from regressing the cumulative excess log stock return from the end of year $t$ to the end of year $t + 3$ on the log price–dividend ratio at the end of year $t$. In addition, $\mu[\cdot]$ denotes the mean and $\sigma[\cdot]$ denotes the standard deviation. The results are the means over 10,000 samples simulated from the numerical solution of the model. Each simulated sample is $83 \times 12$-month long, and is then time-aggregated to the annual frequency.

<table>
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<th>$\gamma$</th>
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<th>1.5</th>
</tr>
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<td>A. $\mu[r_f]$ (%)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
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<td>4.69</td>
<td>2.99</td>
<td>2.19</td>
</tr>
<tr>
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<td>9.29</td>
<td>4.69</td>
<td>2.97</td>
<td>2.16</td>
</tr>
<tr>
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<td>9.32</td>
<td>4.69</td>
<td>2.91</td>
<td>2.08</td>
</tr>
<tr>
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<td>9.48</td>
<td>4.70</td>
<td>2.74</td>
<td>1.84</td>
</tr>
<tr>
<td>10</td>
<td>10.12</td>
<td>4.71</td>
<td>2.45</td>
<td>1.43</td>
</tr>
<tr>
<td>B. $\sigma[r_f]$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2/3</td>
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<td>0.0281</td>
<td>0.0141</td>
<td>0.0094</td>
</tr>
<tr>
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<td>0.0281</td>
<td>0.0141</td>
<td>0.0094</td>
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<td>0.0141</td>
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<tr>
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<td>0.0281</td>
<td>0.0141</td>
<td>0.0094</td>
</tr>
<tr>
<td>10</td>
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<td>0.0281</td>
<td>0.0141</td>
<td>0.0095</td>
</tr>
<tr>
<td>C. $\mu[r - r_f]$ (%)</td>
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<tr>
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<td>D. $\sigma[r]$</td>
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<td></td>
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<tr>
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<td>0.176</td>
<td>0.193</td>
<td>0.200</td>
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<td>0.156</td>
<td>0.176</td>
<td>0.193</td>
<td>0.199</td>
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<td>2</td>
<td>0.156</td>
<td>0.176</td>
<td>0.192</td>
<td>0.198</td>
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<td>0.156</td>
<td>0.175</td>
<td>0.189</td>
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<tr>
<td>10</td>
<td>0.156</td>
<td>0.174</td>
<td>0.185</td>
<td>0.192</td>
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<tr>
<td>E. $\mu[p - d]$</td>
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<td></td>
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<tr>
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<td>5.94</td>
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<td>4.06</td>
<td>4.32</td>
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<td>2.71</td>
<td>3.17</td>
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<td>3.41</td>
</tr>
<tr>
<td>10</td>
<td>2.74</td>
<td>2.88</td>
<td>2.96</td>
<td>2.98</td>
</tr>
</tbody>
</table>
equity premiums in Panel C of Table VI are expected excess log returns, and their values can be negative due to the Jensen’s term \(-(1/2)\text{var}_\tau[r_{t+1}]\). For example, the equity premiums are negative when \(\gamma = 1/\psi\).

The average price–dividend ratio, reported in Panel E of Table VI, varies negatively with the expected stock return, which is the sum of the risk-free rate in Panel A and the equity premium in Panel C. The price–dividend ratio increases with increasing \(\psi\), largely due to the substantial decrease of the risk-free rate. Across different values of \(\gamma\), the risk-free rate is quite flat, and the variation of the price–dividend ratio with \(\gamma\) is opposite to that of the equity premium.

Panels D and F of Table VI report the average volatilities of the stock return and the price–dividend ratio in the simulated data. Both volatilities vary largely with \(\beta_x \propto \eta_d - 1/\psi\), and thus increase with increasing \(\psi\), but remain largely flat with \(\gamma\).

In Model II, the negative relation between the price–dividend ratio and \(\omega^2\) and the positive relation between the equity premium and \(\omega^2\) together generate the predictability of excess stock returns by the price–dividend ratio with negative slopes. Panels G and H of Table VI report the model-implied results for the 3-year-ahead prediction of cumulative excess log stock returns by the log price–dividend ratio. The magnitude of the negative slope and the \(R^2\) depend on how much of the future excess return variation is due to conditional dependence on \(\omega^2\) and how much of the variation in the log price–dividend ratio is driven by \(\omega^2\). For the most part, the slope becomes more negative and the \(R^2\) becomes larger with an increasing \(\psi\). With respect to \(\gamma\), the results are less varied.

<table>
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<td>F. (\sigma[p - d])</td>
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<td>0.103</td>
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<tr>
<td>G. (\mu[\text{slope}(3\text{ yr})])</td>
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<tr>
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<tr>
<td>H. (\mu[R^2(3\text{ yr})])</td>
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</tr>
<tr>
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Table VI. Continued
Overall, the model results point to $\gamma = 10$ and $\psi = 1.5$ in order to achieve a high equity premium, a high return volatility, and a low risk-free rate.\footnote{This is the same as the preferred calibration of Bansal and Yaron (2004). The quantitative results are somewhat different, largely because my calibration is based on the empirical sample period of 1929–2012, longer than that of 1929–98 in Bansal and Yaron (2004).} More detailed simulation results for this preferred calibration are presented in Tables VIII and IX and discussed in Section 3.5.

3.4 Model III (Both Habit and Long-Run risks)
Model III is the general model combining both habit and long-run risks. In the model, habit and long-run risks operate through separate consumption shocks: the former through random walk growth shocks, while the latter through the shocks to expected growth and time-varying volatility, and all shocks are assumed to be mutually uncorrelated. Consequently, the two mechanisms operate in parallel to each other and their effects are complementary.

As in Model I, I need to determine the sensitivity function $\lambda(s_t)$ and the time discount $\delta$, although I only require that the risk-free rate be flat with respect to $s_t$ and matches the target when $x_t = 0$ and $\sigma_r^2 = 1$. The resulting $\lambda(s_t)$ and $\delta$ are almost identical to those for Model I, in particular for cases of $\gamma = 1/\psi$. This is not surprising since the habit and long-run risks mechanisms operate through uncorrelated consumption growth shocks. With habit and the sensitivity function, Model III is able to match the target annual risk-free rate of 0.76%. The volatility of the risk-free rate is very similar to those for Model II with long-run risks only: it is mainly due to the positive dependence of the risk-free rate on $x_t$ and increases with decreasing $\psi$. In the model solution, the relations of key variables—the utility–surplus consumption ratio, the price–dividend ratio, the equity premium, and the return volatility—with state variables are the same as those in Models I and II.

Panel A of Table VII reports the average equity premium in the simulated data for different values of $\gamma$ and $\psi$. These results, when compared with those for Models I and II, confirm that the effects of habit and long-run risks are complementary. For almost all cases, when the equity premium is positive (negative) in Model II, the equity premium of Model III is higher (lower) than that of Model I.\footnote{Note that the time discount parameter $\delta$ is different across three models.} For example, when $\gamma = 2$ and $\psi = 0.5$, the three models generate equity premiums of 5.62%, −1.45%, and 5.37%, respectively. For the case $\gamma = 5$ and $\psi = 1$, the three models generate 4.08%, 0.78%, and 4.51%, respectively. For the case $\gamma = 10$ and $\psi = 1.5$, the three models generate 4.21%, 4.70%, and 6.93%, respectively. Also manifesting the complementary effects of habit and long-run risks are the return volatilities of Model III in Panel B of Table VII. For almost all cases, the return volatilities for Model III are higher than those for Models I and II. Panel C of Table VII indicates that across the $\gamma$ and $\psi$ values, the average price–dividend ratio varies negatively with the equity premium, while the volatility of the price–dividend ratio follows the same pattern as the return volatility.

Model III also generates the negative relation between future excess returns and the price–dividend ratio, since the mechanisms of return predictability in Models I and II are both at work. Panels E and F report the model-implied results for the 3-year-ahead prediction of cumulative excess log stock returns by the log price–dividend ratio. Quantitatively, both the slopes and the $R^2$ values are between those for Model I and Model II—that is, the return prediction results of Model III are somewhat an average of those of Models I and II.
Table VII. Model-implied asset pricing results of Model III (both habit and long-run risks)

This table reports key asset pricing results in the model-implied, simulated data for Model III (both habit and long-run risks). The results are reported for different parameters of risk aversion $\gamma$ and the elasticity of intertemporal substitution $\psi$. Here, $r$ is the log stock return, $r_f$ is the log risk-free rate, $p - d$ is the log price-dividend ratio, and $\text{slope}(3\text{yr})$ and $R^2(3\text{yr})$ are from regressing the cumulative excess log stock return from the end of year t to the end of year $t+3$ on the log price-dividend ratio at the end of year $t$. In addition, $\mu[\cdot]$ denotes the mean and $\sigma[\cdot]$ denotes the standard deviation. The results are the means over 10,000 samples simulated from the numerical solution of the model. Each simulated sample is 83 x 12-month long, and is then time-aggregated to the annual frequency.

<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>$\psi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>0.5</td>
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</table>

A. $\mu[r - r_f]$ (%)

<table>
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<tr>
<th></th>
<th>2/3</th>
<th>1</th>
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<tbody>
<tr>
<td>2/3</td>
<td>28.28</td>
<td>8.13</td>
<td>2.88</td>
<td>1.74</td>
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<tr>
<td>1</td>
<td>15.91</td>
<td>6.25</td>
<td>3.04</td>
<td>1.83</td>
<td></td>
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<tr>
<td>2</td>
<td>8.55</td>
<td>5.37</td>
<td>3.29</td>
<td>2.46</td>
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<tr>
<td>5</td>
<td>7.01</td>
<td>5.66</td>
<td>4.51</td>
<td>4.13</td>
<td></td>
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<tr>
<td>10</td>
<td>4.53</td>
<td>6.48</td>
<td>6.95</td>
<td>6.93</td>
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B. $\sigma[r]$

<table>
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<tr>
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<td>0.270</td>
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<tr>
<td>1</td>
<td>0.313</td>
<td>0.268</td>
<td>0.261</td>
<td>0.263</td>
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<tr>
<td>2</td>
<td>0.249</td>
<td>0.225</td>
<td>0.226</td>
<td>0.229</td>
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<tr>
<td>5</td>
<td>0.254</td>
<td>0.197</td>
<td>0.201</td>
<td>0.205</td>
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<tr>
<td>10</td>
<td>0.280</td>
<td>0.184</td>
<td>0.183</td>
<td>0.186</td>
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</tbody>
</table>

C. $\mu[p - d]$ |

<table>
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<th>2</th>
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<tbody>
<tr>
<td>2/3</td>
<td>1.36</td>
<td>2.61</td>
<td>3.56</td>
<td>4.18</td>
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<tr>
<td>1</td>
<td>1.88</td>
<td>2.88</td>
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<td>2</td>
<td>2.47</td>
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<td>3.83</td>
<td></td>
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<td>5</td>
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<td>10</td>
<td>3.22</td>
<td>2.82</td>
<td>2.72</td>
<td>2.73</td>
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D. $\sigma[p - d]$ |

<table>
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<tbody>
<tr>
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<td>0.536</td>
<td>0.393</td>
<td>0.336</td>
<td>0.327</td>
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<tr>
<td>1</td>
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<td>0.303</td>
<td>0.313</td>
<td></td>
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<tr>
<td>2</td>
<td>0.309</td>
<td>0.256</td>
<td>0.258</td>
<td>0.262</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>0.340</td>
<td>0.195</td>
<td>0.198</td>
<td>0.205</td>
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<tr>
<td>10</td>
<td>0.416</td>
<td>0.172</td>
<td>0.168</td>
<td>0.181</td>
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</table>

(continued)
Table VII. Continued

<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>$\Psi$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.2</td>
</tr>
<tr>
<td>E. $\mu[\text{slope}(3 \text{ yr})]$</td>
<td></td>
</tr>
<tr>
<td>2/3</td>
<td>-1.201</td>
</tr>
<tr>
<td>1</td>
<td>-0.833</td>
</tr>
<tr>
<td>2</td>
<td>-0.492</td>
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<tr>
<td>5</td>
<td>-0.255</td>
</tr>
<tr>
<td>10</td>
<td>-0.216</td>
</tr>
<tr>
<td>F. $\mu[R^2(3 \text{ yr})]$</td>
<td></td>
</tr>
<tr>
<td>2/3</td>
<td>0.523</td>
</tr>
<tr>
<td>1</td>
<td>0.329</td>
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<tr>
<td>2</td>
<td>0.126</td>
</tr>
<tr>
<td>5</td>
<td>0.060</td>
</tr>
<tr>
<td>10</td>
<td>0.042</td>
</tr>
</tbody>
</table>

The quantitative results in Table VII suggest that $\gamma = 5$ and $\Psi = 1$ provide a good match to the empirical equity premium and return volatility. Detailed discussions are presented subsequently in Section 3.5.

So far, the article has discussed the results of all three models given the calibration of the growth rate dynamics and over a matrix of $\gamma$ and $\Psi$ parameters. Taken together, the habit mechanism appears to be a stronger driver of the model results than long-run risks. A comparison of Models I and II indicates that habit generates quantitative results that are large in magnitudes and also spread out over wide ranges, while for long-run risks, the results often cover a narrower range of values. A good example is the model-implied equity premiums: for Model I with habit, the equity premiums are all positive and vary between about 2% to almost 30%; for Model II with long-run risks, the equity premiums are often negative and no higher than 5%. As another example, the return volatility varies between 17% and 42% in Model I, while between 15% and 20% in Model II. Further, the slopes and the $R^2$s of excess return predictive regressions almost always exhibit higher magnitudes in Model I than in Model II. Thus, habit generates not only higher levels of the equity premium, but also stronger time variations.

When the two mechanisms are put together into Model III, the effect of habit is also stronger than that of long-run risks. The quantitative results of Model III are often closer in magnitudes to those of Model I. For example, the equity premiums of Model III in Panel A of Table VII are all positive, and mostly closer to those of Model I than to those of Model II. Indeed, habit is the dominant driver of the equity premium: in the preferred calibration of $\gamma = 5$ and $\Psi = 1$ for Model III, the habit mechanism, through the random walk growth shocks, contributes about 80% of the equity premium, while long-run growth risk contributes 19% and long-run volatility risk 1%.
The quantitative results of Model III vary over $\gamma$ and $\psi$ mostly following the patterns in Model I. For example, regressing the equity premium of Model III on that of Model I and a constant yields a high $R^2$ of 93.3%, while regressing the equity premium of Model III on that of Model II and a constant yields an $R^2$ of only 2.1%. Similarly, the return volatility of Model I can explain 87.5% of the variation of the return volatility of Model III, while the return volatility of Model II can only explain 20.2%. As another example, the slope of the excess return predictive regression of Model I can explain 95.7% of the variation of the slope of Model III while the slope of Model II can only explain 25.7%.

3.5 Preferred Calibrations
For a detailed look at the quantitative implications of the three models at their respective preferred calibrations, Tables VIII and IX present the simulation results and compare with the empirical asset pricing data. All three models provide good match to the empirical equity premium of 4.78% and the empirical return volatility of 19.4% for the aggregate US stock market. This is not surprising since the preferred calibrations are mainly determined by fitting the equity premium and the return volatility in the empirical data.

Model I is constructed to generate a flat risk-free rate of 0.76%, and thus the model-implied risk-free rate matches the empirical estimate exactly. For Model II, the average risk-free rate of 1.43% in the simulated data is higher than the empirical estimate. Still, the empirical estimate is within the 5th-to-95th percentile band. Model III matches the empirical risk-free rate due to the habit component.

The empirical risk-free rates, with a volatility of about 4% as reported in Table VIII, are ex post real rates and contain inflation surprises. The volatility of the ex ante real risk-free rates is most likely much lower. The model-implied risk-free rates are ex ante real rates. The volatilities are 0 by construction in Model I, and about 1% and 2% in Models II and III, respectively, due to the long-run risks component.

The empirical level of the price–dividend ratio is well replicated in Models I and III, both with habit. Model II, with long-run risks only, appears to understate the price–dividend ratio: the 95th percentile of the model-implied average price–dividend ratio is still lower than the empirical value.

A serious challenge to all three models is the volatility of the log price–dividend ratio. For the three models, the average volatilities are 0.139, 0.188, and 0.198, respectively. In contrast, the empirical volatility is 0.382, well above the 95th percentiles of the model-implied volatilities. This echoes the observations in Beeler and Campbell (2012) and Bansal Kiku, and Yaron (2012) that long-run risks models considerably understate the volatility of the price–dividend ratio. Further, the results suggest that habit formation is also subject to a similar weakness. All three models also fall short of accounting for the persistence of the empirical price–dividend ratio. The first-order autocorrelation of the price–dividend ratio in the empirical data is 0.92. To compare, Models I and III generate an average autocorrelation of 0.86, while Model II yields 0.84.

As noted in Beeler and Campbell (2012), the high volatility of the log price–dividend ratio is in part because stock prices are persistently high since the 1990s. Indeed, for the log price–dividend ratio over the sub-period of 1929–89, the volatility is 0.259. In an earlier study, Lettau, Ludvigson, and Wachter (2008) attribute the high aggregate stock prices relative to the fundamental value to a fall in macroeconomic risk, and present a regime-switching model to explain the run-up in asset valuation ratios. Interestingly, over the
The sub-period of 1929–89, the first-order autocorrelation of the log price–dividend ratio is only 0.81. Thus, the high autocorrelation of 0.92 over the full sample period of 1929–2012 may also be related to the regime-switching process proposed in Lettau, Ludvigson, and Wachter (2008).

Table VIII. Empirical and model-implied asset pricing data

The table compares the properties of the annual empirical asset market data for the period 1929–2012 with those of the model-implied, simulated data. Here, \( r \) is the log stock return, \( p - d \) is the log price–dividend ratio, \( \mu(1) \) and \( \mu(2) \) are the first- and second-order autocorrelations, \( \mu[\cdot] \) denotes the mean, and \( \sigma[\cdot] \) denotes the standard deviation. The empirical estimates are obtained with GMM, and standard errors are Newey and West (1987) corrected with five lags. The model-implied results are the means and percentiles over 10,000 samples simulated from the numerical solutions of the models. Each simulated sample is 83 × 12-month long, and is then time-aggregated to the annual frequency.

<table>
<thead>
<tr>
<th>Data</th>
<th>Model I (habit only)</th>
<th>Model II (long-run risks only)</th>
<th>Model III (both)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mu[r - r_f] )</td>
<td>0.117 (0.044)</td>
<td>0.134 0.019</td>
<td>0.072 0.016</td>
</tr>
<tr>
<td>( \sigma[r] )</td>
<td>0.194 (0.019)</td>
<td>0.186 0.162</td>
<td>0.198 0.164</td>
</tr>
<tr>
<td>( \mu[p - d] )</td>
<td>3.29 (0.09)</td>
<td>3.27 3.18</td>
<td>3.19 3.06</td>
</tr>
<tr>
<td>( \sigma[p - d] )</td>
<td>0.382 (0.051)</td>
<td>0.139 0.109</td>
<td>0.198 0.153</td>
</tr>
<tr>
<td>( AC(1)[p - d] )</td>
<td>0.92 (0.04)</td>
<td>0.86 0.58</td>
<td>0.198 0.153</td>
</tr>
<tr>
<td>( AC(2)[p - d] )</td>
<td>0.82 (0.07)</td>
<td>0.68 0.20</td>
<td>0.198 0.153</td>
</tr>
</tbody>
</table>

Table IX. Prediction of excess stock returns by price–dividend ratio

This table compares the empirical results of the return predictive regressions with those from the model-implied, simulated data. The cumulative excess log stock return from the end of year \( t \) to the end of year \( t + J \) is regressed on the log price–dividend ratio at the end of year \( t \). Here, \( J = 1, 3, 5 \) is the horizon. The empirical data are annual for the period 1929–2012. The empirical estimates are obtained with GMM, and standard errors are Newey and West (1987) corrected with five lags. The model-implied results are the means over 10,000 samples simulated from the numerical solutions of the models. Each simulated sample is 83 × 12-month long, and is then time-aggregated to the annual frequency.

<table>
<thead>
<tr>
<th>Horizon (Year)</th>
<th>Data</th>
<th>Model I (habit only)</th>
<th>Model II (long-run risks only)</th>
<th>Model III (both)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \gamma = 5, \psi = 1 )</td>
<td>( \gamma = 10, \psi = 1.5 )</td>
<td>( \gamma = 5, \psi = 1 )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Slope Standard error</td>
<td>Slope</td>
<td>( R^2 )</td>
<td>Slope</td>
<td>( R^2 )</td>
</tr>
<tr>
<td>1</td>
<td>(-0.117 ) (0.044)</td>
<td>0.052</td>
<td>(-0.134 ) 0.019</td>
<td>0.014</td>
</tr>
<tr>
<td>3</td>
<td>(-0.323 ) (0.099)</td>
<td>0.169</td>
<td>(-0.371 ) 0.044</td>
<td>0.032</td>
</tr>
<tr>
<td>5</td>
<td>(-0.486 ) (0.141)</td>
<td>0.293</td>
<td>(-0.562 ) 0.064</td>
<td>0.046</td>
</tr>
</tbody>
</table>
In all three models, the equity premium is predictable by the price–dividend ratio with negative slopes. As reported in Table IX, all the models are able to generate the qualitative features of the empirical results, as both the magnitude of the slope and the $R^2$ rise with the horizon. Quantitatively, Model I provides the best match to the empirical slopes, while the magnitudes of the results are the smallest in Model II. This is consistent with the earlier discussion that the habit mechanism generates stronger time-varying asset pricing implications. All three models fall short of replicating the high empirical $R^2$ values.

Overall, the results are consistent with the conclusions in the existing literature that both habit formation and long-run risks go a long way toward explaining key asset market phenomena. In terms of quantitative magnitudes, the results appear to slightly favor Models I and III, both with the habit formation mechanism.

4. Concluding Remarks

This article presents a model that highlights the distinctive effects of intertemporal substitution on major asset pricing implications. The model builds on the Epstein–Zin utility and incorporates both the habit formation and long-run risks mechanisms. The general model in the article nests the influential models of Campbell and Cochrane (1999) and Bansal and Yaron (2004). This potentially opens up opportunities to test the two models jointly with empirical data in future studies.

I follow the tradition of a large part of the consumption-based asset pricing literature and stay within a partial equilibrium, single good framework. While the parsimonious setup offers the advantage of focusing on the role of intertemporal substitution, there are potential extensions of the model for future research. Specifically, the model directly specifies the consumption growth process, and thus abstracts away from the production technology or the labor choice. A potentially interesting extension is a general equilibrium setting where consumption is endogenously determined by the agent together with the production and labor decisions. The recent study of Dew-Becker (2013) embeds habit formation in the Epstein–Zin preferences in a production economy. Also, this model is formulated in a single good framework, while in general the Epstein–Zin utility is also non-separable across goods. Hence, it would be interesting to extend the model to include, for example, durable goods or housing. These interesting extensions merit separate studies in the future.

Appendices

The appendices derive the log-linear, approximate analytical solution and describe the numerical solution of the model.

For expositional convenience, I use the shorthand notation for the sensitivity function,

$$\lambda_t = \lambda(s_t),$$

(A.1)
and its steady-state value and the derivative at \( s_t = \bar{s} \),
\[
\bar{\lambda} = \lambda(\bar{s}), \quad \bar{\lambda}' = \lambda'(\bar{s}).
\] (A.2)

For the market price of short-run growth risk,
\[
\bar{\xi}_{c,t} = \xi_c(s_t), \quad \bar{\xi}' = \xi'(\bar{s}).
\] (A.3)

In general, a time-varying quantity \( f_t \) is a function of all three state variables \( s_t, x_t, \) and \( \omega_t^2 \), and \( \bar{f} \) denotes its steady-state value when \( s_t = \bar{s}, x_t = 0, \) and \( \omega_t^2 = 1 \).

The approximate solution also utilizes the first-order Taylor expansion around \( s_t = \bar{s} \) and \( \omega_t^2 = 1 \) for the generic form
\[
f(s_t)g(\omega_t^2) \approx f(\bar{s})g(1) + f'(\bar{s})(s_t - \bar{s}) + f(\bar{s})g'(1)(\omega_t^2 - 1).
\] (A.4)

In particular,
\[
f(s_t)(\omega_t^2 - 1) \approx f(\bar{s})(\omega_t^2 - 1).
\] (A.5)

\[
\bar{f}(s_t)(\omega_t^2 - 1) = f(\bar{s})(\omega_t^2 - 1).
\] (A.6)

### A. Utility Function

Denote the log utility–surplus consumption ratio by
\[
y_t = \log \frac{U_t}{C_t - H_t}.
\] (A.7)

Since \( C_t - H_t = C_t S_t \),
\[
\log \frac{C_{t+1} - H_{t+1}}{C_t - H_t} = \log \frac{C_{t+1} S_{t+1}}{C_t S_t} = \Delta C_{t+1} + \Delta S_{t+1}.
\] (A.8)

If \( \gamma \neq 1 \) and \( \psi \neq 1 \), the utility function, rewritten as an equation for the utility–surplus consumption ratio, is
\[
\left( \frac{U_t}{C_t - H_t} \right)^{1 - \frac{1}{\psi}} = 1 - \delta + \delta \left[ \left( \frac{U_{t+1}}{C_{t+1} - H_{t+1}} \right) \left( \frac{C_{t+1} - H_{t+1}}{C_t - H_t} \right)^{1 - \gamma} \right]^{1 - \frac{1}{\psi}}.
\] (A.9)

Let
\[
q_t = \frac{1}{1 - \gamma} \log E_t \left[ e^{(1 - \gamma)(y_{t+1} + \Delta C_{t+1} + \Delta S_{t+1})} \right],
\] (A.10)

then the equation becomes
\[
y_t = \frac{1}{1 - \frac{1}{\psi}} \log \left( 1 - \delta + \delta e^{(1 - \frac{1}{\psi}) q_t} \right).
\] (A.11)

Apply the first-order Taylor expansion around \( q_t = \bar{q} \),
\[
y_t \approx \frac{1}{1 - \frac{1}{\psi}} \log \left( 1 - \delta + \delta e^{(1 - \frac{1}{\psi}) \bar{q}} \right) + \frac{\delta e^{(1 - \frac{1}{\psi}) \bar{q}}}{1 - \delta + \delta e^{(1 - \frac{1}{\psi}) \bar{q}}} (q_t - \bar{q})
\]
\[
= \kappa_{0y} + \kappa_{1y} q_t.
\] (A.12)
Here, $\kappa_0y$ and $0 < \kappa_1 y < 1$ are constants.

If $\gamma = 1$, then the certainty equivalent becomes

$$J(U_{t+1}) = e^{E_t[\log U_{t+1}]}.$$  \hfill (A.13)

If $\psi = 1$, then the aggregator becomes

$$\log U_t = (1 - \delta) \log (C_t - H_t) + \delta \log (J(U_{t+1})).$$  \hfill (A.14)

Hence, if $\gamma = 1$ but $\psi \neq 1$,

$$U_t = \left((1 - \delta)(C_t - H_t)^{1-\frac{1}{\gamma}} \right)^{\frac{1}{1-\psi}},$$  \hfill (A.15)

$$\left(\frac{U_t}{C_t - H_t}\right)^{1-\frac{1}{\gamma}} = 1 - \delta + \delta \left(\exp\left(E_t\left[\log\left(\frac{U_{t+1}}{C_{t+1} - H_{t+1}} - \frac{C_{t+1} - H_{t+1}}{C_t - H_t}\right)\right]\right)\right)^{1-\frac{1}{\gamma}},$$  \hfill (A.16)

$$q_t = E_t[y_{t+1} + \Delta c_{t+1} + \Delta s_{t+1}],$$  \hfill (A.17)

$$y_t = \frac{1}{1 - \psi} \log \left(1 - \delta + \delta e^{(1-\frac{1}{\gamma})q_t}\right).$$  \hfill (A.18)

If $\gamma \neq 1$ but $\psi = 1$,

$$\log U_t = (1 - \delta) \log (C_t - H_t) + \delta \log \left(E_t[(U_{t+1}^{1-\frac{1}{\gamma}})]^{\frac{1}{1-\gamma}}\right),$$  \hfill (A.19)

$$\log \frac{U_t}{C_t - H_t} = \delta \frac{1}{1 - \gamma} \log \left(E_t\left[\frac{U_{t+1}}{C_{t+1} - H_{t+1}} - \frac{C_{t+1} - H_{t+1}}{C_t - H_t}\right]^{1-\gamma}\right),$$  \hfill (A.20)

$$q_t = \frac{1}{1 - \gamma} \log E_t[e^{(1-\gamma)(y_{t+1} + \Delta c_{t+1} + \Delta s_{t+1})}],$$  \hfill (A.21)

$$y_t = \delta q_t.$$  \hfill (A.22)

If $\gamma = \psi = 1$,

$$\log U_t = (1 - \delta) \log (C_t - H_t) + \delta E_t[\log U_{t+1}],$$  \hfill (A.23)

$$\log \frac{U_t}{C_t - H_t} = \delta E_t\left[\log \left(\frac{U_{t+1}}{C_{t+1} - H_{t+1}} - \frac{C_{t+1} - H_{t+1}}{C_t - H_t}\right)\right],$$  \hfill (A.24)

$$q_t = E_t[y_{t+1} + \Delta c_{t+1} + \Delta s_{t+1}],$$  \hfill (A.25)

$$y_t = \delta q_t.$$  \hfill (A.26)

\section*{B. Utility–Surplus Consumption Ratio}

Assume the log-linear approximation

$$y_t \approx V_0 + V_1(s_t - \bar{s}) + V_2x_t + V_3(o_t^2 - 1),$$  \hfill (B.1)
then
\[ y_{t+1} \approx y_0 + V_0(s_{t+1} - \bar{s}) + V_2x_{t+1} + V_3(\omega^2_t + 1) \]

\[ = V_0 + V_1\phi_x(s_t - \bar{s}) + V_2\phi_x x_t + V_3\theta_0(\omega^2 - 1) \]

\[ + V_1\lambda_x x_t + \sigma_x\omega_t v_{x,t+1} + V_2\sigma_x\omega_t v_{x,t+1} + V_3\sigma_x\omega_{x,t+1}, \quad \text{(B.2)} \]

and

\[ y_{t+1} + \Delta c_{t+1} + \Delta s_{t+1} \]

\[ = V_0 + V_1(s_{t+1} - \bar{s}) + V_2x_{t+1} + V_3(\omega^2_{t+1} - 1) \]

\[ + \mu_c + x_t + \sigma_c\omega_t v_{c,t+1} + (\phi_s - 1)(s_t - \bar{s}) + \lambda_c x_t + \sigma_c\omega_t v_{c,t+1} \quad \text{(B.3)} \]

\[ = V_0 + V_1\phi_x(s_t - \bar{s}) + V_2\phi_x x_t + V_3\theta_0(\omega^2 - 1) + \mu_c + x_t + (\phi_s - 1)(s_t - \bar{s}) \]

\[ + (V_1\lambda_x + 1 + \lambda_t)\sigma_x\omega_t v_{x,t+1} + V_2\sigma_x\omega_t v_{x,t+1} + V_3\sigma_x\omega_{x,t+1}. \]

It follows that

\[ E_t[y_{t+1} + \Delta c_{t+1} + \Delta s_{t+1}] \]

\[ = V_0 + V_1\phi_x(s_t - \bar{s}) + V_2\phi_x x_t + V_3\theta_0(\omega^2 - 1) + \mu_c + x_t + (\phi_s - 1)(s_t - \bar{s}). \quad \text{(B.4)} \]

and

\[ \text{var}_t[y_{t+1} + \Delta c_{t+1} + \Delta s_{t+1}] \]

\[ = (V_1\lambda_x + 1 + \lambda_t)^2 \sigma^2 c + V_2^2 \sigma^2 c + V_3^2 \sigma^2 c \]

\[ \approx (2(V_1\lambda_x + 1 + \lambda_t)(V_1\lambda_x + 1) + (V_1\lambda_x + 1 + \lambda_t)^2 \omega^2 + V_2^2 \sigma^2 c + V_3^2 \sigma^2 c). \quad \text{(B.5)} \]

The last line above applies the first-order Taylor expansion of \((V_1\lambda_x + 1 + \lambda_t)^2 \omega^2\) around \(s_t = \bar{s}\) and \(\omega^2_t = 1\).

Substitute

\[ q_t = \frac{1}{1 - \gamma} \log E_t \left[ e^{(1-\gamma)(y_{t+1} + c_{t+1} - s_{t+1})} \right] \]

\[ \approx E_t[y_{t+1} + \Delta c_{t+1} + \Delta s_{t+1}] + \frac{1 - \gamma}{2} \text{var}_t[y_{t+1} + \Delta c_{t+1} + \Delta s_{t+1}] \quad \text{(B.6)} \]

into the equation

\[ y_t \approx \kappa_{0y} + \kappa_{1y} q_t. \quad \text{(B.7)} \]

Collect \(s_t - \bar{s}\) terms on both sides and omit the term containing \(\sigma^2 c \ll 1\),

\[ V_1(s_t - \bar{s}) \approx \kappa_{1y}(V_1\phi_x(s_t - \bar{s}) + (\phi_s - 1)(s_t - \bar{s})), \quad V_1 \approx \frac{\kappa_{1y}(\phi_s - 1)}{1 - \kappa_{1y} \phi_s}. \quad \text{(B.8)} \]

Collect \(x_t\) terms,

\[ V_2x_t \approx \kappa_{1y}(V_2\phi_x x_t + x_t), \quad V_2 \approx \frac{\kappa_{1y}}{1 - \kappa_{1y} \phi_x}. \quad \text{(B.9)} \]
Collect $\omega_t^2$ terms,
\[
V_3 \omega_t^2 \approx \kappa_1 \left( \frac{1 - \gamma}{2} \left( (V_1 x + 1 + \lambda)^2 \sigma_x^2 \omega_t^2 + V_2 \sigma_x^2 \omega_t^2 \right) \right), \tag{B.10}
\]
\[
V_3 \approx \frac{1 - \gamma}{2} \left( (V_1 x + 1 + \lambda)^2 \sigma_x^2 + V_2 \sigma_x^2 \right). \tag{B.11}
\]

**C. Pricing Kernel**

The representative agent maximizes
\[
U_t = \left( \frac{1}{1 - \delta} (C_t - H_t)^{1 - \frac{1}{\phi}} + \delta \left( E_t[U_{t+1}^{1 - \gamma}] \right)^{1 - \gamma} \right)^{\frac{1}{1 - \gamma}}. \tag{C.1}
\]

With wealth $W_t$, the agent chooses consumption $C_t$ and then invests the remaining in the risk-free asset $R_{f,t}$ and risky assets, denoted by an excess return vector $R_{t+1}^e$. The agent chooses a portfolio vector $\omega_t$ for the risky assets. Hence, the budget constraint is
\[
W_{t+1} = (W_t - C_t) (R_{f,t} + \omega_t R_{t+1}^e). \tag{C.2}
\]

The first-order condition with respect to $C_t$ yields
\[
(1 - \delta) (C_t - H_t)^{-\frac{1}{\phi}} = \delta \left( E_t[U_{t+1}^{1 - \gamma}] \right)^{1 - \gamma} E_t \left[ U_{t+1}^{1 - \gamma} \frac{\partial U_{t+1}}{\partial W_{t+1}} (R_{f,t} + \omega_t R_{t+1}^e) \right]. \tag{C.3}
\]

The first-order condition with respect to the portfolio $\omega_t$ yields
\[
0 = \delta \left( E_t[U_{t+1}^{1 - \gamma}] \right)^{1 - \gamma} E_t \left[ U_{t+1}^{1 - \gamma} \frac{\partial U_{t+1}}{\partial W_{t+1}} R_{t+1}^e \right]. \tag{C.4}
\]

The envelope theorem implies that
\[
\frac{\partial U_t}{\partial W_t} = U_t^{\frac{1}{\phi}} (1 - \delta) (C_t - H_t)^{-\frac{1}{\phi}} = (1 - \delta) \left( \frac{U_t}{C_t - H_t} \right)^{\frac{1}{\phi}}. \tag{C.5}
\]

Taken together, the pricing kernel is
\[
M_{t+1} = \delta \left( E_t[U_{t+1}^{1 - \gamma}] \right)^{1 - \gamma} \frac{U_{t+1}}{C_{t+1} - H_{t+1}} \left( \frac{U_{t+1}}{C_{t+1} - H_{t+1}} \right)^{\frac{1}{\phi}}
\]
\[
= \delta \left[ \left( \frac{U_{t+1}}{C_{t+1} - H_{t+1}} \right) \left( \frac{C_{t+1} - H_{t+1}}{C_t - H_t} \right) \left( \frac{U_{t+1}}{C_{t+1} - H_{t+1}} \right)^{1 - \gamma} \right]^{\frac{1}{\phi}}
\]
\[
\times \left( \frac{U_{t+1}}{C_{t+1} - H_{t+1}} \right)^{1 - \gamma} \left( \frac{U_{t+1}}{C_{t+1} - H_{t+1}} \right)^{\frac{1}{\phi}} \tag{C.6}
\].
The above result applies for $c \neq 1$ and any $w$, including $w = 1$. For $c = 1$ and any $w$,

$$M_{t+1} = \delta \left( e^{E_t[U_{t+1}]} \right) \frac{1}{c} \left( C_t - H_t \right)\frac{\frac{U_{t+1}}{C_t} - \frac{H_{t+1}}{H_t}}{\frac{C_t}{C_t} - \frac{H_t}{H_t}}$$

$$= \delta \left( \exp \left( E_t \left[ \left( \frac{U_{t+1}}{C_t} - \frac{H_{t+1}}{H_t} \right) \right] \right) \right) \frac{1}{c} \left( C_t - H_t \right)\frac{\frac{U_{t+1}}{C_t} - \frac{H_{t+1}}{H_t}}{\frac{C_t}{C_t} - \frac{H_t}{H_t}}.$$

(C.7)

D. Market Prices of Risks

The log pricing kernel is

$$m_{t+1} = \log \delta + \frac{\gamma - \frac{1}{\psi}}{1-\gamma} \log E_t \left[ e^{\left(1-\gamma\right)\left(y_{t+1} + \Delta c_{t+1} + \Delta s_{t+1}\right)} \right]$$

$$- \gamma \left(y_{t+1} + \Delta c_{t+1} + \Delta s_{t+1}\right) + \frac{1}{\psi} y_{t+1}.$$  

(D.1)

Here,

$$\frac{\gamma - \frac{1}{\psi}}{1-\gamma} \log E_t \left[ e^{\left(1-\gamma\right)\left(y_{t+1} + \Delta c_{t+1} + \Delta s_{t+1}\right)} \right]$$

$$\approx \left( \gamma - \frac{1}{\psi} \right) \left( E_t[y_{t+1} + \Delta c_{t+1} + \Delta s_{t+1}] + \frac{1-\gamma}{2} \var_t[y_{t+1} + \Delta c_{t+1} + \Delta s_{t+1}] \right).$$  

(D.2)

With the log-linear approximation for $y_t$,

$$E_t[m_{t+1}] = \log \delta + \frac{\gamma - \frac{1}{\psi}}{1-\gamma} \log E_t \left[ e^{\left(1-\gamma\right)\left(y_{t+1} + \Delta c_{t+1} + \Delta s_{t+1}\right)} \right]$$

$$- \gamma E_t[y_{t+1} + \Delta c_{t+1} + \Delta s_{t+1}] + \frac{1}{\psi} E_t[y_{t+1}]$$

$$\approx \log \delta - \frac{1}{\psi} E_t[\Delta c_{t+1} + \Delta s_{t+1}]$$

$$+ \left( \gamma - \frac{1}{\psi} \right) \frac{1-\gamma}{2} \var_t[y_{t+1} + \Delta c_{t+1} + \Delta s_{t+1}]$$

$$= \log \delta - \frac{1}{\psi} (\mu_x + x_t + (\phi_x - 1)(s_t - \tilde{s}))$$

$$+ \left( \gamma - \frac{1}{\psi} \right) \frac{1-\gamma}{2} \left( (V_1 \lambda_t + 1 + \lambda_t) \sigma_x^2 \omega_t^2 + V_2 \sigma_x^2 \omega_t^2 + V_3 \sigma_x^2 \right).$$  

(D.3)
From the results in the preceding section, the derivative with respect to $H$ is

$$
m_t + 1 - E_t[m_t + 1]$$

$$
- \gamma (V_1 \lambda t + 1 + \lambda t) \sigma_e \omega_t \epsilon_{c,t + 1} + V_2 \sigma_x \omega_t \epsilon_{c,t + 1} + V_3 \sigma_{o,0} \epsilon_{o,t + 1}
$$

$$
+ \frac{1}{\psi} (V_1 \lambda t - \sigma_x \omega_t \epsilon_{c,t + 1} + V_2 \sigma_x \omega_t \epsilon_{c,t + 1} + V_3 \sigma_{o,0} \epsilon_{o,t + 1})
$$

$$
= \left( \gamma (V_1 \lambda t + 1 + \lambda t) + \frac{1}{\psi} (-V_1 \lambda t) \right) \sigma_e \omega_t \epsilon_{c,t + 1}
$$

$$
- (\gamma V_2 - \frac{1}{\psi} V_2) \sigma_x \omega_t \epsilon_{c,t + 1} - (\gamma V_3 - \frac{1}{\psi} V_3) \sigma_{o,0} \epsilon_{o,t + 1}
$$

$$
= -\xi c, t \sigma_e \omega_t \epsilon_{c,t + 1} - \xi x \sigma_x \omega_t \epsilon_{c,t + 1} - \xi o \sigma_{o,0} \epsilon_{o,t + 1}.
$$

Then

$$
\text{var}_t[m_t + 1] \approx \left( \gamma (V_1 \lambda t + 1 + \lambda t) + \frac{1}{\psi} (-V_1 \lambda t) \right)^2 \sigma_e^2 \omega_t^2
$$

$$
+ \left( \gamma V_2 - \frac{1}{\psi} V_2 \right)^2 \sigma_x^2 \omega_t^2 + \left( \gamma V_3 - \frac{1}{\psi} V_3 \right)^2 \sigma_{o,0}^2
$$

(E.4)

$$
= \xi c, t \sigma_e^2 \omega_t^2 + \xi x \sigma_x^2 \omega_t^2 + \xi o \sigma_{o,0}^2.
$$

(E.5)

### E. Risk-Free Rate

For the log risk-free rate,

$$
rf_t = -\log E_t[e^{m_t}] = -E_t[m_t + 1] - \frac{1}{2} \text{var}_t[m_t + 1],
$$

(E.1)

both the conditional mean and the conditional variance of $m_t + 1$ are shown in the preceding section.

In general, the risk-free rate is a function of all three state variables

$$
rf_t = rf(s_t - s, x_t, \omega_t^2 - 1).
$$

(E.2)

The sensitivity function $\lambda(s_t)$ is chosen so that the risk-free rate is independent of $s_t$ at $x_t = 0$ and $\omega_t^2 = 1$. Let $rf$ denote the derivative with respect to the $i$th argument, then

$$
rf(s_t - s, 0, 0) = 0.
$$

(E.3)

From the results in the preceding section, the derivative with respect to $x_t$ is

$$
rf(s_t - s, 0, 0) = \frac{1}{\psi}.
$$

(E.4)

Hence, the first-order Taylor expansion of the risk-free rate is

$$
rf_t \approx rf_0 + rf_1(0, 0, 0)(s_t - s) + rf_2(0, 0, 0)x_t + rf_3(0, 0, 0)(\omega_t^2 - 1)
$$

$$
= F_0 + \frac{1}{\psi} x_t + F_3(\omega_t^2 - 1).
$$

(E.5)

Here, $F_0$ and $F_3$ are constants.
F. Price–Dividend Ratio

The stock dividend growth is

$$\log \frac{D_{t+1}}{D_t} = \Delta d_{t+1} = \mu_d + \eta_d x_t + \sigma_d \omega_t e_{d,t+1}. \quad (F.1)$$

Assume the log-linear approximation

$$\log P_t = z_t \approx A_0 + A_1 (s_t - \bar{s}) + A_2 x_t + A_3 (\omega_t^2 - 1). \quad (F.2)$$

Apply the Taylor expansion

$$\log(1 + e^{z_t}) \approx \log(1 + e^\bar{z}) + \frac{e^\bar{z}}{1 + e^\bar{z}} (z_t - \bar{z}) = \kappa_0 + \kappa_1 z_{t+1}, \quad (F.3)$$

in which \(\kappa_0\) and \(0 < \kappa_1 < 1\) are constants, to the log return,

$$r_{t+1} = \log(1 + e^{z_{t+1}}) + \Delta d_{t+1} - z_t \approx \kappa_0 + \kappa_1 z_{t+1} + \Delta d_{t+1} - z_t. \quad (F.4)$$

The pricing equation

$$1 = E_t[e^{m_{t+1} + r_{t+1}}] \quad (F.5)$$

implies

$$e^{z_t} \approx E_t\left[e^{m_{t+1}} e^{\kappa_0 + \kappa_1 z_{t+1} + \Delta d_{t+1}}\right]$$

$$\approx E_t[e^{m_{t+1}}] E_t\left[e^{\kappa_0 + \kappa_1 z_{t+1} + \Delta d_{t+1}}\right] e^{\text{cov}[m_{t+1}, \kappa_0 + \kappa_1 z_{t+1} + \Delta d_{t+1}]]. \quad (F.6)$$

The last line above applies the result that, for normal random variables \(X\) and \(Y\),

$$E[e^{X+Y}] = E[e^X] E[e^Y] e^{\text{cov}[X,Y]]. \quad (F.7)$$

From the risk-free rate,

$$E_t[e^{m_{t+1}}] = e^{-\eta_d} \approx e^{-f_0 - \frac{1}{2} \sigma_d^2 (\omega_t^2 - 1)}. \quad (F.8)$$

For

$$\kappa_0 + \kappa_1 z_{t+1} + \Delta d_{t+1}$$

$$= \kappa_0 + \kappa_1 A_0 + \kappa_1 A_1 (s_{t+1} - \bar{s}) + \kappa_1 A_2 x_{t+1} + \kappa_1 A_3 (\omega_{t+1}^2 - 1) + \Delta d_{t+1}$$

$$= \kappa_0 + \kappa_1 A_0 + \kappa_1 A_1 \phi (s_t - \bar{s}) + \kappa_1 A_2 \sigma_x \phi \omega_t e_{x,t+1} +$$

$$+ \kappa_1 A_3 \phi (\omega_t^2 - 1) + \kappa_1 A_3 \sigma_{\omega_t} e_{\omega,t+1} +$$

$$+ \mu_d + \eta_d x_t + \sigma_d \omega_t e_{d,t+1}, \quad (F.9)$$

it follows that

$$E_t[\kappa_0 + \kappa_1 z_{t+1} + \Delta d_{t+1}]$$

$$= \kappa_0 + \kappa_1 A_0 + \kappa_1 A_1 \phi (s_t - \bar{s}) + \kappa_1 A_2 \phi x_t + \kappa_1 A_3 \phi (\omega_t^2 - 1) + \mu_d + \eta_d x_t, \quad (F.10)$$
and
\[
\begin{align*}
\text{var}[\kappa_0 + \kappa_1 z_{t+1} + \Delta d_{t+1}] &= \kappa_1^2 A_j^2 \sigma_e^2 \sigma_v^2 + \kappa_1^2 A_j^2 \sigma_s^2 \sigma_v^2 + \kappa_1^2 A_j^2 \sigma_s^2 \sigma_v^2 + \sigma_s^2 \sigma_v^2 + 2\chi \kappa_1 A_1 \xi \sigma_e \sigma_d \sigma_v^2 \\
&= \kappa_1^2 A_j^2 \chi^2 \sigma_s^2 \sigma_v^2 + 2\chi \kappa_1 A_1 \xi \sigma_s^2 \sigma_v^2 + \kappa_1^2 A_j^2 \sigma_s^2 \sigma_v^2 + \sigma_s^2 \sigma_v^2 \\
&\quad + 2\chi \kappa_1 A_1 \sigma_e \sigma_d (\chi \sigma_s^2 + \chi^2 (s_t - \bar{s})).
\end{align*}
\]
(11)

The last term in the above results from the correlation \(\text{corr}[\kappa_{t+1}, \kappa_{t+1}] = \chi\). In addition, first-order Taylor expansions are applied to \(\hat{s}_t^2 \sigma_s^2\) and \(\hat{\lambda}_t \sigma_s^2\) around \(s_t = \bar{s}\) and \(\sigma_v^2 = 1\). Finally,
\[
\log E_t \left[ e^{\kappa_0 + \kappa_1 z_{t+1} + \Delta d_{t+1}} \right] \approx E_t [\kappa_0 + \kappa_1 z_{t+1} + \Delta d_{t+1}] + \frac{1}{2} \text{var}[\kappa_0 + \kappa_1 z_{t+1} + \Delta d_{t+1}],
\]
(12)
and
\[
\begin{align*}
\text{cov}[m_{t+1}, \kappa_0 + \kappa_1 z_{t+1} + \Delta d_{t+1}] &= -\xi \kappa_1 \kappa_1 A_1 \xi \sigma_s^2 \sigma_v^2 - \xi_\sigma \sigma_\sigma \sigma \sigma_v^2 - \xi_\kappa \kappa \kappa_1 A_1 \sigma_s^2 \sigma_v^2 - \xi_\sigma \sigma \sigma \sigma_v^2 \\
&\approx -\kappa_1 A_1 (\xi \kappa \kappa_1 A_1 \sigma_s^2 \sigma_v^2 + (\xi_\kappa \kappa_1 A_1 \sigma_s^2 \sigma_v^2 + \xi_\sigma \sigma \sigma \sigma_v^2)) \sigma_v^2 \\
&\quad - \chi (\xi \kappa \kappa_1 A_1 \sigma_s^2 \sigma_v^2 + \xi_\sigma \sigma \sigma \sigma_v^2) \sigma_v^2 - \xi_\sigma \sigma \sigma \sigma_v^2 \\
&\quad - \chi (\xi \kappa \kappa_1 A_1 \sigma_s^2 \sigma_v^2 + \xi_\sigma \sigma \sigma \sigma_v^2) \sigma_v^2 - \xi_\sigma \sigma \sigma \sigma_v^2.
\end{align*}
\]
(13)

On both sides of the pricing equation, collect \(s_t - \bar{s}\) terms and omit terms containing \(\sigma_v^2 \ll 1\),
\[
A_1 (s_t - \bar{s}) \approx \kappa_1 A_1 \phi_x (s_t - \bar{s}) + 2\chi \kappa_1 A_1 \xi (s_t - \bar{s}) \sigma_e \sigma_d - \chi (s_t - \bar{s}) \sigma_e \sigma_d,
\]
(14)
\[
A_1 \approx -\frac{\chi}{\xi} \frac{\xi \kappa \kappa_1 A_1 \sigma_s^2 \sigma_v^2}{1 - \kappa_1 \phi_x}. \tag{15}
\]

Numerically it can be confirmed that \(2\chi \kappa_1 A_1 \sigma_s^2 \sigma_v^2 \ll 1 - \kappa_1 \phi_x\). Consequently,
\[
A_1 \approx -\frac{\chi}{\xi} \frac{\xi \kappa \kappa_1 A_1 \sigma_s^2 \sigma_v^2}{1 - \kappa_1 \phi_x}. \tag{16}
\]
Collect \(x_t\) terms,
\[
A_2 x_t = -\frac{1}{\psi} x_t + \kappa_1 A_2 \phi_x x_t + \eta_d x_t, \quad A_2 = \frac{\eta_d - \frac{1}{\psi}}{1 - \kappa_1 \phi_x}. \tag{17}
\]
Collect \(\sigma_v^2\) terms,
\[
A_3 \sigma_v^2 \approx -F_3 \sigma_v^2 + \kappa_1 A_3 \phi_v \sigma_v^2 \\
+ \frac{1}{2} \left( \kappa_1^2 A_1^2 \sigma_s^2 \sigma_v^2 + \kappa_1^2 A_j^2 \sigma_s^2 \sigma_v^2 + \sigma_s^2 \sigma_v^2 + 2\chi \kappa_1 A_1 \sigma_s \sigma_d \sigma_v^2 \right) \\
- \kappa_1 A_1 \sigma_s^2 \sigma_v^2 \\
- \chi \sigma_s \sigma_d \sigma_v^2 - \xi \kappa \kappa_1 A_1 \sigma_s^2 \sigma_v^2,
\]
(18)
\[
A_3 \approx -\frac{F_3}{1 - \kappa_1 \phi_v} \left( \frac{1}{2} \left( \kappa_1^2 A_1^2 \sigma_s^2 \sigma_v^2 + \kappa_1^2 A_j^2 \sigma_s^2 \sigma_v^2 + \sigma_s^2 \sigma_v^2 + 2\chi \kappa_1 A_1 \sigma_s \sigma_d \sigma_v^2 \right) \\
- \kappa_1 A_1 \sigma_s^2 \sigma_v^2 \\
- \chi \sigma_s \sigma_d \sigma_v^2 - \xi \kappa \kappa_1 A_1 \sigma_s^2 \sigma_v^2 \right). \tag{19}
\]

G. Numerical Solution

The models in the article are solved numerically using a fixed-point iteration method. As a generic illustration of the method, suppose the solution quantity $z$ is a function of the state variable $s$, and is defined in the equation

$$zs_t(\cdot) = FE_t[G(z(s_{t+1}))],$$  \hspace{1cm} (G.1)

in which $F$ and $G$ are functions, and $E_t[\cdot]$ is time-$t$ conditional expectation. Represent $z$ by cubic splines on a grid of $s$. Beginning with the initial guess $z^0$, for each $z^k$, $z^{k+1}$ is computed using

$$z^{k+1}(s_t) = F\left(E_t\left[G\left(z^k(s_{t+1})\right)\right]\right),$$  \hspace{1cm} (G.2)

The expectation is evaluated using the Gauss–Hermite quadrature with twenty nodes. The iteration terminates when the difference between $z^k$ and $z^{k+1}$ is less than $10^{-8}$. If $s$ is a multi-dimensional state vector, then $z$ is represented using multi-dimensional cubic splines on a multi-dimensional grid.

In its original form, the Gauss–Hermite quadrature is a set of nodes $x_i$ and weights $w_i$ used to evaluate the integral

$$\int_{-\infty}^{+\infty} f(x) e^{-x^2} dx \approx \sum_i w_i f(x_i).$$  \hspace{1cm} (G.3)

With a change of variable, this quadrature can be used to evaluate the expectation with respect to normal random variables. Specifically, if $y \sim N(\mu, \sigma^2)$, then

$$E[y] = \int_{-\infty}^{+\infty} b(y) \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(y - \mu)^2}{2\sigma^2}\right) dy.$$  \hspace{1cm} (G.4)

Let $y = \sqrt{2}\sigma x + \mu$, then

$$E[y] = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{\pi}} b\left(\sqrt{2}\sigma x + \mu\right) e^{-x^2} dx \approx \sum_i \frac{w_i}{\sqrt{\pi}} b\left(\sqrt{2}\sigma x_i + \mu\right).$$  \hspace{1cm} (G.5)

To evaluate the expectation with respect to multiple independent normal random variables $y_n \sim N(\mu_n, \sigma_n^2), n = 1, 2, \ldots$,

$$E[b(y_1, y_2, \ldots)] \approx \sum_{i=1}^n \frac{w_i}{\sqrt{\pi}} \frac{w_j}{\sqrt{\pi}} \ldots b\left(\sqrt{2}\sigma_1 x_i + \mu_1, \sqrt{2}\sigma_2 x_j + \mu_2, \ldots\right).$$  \hspace{1cm} (G.6)

For the models in the current paper, I determine the utility–surplus consumption ratio using Equation (16). Then I compute the pricing kernel and solve for the price–dividend ratio using Equation (30).

For Models I and III, I also need to determine the sensitivity function $\lambda(s_t)$ so that the risk-free rate is flat with respect to $s_t$, and the time discount factor $\delta$ so that the risk-free rate matches the target. For the power utility with $\gamma = 1/\psi$, $\lambda(s_t)$ is known analytically as in Campbell and Cochrane (1999), and $\delta$ is also easily imputed. For general cases, $\lambda(s_t)$ and $\delta$ are computed numerically. I represent $\lambda(s_t)$ as cubic splines over a grid of $s_t - \bar{s}$, and impose the restrictions on the level and the derivative of $\lambda(s_t)$ at $s_t = \bar{s}$. The approximate analytical solutions indicate that higher $\delta$ and/or higher $\lambda(s_t)$ imply a lower risk-free rate. This suggests the following iterative approach. I begin with initial guesses of $\lambda(s_t)$ and $\delta$, etc.
and solve for the utility-surplus consumption ratio. From the solution, I compute the risk-free rate for each \( s_t \) and the average across all \( s_t \) values. If the average risk-free rate is higher (lower) than the target risk-free rate, then I increase (decrease) \( \delta \). If the risk-free rate at \( s_t \) is higher (lower) than the average, then I increase (decrease) \( \lambda(s_t) \). I repeat the procedure until the resulting risk-free rate is constant at the target value for all \( s_t \) values. In practice, I stop the iteration when both the variation in the risk-free rate across different \( s_t \) values and the deviation of the average risk-free rate from the target are less than 0.01% when annualized.

A number of papers indicate that it is a challenge to obtain accurate numerical solutions for the habit formation model of Campbell and Cochrane (1999).\(^{13}\) To check the accuracy of the numerical method used in my study, I apply the method to the habit formation model as calibrated in Campbell and Cochrane (1999) and the long-run risks model as calibrated in Bansal and Yaron (2004).

For the Campbell and Cochrane (1999) model, the sensitivity function can be obtained analytically. Still, I use the numerical procedure as described above. The numerically determined sensitivity function is essentially the same as the analytical result. For example, the long-run mean of the surplus ratio, \( e^\lambda \), is 0.570 in the analytical result and 0.571 in the

\(^{13}\) For example, Calin et al. (2005), Wachter (2005), Chen, Collin-Dufresne, and Goldstein (2003, 2009).
numerical result. I solve the model using different grids of $s_t - \bar{s}$. All the grids span between the left-most value $s_t - \bar{s} = L$ and the right-most value $s_t - \bar{s} = 0.5$. The grids differ in $L$ and the density of grid points. Panel A of Table X reports the key summary statistics of the simulated asset pricing data based on the numerical solutions. As $L$ becomes more negative to lengthen the grid, there are considerable decreases in the equity premium and the return volatility; in the meantime, the price–dividend ratio rises while its volatility decreases. The results largely flatten out as $L$ approaches $-100$. Further increasing the density of the grid generates essentially the same results.

These results are consistent with those in existing studies. Using a 17-point grid over the range from $s_t - \bar{s} = -2$ to 0.5, Campbell and Cochrane (1999) obtain an equity premium of 6.52% and a return volatility of 0.200. In a thorough study of the numerical solution of the Campbell and Cochrane (1999) model, Wachter (2005) shows that both the risk premium and the return volatility of the consumption claim decrease as the grid density increases and the left-most grid point becomes more negative.

Based on the observations of Panel A of Table X, I use the grid of $[-100 : 0.5 : -5) \cup [-5 : 0.01 : 0.5]$ for $s_t - \bar{s}$ when solving for Models I and III.

For the model and calibration in Bansal and Yaron (2004), Panel B of Table X reports the key summary statistics of the simulated asset pricing data based on the numerical solutions using different grids. For both state variables, $x_t$ and $x_{2t}$, the unconditional distributions are normal with standard deviations of $\sigma_x/\sqrt{1 - \phi_x^2}$ and $\sigma_{x2}/\sqrt{1 - \phi_{x2}^2}$, respectively. The grids span between $-3$ and 3 standard deviations, and so on. The simulation results are fairly close to those reported in Bansal and Yaron (2004) and Beeler and Campbell (2012), and are essentially identical for different grids. Based on these observations, I use the grid of $[-4 : 0.1 : 4]$ standard deviations for $x_t$ and $\omega_{x}^2$ when solving for Models II and III.

References


