Building on recent work of Masser concerning algebraic values of the Riemann zeta function, we prove two general results about the scarcity of algebraic points on the graphs of certain restrictions of certain analytic functions. For any of the graphs to which our results apply and any positive integer $d$, we show that there are at most $C(\log H)^{\frac{3}{2}+\varepsilon}$ algebraic points of degree at most $d$ and multiplicative height at most $H$ on that graph. In particular, we obtain this conclusion for any restriction of $\Gamma(z)$ or $\zeta(\pi z)$ to a compact disk, answering questions from Masser’s paper, the latter having been suggested by Pila. As in Masser’s original work, the constant $C$ may be effectively computed from certain data associated with the function in question.

1 Introduction

Recently, there have been several papers studying the distribution of rational points on transcendental subsets of Euclidean spaces. An important general result in this direction, due to Pila and Wilkie [19], establishes an $H^\varepsilon$ bound on the number of rational points of multiplicative height at most $H$ lying on the transcendental part of a set $X \subseteq \mathbb{R}^n$ definable in an o-minimal expansion of the real field. Pila [17] extends this result from rational points to algebraic points of bounded degree. Examples of functions definable...
in such structures include the restriction of the Euler gamma function to the interval \((0, \infty)\) and the restriction of the Riemann zeta function to the interval \((1, \infty)\) \([6,\text{ Corollaries 9.4 and 9.5}]\). The methods involved here go back to the Bombieri and Pila \([3]\) determinant method, which Pila \([14]\) used to prove that the above bound holds for the graph of the restriction of a real analytic function to a compact interval on which it is transcendental.

In a substantial improvement for the particular case of the Riemann zeta function, Masser \([12]\) recently proved the following result. (Note that \(H(z)\) denotes the absolute multiplicative height of an algebraic number \(z\), so \(H(z) = e^{h(z)}\), where \(h(z)\) is defined on \([24, \text{ p. 75}]\). We use \(H\) on its own to denote a number which acts as a bound for the height.)

**Theorem 1.1** (Masser). For any positive integer \(d\), there is an effective \(C > 0\) such that, for all \(H > e^e\), there are at most \(C\left(\frac{\log H}{\log \log H}\right)^2\) real \(z \in (2, 3)\) such that \([\mathbb{Q}(z, \zeta(z)) : \mathbb{Q}] \leq d\) and \(\max\{H(z), H(\zeta(z))\} \leq H\). \(\square\)

In fact, he can replace the interval \((2, 3)\) with any compact disk centered at 0, with \(C\) then depending effectively on the radius of this disk. Masser’s improvement consists not just in the quality of the bound but also in the effectiveness of the constant. This is certainly not known for the constant in the Pila–Wilkie result, applied to the structure \((\bar{\mathbb{R}}, \zeta|_{(1, \infty)})\) say. Similar results with a power of log bound have been proved, in \([4, 8, 18]\), for certain subsets of \(\mathbb{R}^n\), but generally with a much larger exponent.

In this paper, we answer questions from Masser’s paper \([12]\). We prove results similar to his but with the function \(\zeta\) replaced by \(\Gamma\), or, answering a question of Pila’s, \(\frac{\zeta(z)}{\pi^z}\), although we must settle for an exponent of \(3 + \varepsilon\) rather than 2. We do this by proving two general results which apply to several examples. For the Weierstrass zeta functions also suggested by Masser, Margaret Thomas and the second author have shown that the method of \([8, 16]\) is applicable, thanks to some work of Macintyre’s \([11]\), though it leads to an exponent quite a bit bigger than 3.

Our first result is the following. It is given in greater generality in Section 2, where it is also proved.

**Theorem 1.2.** Let \(r > 0\). Let \(f\) be analytic on the closed disk with center 0 and radius \(5r\). Assume that \(f(0)\) is a nonzero integer power of \(\pi\). For any positive integer \(d\), there is an effective \(C > 0\) such that, for all \(H > e^e\), there are at most \(C(\log H)^3 \log \log H\) complex numbers \(z\) such that \(|z| \leq r\), \([\mathbb{Q}(z, f(z)) : \mathbb{Q}] \leq d\) and \(\max\{H(z), H(f(z))\} \leq H\). \(\square\)
For the proof, we follow the general outline of Masser’s argument. Indeed we quote [12, Proposition 2] and some data from its proof which uses a generalization of Siegel’s lemma rather than the determinant method. Masser then uses his own zero estimate for the zeta function to finish his proof of Theorem 1.1. For the functions in our theorem, such a zero estimate, where the polynomial can have arbitrary complex coefficients, is not generally possible. This is discussed after Remark 2.8. However, we may use the fact that the polynomials produced by the generalized Siegel’s lemma have integer coefficients. We were unable to prove a zero estimate under this restriction on the polynomials, but we also know that the coefficients are not too large and this allows us to count zeroes using a standard result from complex analysis (Fact 2.4) and the transcendence measures which are known for nonzero integer powers of π. It is noteworthy that we are using not a transcendence assumption on f but rather the assumption that f(0) is transcendental and has a sufficiently good transcendence measure.

Theorem 2.7, the more general version of Theorem 1.2, is expressed in terms of a transcendence measure of a particular form. This is based on what is known for π. We could replace it with known transcendence measures for e or log α, for algebraic α, at the expense, perhaps, of a larger exponent. It is known that almost all (in the sense of Lebesgue measure) transcendental numbers have transcendence measures of a form that is better than the one we are using. So the transcendence measure assumption could be varied. However, it cannot be omitted completely. Examples due to Bombieri and Pila (see [15]) or to Surroca [21] show that the bound in Pila’s result [14] cannot be improved to that extent.

As an example of a function to which Theorem 1.2 applies we could take $f(z) = 6(z + 1)ζ(z + 2)$ with $f(0) = π^2$ and any choice of $r > 0$. Or we could take $f(z) = 6(z - 2)ζ(z - 1)\frac{ζ(z - 1)}{\pi}$ with $f(0) = π$ and again any choice of $r > 0$. So we recover a weakening of Masser’s result and also answer Pila’s question for $\frac{ζ(z)}{πz}$. In both of these cases, we can actually remove the log log H factor in the bound by, as Masser does in [12], making the radius large and using the growth of the functions involved.

For any appropriately restricted analytic function, Surroca has proved a $C (\log H)^2$ bound [21] but under the (necessary) proviso that the bound only applies at some infinite sequence of H.

Using $Γ\left(\frac{1}{2}\right) = \sqrt{π}$, we could also get a result for a restriction of the gamma function using the method of Theorem 1.2. But in fact we can omit the transcendence measure condition at the expense of making some growth assumptions and omitting from the count those rational points with second coordinate zero. A more general version of the following result is stated and proved in Section 3.
Theorem 1.3. Let $f$ be an entire function on $\mathbb{C}$. Let $r, a, b, s, t \in \mathbb{R}$ be such that $r, a, s, t > 0$ and $b > 1$. Suppose

1. $|f(z)| \leq s|z|^{t|z|}$ for all $|z| \geq r$
2. $|f(x)| \leq ab^{-x}$ for all $x \geq r$.

For any positive integer $d$, there is an effective $C > 0$ such that, for all $H > e^e$, there are at most $C (\log H)^3 (\log \log H)^3$ real $x > 0$ such that $f(x) \neq 0$, $[\mathbb{Q}(x, f(x)) : \mathbb{Q}] \leq d$ and $\max[H(x), H(f(x))] \leq H$. □

Note that there is no upper bound on $x$. We also do not assume directly that $f$ is transcendental, though it does follow from our assumptions that if this is not the case then $f$ is identically zero. For this theorem, we again use the generalized Siegel’s lemma, via [12, Proposition 2], and Fact 2.4, in this case relying on the growth conditions to find a suitable point at which to center a disk which is chosen in terms of $H$ rather than, as in Theorem 1.2, fixed throughout.

For examples of functions to which this theorem applies, we can take $f(z) = (z - 1)(\zeta(z) - 1)$, $f(z) = (z - 1)^{\zeta(z)}/z^2$, or $f(z) = 1/\Gamma(z)$. In fact, for the first and third of these examples, we can do better thanks to Masser’s zero estimate for the zeta function and a similar result recently obtained by Besson [2] for the gamma function. Indeed Besson obtains Theorem 1.1 with $\Gamma$ in place of $\zeta$ (including the sentence after Theorem 1.1), though with the very slightly weaker bound of $C (\log H)^3/\log \log H$. We give further examples in Section 4.

Masser points out that the expectation is that there are no rational $z \in (2, 3)$ such that $\zeta(z)$ is also rational. So his exponent 2 could be considered to be 2 away from optimal. Similarly, as is discussed in the second paragraph of Section 4, our exponent in Theorem 1.3 is at most $2 + \varepsilon$ off optimal. The function $f(z) = e^{-z} \sin(\pi z)$ shows that it is necessary, in Theorem 1.3, to consider only those $x$ for which $f(x) \neq 0$.

In addition to what is given before Theorem 1.1, we mention two further items of notation. For any polynomial $P$ with integer coefficients, we use $|P|$ to denote the maximum of the absolute values of these coefficients. We use $D(r)$ to denote the open disk in $\mathbb{C}$ with center 0 and radius $r$. Then $\overline{D(r)}$ is its closure.

2 Functions Having a Value with Good Transcendence Measure

In this section, we state and prove a more general version of Theorem 1.2. The first lemma is a routine application of the statement and proof of [12, Proposition 2].
Lemma 2.1. Let $r, \delta > 0$. Let $g$ be an analytic function on an open set containing $\overline{D(4r(1+\delta))}$. Let $d$ be a positive integer. There exists $C_1 > 0$ such that, for all $H > e^e$, there is a nonzero polynomial $P(X, Y)$ satisfying the following conditions:

(i) for all $z \in \overline{D(r)}$, if $[\mathbb{Q}(z, g(z)) : \mathbb{Q}] \leq d$ and $\max\{H(z), H(g(z))\} \leq H$, then $P(z, g(z)) = 0$,

(ii) $P(X, Y)$ has degree at most $T = C_1 \log H$ and integer coefficients with

$$|P| \leq 2^{\frac{3}{2}}(T + 1)^2 H^T.$$  

\[\square\]

Proof. Let $Z = 2r(1+\delta)$, $A = \frac{1}{2r}$, and $M = \max\{\max\{|z|, |g(z)|| : |z| \leq 2Z\}$. Let

$$C_1 = \max\left\{ \sqrt{8d}, e, \frac{d(144d + 16 \log (M + 1))}{\log (1 + \delta)} \right\}.$$  

Let $H > e^e$ and $T = C_1 \log H$. Let $\mathcal{Z}$ be the set of all $z \in \overline{D(r)}$ such that $[\mathbb{Q}(z, g(z)) : \mathbb{Q}] \leq d$ and $\max\{H(z), H(g(z))\} \leq H$. Then the inequalities

$$T \geq \sqrt{8d} \text{ and } (AZ)^T > (4T)^{\frac{\delta^2}{M}} (M + 1)^{16d} H^{48d^2}$$

are true.

It is shown in [12, proof of Proposition 2] that there then exists a nonzero polynomial $P(X, Y)$ satisfying condition (i) and having degree at most $T$. Assuming the cardinality of $\mathcal{Z}$ is at least $\frac{T^2}{8d^2}$, it is shown (see [12, line (12), p. 2044]) that $P$ may be chosen to satisfy the whole of condition (ii). If the cardinality, $S$ say, of $\mathcal{Z}$ is less than $\frac{T^2}{8d^2}$, then we can repeat the first part of the proof in [12]. Fix the points $z_1, \ldots, z_S$ in $\mathcal{Z}$ and apply [7, Lemme 1.1, p. 98], exactly as on [12, p. 2044]. The solvability condition is still fine, as now we have fewer equations. Similarly, the required bound on the coefficients ([12, line (12), p. 2044]) still holds. And now there are no more points, so we are done, and need not extrapolate as in the second part of the proof in [12]. Finally, if there are no points at all, then we can take $P = 1$.  

\[\square\]

Remark 2.2. Note that $C_1$ depends only on $d$, $M$, and $\delta$ and may be effectively computed from them. And $M$ depends on $g$, $r$, and $\delta$.  

We shall be using the notion of a transcendence measure. We recall the definition (as used in [23], though for convenience we insist on $S \geq e$).
**Definition 2.3.** Let $a \in \mathbb{C}$. A transcendence measure for $a$ is a real-valued function $\varphi(x, y)$ such that, for every $S \geq e$ and nonzero $Q(X) \in \mathbb{Z}[X]$ with degree at most $S$ and such that $|Q| \geq 16$, $\log |Q(a)| \geq -\varphi(S, \log |Q|)$.

The next lemma shows how, given a transcendence measure for $g(0)$, we can estimate the number of zeroes of polynomials in $z$ and $g(z)$ when $|z| \leq r$. In its proof, we make use of the following well-known fact. (It is an immediate consequence of [10, Theorem 1.1, p. 340]; see also [22, p. 171].)

**Fact 2.4.** Let $r' > r > 0$. Let $f$ be analytic on an open set containing $D(r')$, nonconstant on $D(r')$ and such that $f(0) \neq 0$. Let $M = \max\{|f(z)| : |z| \leq r'|$. Then there are at most

$$\left(\frac{1}{\log \frac{r'}{r}}\right) \left(\log \frac{M}{|f(0)|}\right)$$

complex numbers $z \in D(r)$ such that $f(z) = 0$.

**Lemma 2.5.** Let $r' > r > 0$. Let $g$ be analytic on an open set containing $D(r')$. Suppose that $g(0)$ is transcendental and $\varphi(x, y)$ is a transcendence measure for $g(0)$. Let $T \geq e$. Let $P(X, Y) \in \mathbb{Z}[X, Y]$ be nonzero with degree at most $T$ and such that $|P| \geq 16$. Assume that $P(0, Y)$ is not the zero polynomial in the variable $Y$. Let $M_p = \max\{|P(z, g(z))| : |z| \leq r'|$. There are at most

$$\frac{1}{\log \frac{r'}{r}} (\log M_p + \varphi(T, \log |P|))$$

complex numbers $z$ such that $|z| \leq r$ and $P(z, g(z)) = 0$.

**Proof.** We consider the function $P(z, g(z))$. This function is analytic on an open set containing $D(r')$. Also it is not identically zero on $D(r')$, since $P(0, g(0))$ is the result of evaluating a nonzero polynomial with rational coefficients at a transcendental value. By Fact 2.4, the number of $z \in \mathbb{C}$ such that $|z| \leq r$ and $P(z, g(z)) = 0$ is at most

$$\left(\frac{1}{\log \frac{r'}{r}}\right) \left(\log \frac{M_p}{|P(0, g(0))|}\right).$$

By assumption, we have

$$\log |P(0, g(0))| \geq -\varphi(T, \log |P|).$$
So the number of \( z \in \mathbb{C} \) such that \(|z| \leq r \) and \( P(z, g(z)) = 0 \) is at most
\[
\frac{1}{\log r} \left( \log M_p + \psi(T, \log |P|) \right).
\]

We recall an important example of a transcendence measure.

**Fact 2.6.** The function \( \varphi(x, y) = 240x(y + x \log x)(1 + \log x) \) is a transcendence measure for \( \pi \).

This is given by Waldschmidt [23] where he notes that it was already proved in [5] that there is a constant \( E > 0 \) such that \( Ex(y + x \log x)(1 + \log x) \) is a transcendence measure for \( \pi \). For the main result of this section, we consider transcendence measures of this form. It should be clear that one could obtain the same conclusion, or similar conclusions, by using different forms of transcendence measure.

**Theorem 2.7.** Let \( q \in \mathbb{Q} \) and \( r, \delta > 0 \). Let \( f \) be a function that is analytic on an open set containing the closed disk with center \( q \) and radius \( 4r(1 + \delta) \). Let \( d \) be a positive integer and \( E > 0 \). Suppose the function \( \varphi(x, y) = Ex(y + x \log x)(1 + \log x) \) is a transcendence measure for \( f(q) \). There exists \( C > 0 \) such that, for all \( H > e^e \), there are at most \( C (\log H)^{\frac{3}{2}} \log \log H \) complex numbers \( z \) such that \(|z - q| \leq r\), \([\mathbb{Q}(z, f(z)) : \mathbb{Q}] \leq d \) and \( \max\{H(z), H(f(z))\} \leq H \).

**Proof.** Let \( g(z) = f(z + q) \) whenever \( f(z + q) \) is defined. Then \( g \) is as in Lemma 2.1. Let \( C_1 \) be as in Lemma 2.1 and \( H > e^e \). We may assume \( C_1 \geq 1 \). Let \( P(X, Y) \) be as in Lemma 2.1. Let \( r' = 2r \) and \( T = C_1 \log H \). Let \( M_p \) be as in Lemma 2.5. We have \( T \geq e \) and we may assume \(|P| \geq 16 \) (since otherwise we could use \( 16P \) instead which would satisfy the conditions given in Lemma 2.1 on account of the size of \( H \) and \( C_1 \)). It follows, by Lemma 2.5, that there are at most
\[
\frac{1}{\log 2} \left( \log M_p + ET(\log |P| + T \log T)(1 + \log T) \right)
\]
complex numbers \( z \) such that \(|z| \leq r\), \([\mathbb{Q}(z, f(z)) : \mathbb{Q}] \leq d \) and \( \max\{H(z), H(f(z))\} \leq H \).

Let \( M \) be as in Lemma 2.1. Let \( r_1 = \max\{r, 1\} \) and \( M_1 = \max\{M, 1\} \). We have
\[
M_p \leq (T + 1)^2 |P|(2r_1)^TM_1^T
\]
and so
\[
\log M_p \leq T(2 + \log |P| + \log(2r_1) + \log M_1).
\]
Therefore,

\[
\frac{1}{\log 2} (\log M_p + ET(\log |P| + T \log T)(1 + \log T)) \leq \frac{T \log T}{\log 2} ((1 + 2E) \log |P| + 2 + \log (2r_t) + \log M_t + 2ET \log T).
\]

We have

\[
|P| \leq 2^{\frac{1}{3}} (2C_1 \log H)^2 H^{C_1 \log H}
\]

and so

\[
\log |P| \leq 6C_1 (\log H)^2.
\]

We have

\[
T = C_1 \log H, C_1 \geq e \quad \text{and} \quad \log H \geq e
\]

and so

\[
\log T \leq 4(\log C_1) \log \log H.
\]

Therefore,

\[
\frac{T \log T}{\log 2} ((1 + 2E) \log |P| + 2 + \log (2r_t) + \log M_t + 2ET \log T) \leq \frac{C_2^3(4C_1 \log C_1)((1 + 2E)6C_1 + 2 + \log (2r_t) + \log M_t + 2E)}{\log 2} (\log H)^3 \log \log H.
\]

Let

\[
C_2 = \frac{C_2^3(4C_1 \log C_1)((1 + 2E)6C_1 + 2 + \log (2r_t) + \log M_t + 2E)}{\log 2}.
\]

Then there are at most \( C_2 (\log H)^3 \log \log H \) complex numbers \( z \in D(r) \) such that \([Q(z, g(z)) : \mathbb{Q}] \leq d\) and \( \max\{H(z), H(g(z))\} \leq H \).

For all algebraic \( z \in \mathbb{C} \), we have

\[
H(z - q) \leq 2H(-q)H(z) = 2H(q)H(z)
\]

by Property 3.3 on [24, p. 75]. So there are at most \( C_2 (\log (2H(q)H))^3 \log \log (2H(q)H) \) complex numbers \( z \) such that \( |z - q| \leq r \), \([Q(z, f(z)) : \mathbb{Q}] \leq d\) and \( \max\{H(z), H(f(z))\} \leq H \).
We have \( \log(3H(q)) \geq 1 \) and \( \log H \geq e \). Therefore,

\[
(\log(2H(q)H))^3 \leq (2\log(3H(q)))^3(\log H)^3.
\]

Also

\[
\log(2H(q)H) \leq 4\log(3\log(3H(q)))\log\log H.
\]

Let \( C = C_2(2\log(3H(q)))^34\log(3\log(3H(q))) \). Then there are at most \( C(\log H)^3 \log\log H \) complex numbers \( z \) such that \( |z - q| \leq r \), \( [\mathbb{Q}(z, f(z)) : \mathbb{Q}] \leq d \) and \( \max\{|H(z), H(f(z))| \leq H \}. \) Since \( C \) clearly does not depend on \( H \), this completes the proof. ■

**Remark 2.8.** The constant \( C \) depends only on \( d, \delta, r, M, E, \) and \( H(q) \) and may be computed effectively from them. □

One might hope that it is possible to obtain a zero estimate like [12, Proposition 1] for any function satisfying the conditions of Theorem 2.7, allowing us to follow Masser’s argument more closely in this general setting. However, this is not the case.

Let \( N_k = 2^{2^k} \) and \( f(z) = \pi + \sum_{k \geq 1} z^{N_k} \). Then \( f \) defines an analytic function on \( |z| < 1 \) with \( f(0) = \pi \). We first show that \( f \) is transcendental by showing that it has the unit circle as a natural boundary. So, fix a positive integer \( M \) and an \( N_M \)th root of unity, \( \zeta \) say. Then for \( r \in (0, 1) \) we have

\[
f'(r\zeta) = \sum_{k=0}^{M-1} N_k (r\zeta)^{N_k-1} + \frac{1}{\zeta} \sum_{k \geq M} N_k r^{N_k-1}.
\]

So \( |f'(r\zeta)| \to \infty \) as \( r \to 1 \). As \( M \) varies, these roots of unity are dense in the unit circle, and so \( f \) is indeed transcendental over \( \mathbb{C}(z) \). Now suppose that \( c \) and \( \kappa \) are such that for any nonzero polynomial \( P \in \mathbb{Z}[X, Y] \) of degree at most \( T \) the function \( P(z, f(z)) \) has at most \( cT^\kappa \) zeroes, counted with multiplicity. Consider \( P(X, Y) = Y - \pi - \sum_{k=1}^{K-1} X^{N_k} \). Then \( P(z, f(z)) \) has a zero of order at least \( N_K \) at 0. But \( \frac{N_k}{N_{k+1}} \to \infty \) as \( K \to \infty \), and this contradicts the zero estimate. So \( f \) does not satisfy any such zero estimate.

It follows easily from Fact 2.6 that any non-zero integer power of \( \pi \) has a transcendence measure of the form considered in Theorem 2.7. Therefore, Theorem 2.7 (together with Remark 2.8) implies Theorem 1.2, with \( \delta = \frac{1}{4} \) and \( q = 0 \).

### 3 Functions Satisfying Certain Growth Conditions

In this section, we state and prove a more general version of Theorem 1.3. The first lemma ensures that the input values of interest to us will not be too large.
Lemma 3.1. Let \( f \) be an entire function on \( \mathbb{C} \). Let \( r \geq 0 \). Suppose \( \theta, \phi, a, b \in \mathbb{R} \) are such that \( \theta \leq \phi \), \( a > 0 \), \( b > 1 \) and \( |f(z)| \leq ab^{-|z|} \) for all \( z \in \mathbb{C} \) such that \( \theta \leq \arg z \leq \phi \) and \( |z| \geq r \).

Let \( d \) be a positive integer. Let \( C_1 \geq \max\{r, \frac{|\log a + d\log b|}{\log b}\} \). For all \( H > e \), if \( z \in \mathbb{C} \) is such that \([Q(z, f(z)) : \mathbb{Q}] \leq d\), \( \max\{\arg H(z), H(f(z))\} \leq H \), \( \theta \leq \arg z \leq \phi \) and \( f(z) \neq 0 \), then \( |z| \leq C_1 \log H \).  

Proof. Let \( H > e \). Let \( z \in \mathbb{C} \) be such that \([Q(z, f(z)) : \mathbb{Q}] \leq d\), \( \max\{\arg H(z), H(f(z))\} \leq H \), \( \theta \leq \arg z \leq \phi \) and \( f(z) \neq 0 \). The result is immediate if \( |z| < r \). Assume \( |z| \geq r \). We have \( H(f(z)) \leq H \) and so \( |f(z)| \geq H^{-d} \), by (3.13) on [24, p. 82]. Then \( ab^{-|z|} \geq H^{-d} \) and so \( |z| \leq \frac{\log a + d\log H}{\log b} \leq C_1 \log H \).  

The following is a variation on the theme of Lemma 2.1 and so again uses the statement and proof of Proposition 2 in [12].

Lemma 3.2. Let \( f \) be an entire function on \( \mathbb{C} \). Let \( r \geq 0 \). Suppose \( s, t > 0 \) are such that \( |f(z)| \leq s|z|^{|z|} \) for all \( z \in \mathbb{C} \) with \( |z| \geq r \).

For all \( C_1 \geq e \), there exists \( C_2 > 0 \) such that, for all \( H > e^n \), there is a nonzero polynomial \( P(X, Y) \) satisfying the following conditions:

(i) for all \( z \in \mathbb{C} \), if \( |z| \leq C_1 \log H \), \([Q(z, f(z)) : \mathbb{Q}] \leq d\) and \( \max\{\arg H(z), H(f(z))\} \leq H \), then \( P(z, f(z)) = 0 \),

(ii) \( P(X, Y) \) has degree at most \( T = C_2(\log H)\log \log H \) and integer coefficients with

\[
|P| \leq 2^{\frac{3}{2}}(T + 1)^2H^T. 
\]

Proof. Let \( C_1 \geq e \), \( M_r = \max\{|z|, |f(z)| : |z| \leq r\} \), \( s_1 = \max\{s, 1\} \), and \( t_1 = \max\{t, 1\} \). Let \( H > e^n \), \( Z = 2eC_1 \log H \) and \( A = \frac{1}{ZC_1 \log H} \). Let \( Z \) be the set of all \( z \in D(C_1 \log H) \) such that \([Q(z, f(z)) : \mathbb{Q}] \leq d\) and \( \max\{\arg H(z), H(f(z))\} \leq H \). Let \( M = \max\{|z|, |f(z)| : |z| \leq 2Z\} \).

We have

\[
M \leq \max\{4eC_1 \log H, M_r, s(4eC_1 \log H)^{4t_1C_1 \log H}\}. 
\]

It follows that \( 16d\log(M + 1) \leq (C_2 - 144d^2) \log H \log \log H \) for some positive \( C_2 \) that does not depend on \( H \). Let \( T = C_2(\log H)\log \log H \). It follows that the inequalities

\[
T \geq \sqrt{8d} \quad \text{and} \quad (AZ)^{4eC_2} > (4T)^{\frac{32d^2}{e}}(M + 1)^{16d}H^{8d^2
\]

are true.

We conclude as in Lemma 2.1.
Theorem 3.3. Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function. Suppose $r, a, b, s, t, \theta, \phi \in \mathbb{R}$ are such that $a, s, t > 0$, $r \geq 0$, $b > 1$, $\theta \leq 0 \leq \phi$ and the following two conditions are satisfied:

1. $|f(z)| \leq s|z|^d$ for all $z \in \mathbb{C}$ with $|z| \geq r$,
2. $|f(z)| \leq ab^{-|z|}$ for all $z \in \mathbb{C}$ such that $\theta \leq \arg z \leq \phi$ and $|z| \geq r$.

Let $d$ be a positive integer. There exists $C > 0$ such that, for all $H > e^e$, there are at most $C (\log H)^2 (\log \log H)^3$ complex numbers $z$ such that the following conditions are satisfied:

3. $\theta \leq \arg z \leq \phi$,
4. $f(z) \neq 0$,
5. $[\mathbb{Q}(z, f(z)) : \mathbb{Q}] \leq d$,
6. $\max(H(z), H(f(z))) \leq H$. \hfill $\Box$

Proof. Let $C_1$ and $C_2$ be as in Lemmas 3.1 and 3.2 and such that $C_1 \geq e$. Let $H > e^e$. It follows that there is a nonzero polynomial $P(X, Y)$ satisfying the following conditions:

(i) for all $z \in \mathbb{C}$, if $z$ satisfies (3)–(6) then $P(z, f(z)) = 0$,
(ii) $P(X, Y)$ has degree at most $T = C_2 (\log H) \log \log H$ and integer coefficients with

$$|P| \leq 2^3 (T + 1)^2 H^T.$$ 

Let $R(X) = P(X, 0)$. Let $Q(X, Y) = P(X, Y) - R(X)$. If $P(X, Y) = R(X)$, then we can choose $C = C_2$, since the equation $R(X) = 0$ has at most $C_2 (\log H) \log \log H$ solutions. We assume $P(X, Y) \neq R(X)$ and so $Q(X, Y)$ is nonzero. Since we are not considering $z$ for which $f(z) = 0$, we may assume that $Y$ is not a factor of every term of $P(X, Y)$ and therefore that $R(X)$ is nonzero.

We now set about obtaining some $x_0$ on the positive real axis with the property that $|Q(x_0, f(x_0))| \leq \frac{1}{2}$ and $|R(x_0)| \geq 1$. Let $K = \max\{1, r, \frac{\log a}{\log b}\}$ and $x > 0$. Then $|f(x)| \leq ab^{-x} \leq 1$ provided $x \geq K$. Since $Y$ divides $Q(X, Y)$, we then have $|Q(x, f(x))| \leq \frac{1}{2}$ provided both $(T + 1)^2 |P| a b^{-x} x^T \leq \frac{1}{2}$ and $x \geq K$. We have

$$(T + 1)^2 |P| a b^{-x} x^T \leq \frac{1}{2}$$

if and only if

$$x \log b \geq \log(2(T + 1)^2 |P|a) + T \log x,$$
which holds provided
\[ x \log b \geq \max\{2 \log(2(T + 1)^2|P|a), 2T \log x\}. \]

It follows that
\[ |Q(x, f(x))| \leq \frac{1}{2} \]
provided
\[ x \geq \max \left\{ \frac{2 \log(2(T + 1)^2|P|a)}{\log b}, e, \left( \frac{2T}{\log b} \right)^{\frac{3}{2}}, K \right\}. \]

We investigate this condition further. In what follows, \( C \) is a positive constant (not depending on \( H \)) that may be different at each appearance.

We have
\[ \log |P| \leq C (\log H)^2 \log \log H \]
and
\[ \log T \leq C \log \log H. \]

Therefore,
\[ \frac{2 \log(2(T + 1)^2|P|a)}{\log b} \leq C (\log H)^2 \log \log H. \]

We also have
\[ \left( \frac{2T}{\log b} \right)^{\frac{3}{2}} \leq C (\log H)^2 \log \log H. \]

So \( |Q(x, f(x))| \leq \frac{1}{2} \) provided \( x \geq C (\log H)^2 \log \log H \).

We have \( C (\log H)^2 \log \log H \geq T + 1 \). It follows that there is an integer in the interval
\[ [C (\log H)^2 \log \log H, 2C (\log H)^2 \log \log H] \]
at which \( R(X) \) does not vanish. Choose \( x_0 \) to be such an integer. Then \( |Q(x_0, f(x_0))| \leq \frac{1}{2} \) and \( |R(x_0)| \geq 1 \) as desired. It follows that \( |P(x_0, f(x_0))| \geq \frac{1}{2} \).

We have \( C_1 (\log H) \leq C (\log H)^2 \log \log H \) and so, by Lemma 3.1, for all \( z \in \mathbb{C} \), if \( z \) satisfies conditions (3)--(6), then \( z \) lies in the closed disk with center \( x_0 \) and radius \( 2x_0 \).
On the closed disk with center $x_0$ and radius $4x_0$ we have

$$|P(z, f(z))| \leq (T + 1)^2|P| |Cx_0|^{C_0T}.$$  

It follows, by Fact 2.4, that the set of all $z \in \mathbb{C}$ which satisfy conditions (3)–(6) has cardinality at most $\log|P| + C T x_0 \log x_0$. This is at most $C (\log H)^3 (\log \log H)^3$, as required. 

\[\square\]

Remark 3.4. The constant $C$ depends only on $r, a, b, s, t$, and $d$ and may be computed effectively from them. 

The log log $H$ factor may be removed in the statement of Theorem 2.7 if one additionally assumes that $f$ is entire and satisfies condition (1) of Theorem 3.3. The argument involves a further variant of Lemma 2.1. In fact, it is the variant of Lemma 2.1 which Masser uses in [12] in which it is possible to take $T = C_1 \frac{\log H}{\log \log H}$ (for an effective $C_1 > 0$ which is possibly different from the $C_1$ we obtain in Lemma 2.1). The relevant argument may be found in [12, Section 4].  

Similarly, the $(\log \log H)^3$ factor may be removed in the statement of Theorem 3.3 if we restrict our attention to algebraic points on the graph of the restriction of $f$ to $D(R)$ for some $R > 0$. One uses the argument from [12] mentioned above to get $T = C_2 \frac{\log H}{\log \log H}$ (for a possibly different effective $C_2 > 0$) and then runs through the proof of Theorem 3.3 with this new value.

4 Examples

In this section, we apply our theorems to some examples. Let $f_1(z) = (z - 1)\zeta(z)$, where $\zeta$ is the Riemann zeta function. We know that $f_1(2) = \frac{\pi^2}{6}$. When $x \geq e$ and $y \geq 0$, we have

$$2^{40}(2x)((y + x \log 6) + 2x \log (2x))(1 + \log (2x))$$

$$+ x \log 6 \leq 2^{44}(1 + \log 6)x(y + x \log x)(1 + \log x).$$

It is then clear from Fact 2.6 that the function

$$\varphi_1(x, y) = 2^{44}(1 + \log 6)x(y + x \log x)(1 + \log x)$$

is a transcendence measure for $\frac{\pi^2}{6}$ (see Definition 2.3). Since $f_1(z)$ is entire, we can apply Theorem 2.7 to the restriction of $f_1$ to any compact disk centered at 2. Although an
interesting application of Theorem 2.7, this is just a weaker version of what Masser obtains in [12]. Specifically, he gets \( C(\log H)^2 \) where we have \( C(\log H)^{3+\varepsilon} \) (though we could remove the \(+\varepsilon\) in the light of our comments at the end of Section 3 and what is known about the growth of \( \zeta \)). He does this by proving a zero estimate for \( \zeta \) [12, Proposition 1].

Masser reports that Pila suggested replacing \( \zeta \) with \( \zeta(\pi) \). Unlike \( \zeta \), this function is known to have rational points with positive input value. Indeed, the heights of the rational values \( \frac{\zeta(2n)}{\pi^n} \), as \( n \) runs through the positive integers, are known to grow slowly enough so that a bound of \( C(\log H)^{1-\delta} \) is not possible for the number of rational points of height at most \( H \) on the graph of \( \frac{\zeta(\pi)}{\pi^2} \) restricted to \([2, \infty)\), for any \( \delta > 0 \). We thank Masser for alerting us to this fact which implies that the bound in our Theorem 3.3 could not be lowered to \( C(\log H)^{1-\delta} \).

Masser's zero estimate is specific to \( \zeta \). We do not know whether such a result holds also for \( \frac{\zeta(\pi)}{\pi^2} \). However, our technique works just as well for \( \frac{\zeta(\pi)}{\pi^2} \) as for \( \zeta \). Let \( f_2(z) = \frac{(x-1)\zeta(x)}{\pi^x} \). We know that \( \zeta(-1) = -\frac{1}{12} \). Therefore, \( f_2(-1) = \pi \frac{240}{6} \). It follows from Fact 2.6 that the function

\[
\phi_2(x, y) = 2^{40}(1 + \log 6)x(y + x\log x)(1 + \log x)
\]

is a transcendence measure for \( \pi \frac{240}{6} \). Since \( f_2 \) is entire, Theorem 2.7 applies to any restriction of \( f_2 \) to a disk centered at \(-1\). This answers Pila's question with a bound which is only a little worse than Masser's original one for \( \zeta \). One can calculate a bound for the constant using the specific formulas given in Section 2. We find that there are at most \( 6 \cdot 10^{37}(\log H)^3 \log \log H \) rational points of multiplicative height at most \( H \) on the graph of the restriction of \( f_2 \) to the disk of radius 4 centered at \(-1\). This calculation uses the fact that \( R^{Sl} \) is a bound for \(|(z-1)\zeta(z)|\) when \(|z| \leq R \). For this, we refer the reader to Proposition 1.4 on [1, p. 6]. We also note that, in [1], Besson estimates the constant from [12].

Both \( \zeta \) and \( \frac{\zeta(\pi)}{\pi^2} \) give rise to examples of Theorem 3.3 too. Let \( f_1^\ast(z) = (z-1)(\zeta(z) - 1) \). Fix some \( \theta \in (-\frac{\pi}{2}, 0] \) and \( \phi \in [0, \frac{\pi}{2}) \). Then there are known to exist effective \( r, a, s, t > 0 \) and \( b > 1 \) such that the assumptions of Theorem 3.3 hold for both \( f_1^\ast \) and \( f_2 \) (condition (2) follows easily from the formula \( \zeta(z) = \sum \frac{1}{n^z} \) for \( Re(z) > 1 \)). Therefore, we obtain the conclusion of Theorem 3.3 for \( f_1^\ast \) and \( f_2 \) and so also for \( \zeta \) and \( \frac{\zeta(\pi)}{\pi^2} \). In the case of \( \zeta \) one could, in fact, use Masser's zero estimate to get a bound of \( C(\log H)^2(\log \log H)^2 \) and for that one does not have to exclude the points where \( f_1^\ast(z) = 0 \). For any \( q \in \mathbb{Q} \), \( f_1^\ast(z - q) \) and \( f_2(z - q) \) are clearly also functions to which Theorem 3.3 applies (with the same values of \( \theta \) and \( \phi \) as before). For \( f_2(z + 1) \), the
restriction \( f_2(z + 1) \neq 0 \) is not significant since \( \zeta(z) \) is known to have no zeroes with real part \( > 1 \) and \( f_2(1) = \frac{1}{\pi} \).

The function \( \Gamma \) is also asked about in [12]. Let \( f_3(z) = \frac{1}{\Gamma(z)} \). Again fix \( \theta \in (-\frac{\pi}{2}, 0] \) and \( \phi \in [0, \frac{\pi}{2}) \). There are known to exist effective \( r, a, s, t > 0 \) and \( b > 1 \) such that \( f_3 \) satisfies the assumptions of Theorem 3.3 (see, e.g., the material on Binet’s result on [25, p. 249] and use the formula \( \Gamma(z + 1) = z\Gamma(z) \) to cope with values whose real part is not positive). Therefore, we obtain the conclusion of Theorem 3.3 for \( f_3 \) and so also for \( \Gamma \).

Since \( f_3 \) has no zeroes in the right half plane, we are not omitting any algebraic points. (It also follows, using the observations at the end of Section 3, that, for any restriction of \( \Gamma \) to a compact disk in the right half plane and for any positive integer \( d \), there exists an effective \( C > 0 \) such that, for all \( H > e^d \), there are at most \( C (\log H)^3 \) complex numbers \( z \) in this disk such that \([\mathbb{Q}(z, \zeta(z)) : \mathbb{Q}] \leq d \) and \( \max|H(z), H(\zeta(z))| \leq H \).) Let \( q \in \mathbb{Q} \). One may replace \( f_3(z) \) with \( f_3(z - q) \) in this argument, but \( f_3(z - q) \) might have zeroes in the right half plane. However, it will only have finitely many and an effective bound is known. Therefore, the assumption that our disk lie in the right half-plane is unnecessary.

By proving a zero estimate for \( \Gamma \) similar (though not identical) to Masser’s one for \( \zeta \), Besson [2] obtains, for any \( R > 0 \), a bound of the form \( C (\log H)^2 \) for the restriction of \( \Gamma \) to \( \overline{D(R)} \) (with effective \( C > 0 \)). Analogous to the case of \( \zeta \), we could use his zero estimate to improve the bound in Theorem 3.3 to \( C (\log H)^2 (\log \log H)^3 \) in the case of \( f_3 \), and therefore also \( \Gamma \), when \( \theta \) and \( \phi \) are as above. Without knowing Besson’s zero estimate, we could at least remove the \( (\log \log H)^3 \) factor in this case by exploiting the extra speed with which \( f_3(z) \) approaches 0 when \( \theta \leq \arg z \leq \phi \) and \( |z| \to \infty \).

Dedekind zeta functions give further examples of functions to which Theorem 3.3 can be applied. Recall that for a number field \( K \) the zeta function is defined by

\[
\zeta_K(z) = \sum_I \frac{1}{N(I)^z}
\]

for \( z \) with real part \( > 1 \), where the sum is over ideals \( I \) in the integers of \( K \) and \( N(I) \) is the norm of \( I \). These functions have meromorphic continuations with simple poles at \( z = 1 \) and so the functions \( f_K(z) = (z - 1)(\zeta_K(z) - 1) \) are entire. Let \( n \) be the degree of \( K \) over \( \mathbb{Q} \). By well-known properties of \( \zeta_K \) [13, Corollary 3, p. 326], \( |\zeta_K(z)| \leq (\zeta(\Re(z)))^n \) for \( \Re(z) > 1 \), with the Riemann zeta function on the right-hand side. Fix some \( \theta \in (-\frac{\pi}{2}, 0] \) and \( \phi \in [0, \frac{\pi}{2}) \). It follows that there exist \( a, r > 0 \) and \( b > 1 \) such that \( f_K \) satisfies condition (2) of Theorem 3.3. By [13, Theorem 7.1, p. 326–327] the function

\[
\Psi(z) = z(z - 1)A^2 \Gamma(z/2)^r \Gamma(z)^s \zeta_K(z)
\]
is such that $\Psi(z) = \Psi(1 - z)$, where $r_1$ and $2r_2$ are the number of real and complex embeddings of $K$, respectively, and $A$ is a certain constant. Using standard estimates for $\Gamma$, we have that there exist $r, s, t > 0$ such that the growth condition (1) of Theorem 3.3 holds for $\Psi$ on the half plane with real part greater than 1. By the functional equation this growth condition holds on the half plane with real part less than 0. And by the same theorem in [13], this function is bounded on the missing vertical strip. So there are $r, s, t > 0$ such that condition (1) holds for $\Psi$. Then, using the estimates for $1/\Gamma$, there are $r, s, t > 0$ such that condition (1) holds for $(z - 1)\zeta_K(z)$ and so for $f_K$. So Theorem 3.3 applies to $f_K$ (with $\theta$ and $\phi$ as above). Using these functions, we can also find further examples taking infinitely many rational values. For Klingen [9] and Siegel [20] proved that if $K$ is totally real then, for each positive integer $m$, the numbers

$$\frac{\zeta_K(2m)\sqrt{D}}{\pi^{2nm}}$$

are rational, where $D$ is the discriminant of $K$.

Theorem 3.3 is quite general, requiring only an entire function and the satisfaction of certain growth conditions. There are several interesting examples and then any product of examples and any linear combination of examples are again examples. However, the growth conditions are not always met and, perhaps surprisingly, in many cases of interest one has a rational $q$ such that $f(q)$ has a suitable transcendence measure, allowing one to use Theorem 2.7 or a variant thereof obtained by slightly changing the form of transcendence measure which one will accept.

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