AN INTERPOLATION SERIES ASSOCIATED WITH THE BESSEL–HANKEL TRANSFORM

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1. Introduction

The orthogonal functions

\[ w_n(x) = \frac{\sin \pi(x-n)}{\pi(x-n)}, \quad n = 0, \pm 1, \ldots, \quad x \in (-\infty, \infty), \]

of Whittaker's "cardinal" series \( \sum_{n} a_n w_n(x) \) possess several well known properties [5]; they form a complete orthonormal set in the Hilbert function space known as the Paley–Wiener functions, and \( w_n(m) = \delta_{nm} \) (Kronecker's Symbol) for all integers \( n \) and \( m \). This means that the cardinal series is not only an orthogonal expansion for the Paley–Wiener functions, but it also provides a process for interpolation at the integers since the series reduces formally to \( a_m \) when \( x \) is an integer \( m \). In the present note we shall consider further sets of this type and in particular a set involving Bessel functions.

Cardinal series interpolation has important applications in information theory, where it was introduced by C. E. Shannon [11]. It is, for example, important for the electrical engineer to know that a certain type of transmitted signal, a function of time, lies in a subspace (the Paley–Wiener functions) of \( L^2(-\infty, \infty) \), and that this subspace possesses an orthogonal basis with respect to which the "coordinates" of the signal are actually values taken by the signal at certain instants of time. It was with this application in mind that H. P. Kramer introduced a generalisation of the cardinal series in a lemma which we adopt as the starting point for the present discussion.

LEMMA 1 (Kramer [7]). Let \( (a, b) \) be a finite interval of \( \mathbb{R} \) (the real numbers). Let \( K(x, t) \in L^2(a, b) \) for each \( x \in \mathbb{R} \) and suppose that the sequence of real numbers \( \{x_n\} \) (where \( n \) runs over some indexing set of integers) is such that \( \{K(x_n, t)\} \) forms a complete orthogonal set (COS) in \( L^2(a, b) \). If

\[ f(x) = K u = \int_{a}^{b} K(x, t) u(t) \, dt, \quad (1) \]

for some \( u \in L^2(a, b) \), then

\[ f(x) = \lim_{N \to \infty} \sum_{|n| \leq N} f(x_n) s_n(x), \quad (2) \]

pointwise on \( \mathbb{R} \), where

\[ s_n(x) = \frac{1}{b} \int_{a}^{b} K(x, t) \overline{K(x_n, t)} \, dt / \int_{a}^{b} |K(x_n, t)|^2 \, dt. \]

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If \( K(x, t) \) is taken to be the Fourier kernel \((2\pi)^{-\frac{1}{2}} e^{-ixt}\) on \([-\pi, \pi]\), and \( x_n \) to be \( n \), \( n = 0, \pm 1, \ldots \), then \( s_n \) becomes \( w_n \) and (2) is the cardinal series. Before looking at the specific case where \( K(x, t) \) is taken to be the Bessel-Hankel kernel \((xt)^\nu J_\nu(xt)\) on \([0, 1]\), some preliminary results will be obtained in which certain general boundedness assumptions are made on \( K \), the integral transform (1), under which the range of \( K \) is a Hilbert space with reproducing kernel (to be defined). This is the space \( \mathcal{G} \) of Paley-Wiener functions in the Fourier kernel case. Since both the Fourier and the Bessel-Hankel operators are unitary, we are interested in the case where \( K \) is taken to be an isometry, e.g., Lemma 3 (ii), and the subsequent remarks. Some well known properties of the space \( \mathcal{G} \) are immediate consequences of the reproducing kernel theory. For example, Lemma 3 (i) may be compared with [5; p. 339]; further, the elements of \( \mathcal{G} \) satisfy Bateman's integral equation [5; p. 340] which is, in fact, just the reproducing equation in \( \mathcal{G} \).

Our main results are contained in §3 where Bessel-Hankel transforms replace the Fourier transform treated in [5]. In Theorem 1 interpolation formulae (cf. [7; p. 72]) for the Hilbert space analogous to \( \mathcal{G} \) are given and in Theorem 2 a comparison of these two spaces is made. As an application of Theorem 1 we deduce certain expansion formulae for Bessel functions.

Before proceeding, we remark that one may obtain Riesz-Fischer and Parseval theorems of \( L^2 \) type which include Lemma 1 by making routine generalisations of theorems known for the Fourier case [4, 13]. Also, results such as Lemmas 1 and 4 are easily modified to give similar results in terms of biorthogonal sets.

In what follows \(<,>\) will denote inner product; a subscript is attached where some range of integration is to be stressed.

### 2. Preliminaries

Let \( X \) denote a subset of \( \mathbb{R} \) which contains \((a, b)\) and is of finite or infinite Lebesgue measure. Let \( K(x, t) \) be as in Lemma 1, and further assume that it is defined and measurable on \( X \times (a, b) \). Put

\[
Ku = \int_a^b K(x, t) u(t) \, dt, \quad u \in L^2(a, b).
\]

\( K \) is one-to-one, for if \( Ku = 0 \), then \( u \) is orthogonal to every \( K(x_n, t) \) and hence \( u \) is null. Let \( \mathcal{N} \) denote the range of \( K \); then our basic assumption concerning \( K \) is that \( \mathcal{N} \subset L^2(X) \). For this it is necessary and sufficient [1; p. 237] that there exist an absolute constant \( C \) such that

\[
\left\| \int_a^b K(x, t) u(t) dt \right\| v(x) dx \leq C \| u \| \| v \|,
\]

for any \( u \in L^2(a, b) \), \( v \in L^2(X) \), the integrals to be taken in the given order. It follows at once that \( K \) is bounded.

**Definition.** Let \( \mathcal{H} \) be a class of complex-valued functions defined on a set \( X \) which forms a Hilbert space under the norm of \( L^2(X) \); \( g(s, x) \) is said to be a reproducing kernel (r.k.) for \( \mathcal{H} \) if
(i) \( g(., x) \in \mathcal{H} \quad \forall x \in X, \)

(ii) \( f(x) = \langle f, g(., x) \rangle \quad \forall f \in \mathcal{H}, x \in X \) (the reproducing equation).

**Properties.** We shall require the following properties of spaces with r.k. (see, e.g. [9]).

(a) \( \mathcal{H} \) has r.k. if and only if the evaluation functional \( l_x(f) = f(x) \) is bounded for every \( x \in X \).

(b) If \( g \) exists for \( \mathcal{H} \), it is unique.

(c) \( g(s, x) = \langle g(., s), g(., x) \rangle \); in particular \( g(x, x) = \|g(., x)\|^2 \).

(d) If the sequence \( \{f_n\} \) converges strongly to \( f \) in \( \mathcal{H} \) with r.k. \( g \), then \( \{f_n\} \) converges pointwise on \( X \) to \( f \). The convergence is uniform over any subset of \( X \) for which \( g(x, x) \) is bounded.

(e) Let \( \{\phi_n\} \) be any complete orthonormal set (CONS) in \( \mathcal{H} \). Then

\[
g(., x) = \sum \phi_n(.) \bar{\phi}_n(x)
\]

strongly, \( \forall x \in X \).

**Remark.** A sufficient condition for \( \mathcal{H} \) to have r.k. is that all functions of \( \mathcal{H} \) be continuous. This is proved by considering the linear functionals \( T_f = \langle f, h_x \rangle \) where \( \int h_x = 1 \) and \( h_x \) has support on \( |x-s| < \varepsilon \); it follows from the Banach–Steinhaus principle and property (a) that \( \mathcal{H} \) has r.k. On the other hand, O. Lehto [8] has given an example of a space with r.k. containing discontinuous functions.

Let \( s_n \) be as in Lemma 1. From the boundedness of \( K \) it follows that every function in \( \mathcal{N} \) has a norm-convergent expansion in the \( s_n \). If \( K \) is also an isometry then \( \{s_n\} \) is a COS in \( \mathcal{N} \) and clearly \( K^{-1} \) is a bounded transformation of \( \mathcal{N} \) onto \( L^2(a, b) \).

**Lemma 2.** If \( K^{-1} \) is bounded, then \( \mathcal{N} \) is a Hilbert space with r.k.

**Proof.** Since \( K^{-1} \) is bounded, \( \mathcal{N} \) is a closed and hence complete subspace of \( L^2(X) \). For every \( f \in \mathcal{N} \) we have

\[
f = \int_a^b K(x, t) u(t) dt, \quad u \in L^2(a, b).
\]

Upon writing \( u = K^{-1} f \) and applying Schwarz' inequality, the result follows by property (a).

**Lemma 3.** Let \( \mathcal{N} \) have r.k. \( g \). Then

(i) if \( \{\phi_n\} \) is any CONS in \( \mathcal{N} \), the Fourier series of \( f \in L^2(X) \) with respect to \( \{\phi_n\} \) converges pointwise to \( \langle f, g(., x) \rangle \) (the orthogonal projection of \( f \) on \( \mathcal{N} \)).

(ii) If \( K \) is an isometry, then

\[
g(s, x) = \langle K(s, .), K(x, .) \rangle_{(a, b)}.\]

**Proof.** (i) With \( a_n = \langle f, \phi_n \rangle \),

\[
\left| \sum_{n=1}^N a_n \phi_n(x) - \langle f, g(., x) \rangle \right| = \left| \langle f, \sum_{n=1}^N \phi_n(.) \bar{\phi}_n(x) - g(., x) \rangle \right|.
\]

The result follows by property (e) and Schwarz' inequality.
(ii) If \( f \in \mathcal{N} \),
\[
    f = K u = \int_a^b K(x, t) u(t) \, dt, \quad u \in L^2(a, b),
\]
but since \( K \) is an isometry this expression is equal to the inner product
\[
    \langle f, KK(x, \cdot) \rangle_x.
\]
Then by uniqueness of \( g \),
\[
    g(s, x) = KK(x, \cdot)(s) = \int_a^b K(s, t) K(x, t) \, dt.
\]

Remarks. Suppose that for every \( x \in X \), \( K(x, \cdot) \) can be extended to \( X \) in such a manner that
\[
    S v = (K(x, \cdot)v) \, dt
\]
is a bounded operator mapping \( L^2(X) \) onto itself; in general the integral is supposed to exist as a limit in the mean (l.i.m.). Let \( P \) denote the operation of truncating \( u \in L^2(X) \) so that it is null outside \([a, b]\). Then \( Q = S P S^{-1} \) is a bounded linear idempotent, and hence an orthogonal projection on \( L^2(X) \) whose range \( \mathcal{N} \) is closed. \( \mathcal{N} \) is also the range of \( K = SP, K^{-1} \) is bounded and \( \mathcal{N} \) has r.k. When \( S \) is unitary the image of a CONS in \( L^2(X) \) is again a CONS.

Definition. Let \( \mathcal{H} \) be a separable Hilbert space of functions on \( X \) with r.k. \( g \) and let \( \{x_n\} \) be a sequence of points in \( X \) indexed by some set \( I \) of integers. We call a set \( \{\phi_n\}, n \in I \), of functions in \( \mathcal{H} \) an interpolatory set with respect to \( \{x_n\} \) if
\[
    \phi_n(x_m) = \delta_{nm} \quad \text{for each } n \text{ and } m \text{ in } I.
\]

Lemma 4. (i) With the notations of the above definition, let \( \{x_n\} \) be such that \( \{g(\cdot, x_n)\} \) forms a CONS in \( \mathcal{H} \). Then for each \( f \) in \( \mathcal{H} \) we have the interpolation formula \( f = \sum f(x_n) \sigma_n \), where \( \sigma_n = g(\cdot, x_n)/g(x_n, x_n) \).

(ii) Conversely, let \( \{\phi_n\} \) be a CONS in \( \mathcal{H} \) and \( \{x_n\} \) be such that either (a) we have \( f = \sum f(x_n) \phi_n \) for each \( f \) in \( \mathcal{H} \), or (b) \( \{\phi_n\} \) is an interpolatory set with respect to \( \{x_n\} \). Then \( \phi_n = g(\cdot, x_n)/g(x_n, x_n) \).

Proof. (i) \[
    g(x_m, x_n) = \langle g(\cdot, x_m), g(\cdot, x_n) \rangle \quad \text{by property (c)},
\]
\[
    = \delta_{nm} g(x_n, x_n);
\]
so orthogonality implies that \( \{\sigma_n\} \) is an interpolatory set with respect to \( \{x_n\} \). The result now follows from the completeness of \( \{g(\cdot, x_n)\} \) and property (d).

(ii) In case (a) the completeness yields
\[
    f(x_n) = \langle f, \phi_n \rangle \quad \forall f \in \mathcal{H};
\]
hence, by uniqueness of $g$, $\|\phi_n\|^2 \phi_n = g(\cdot, x_n)$. Taking norms, $\|\phi_n\|^{-1} = g(x_n, x_n)^\dagger$ so that $\phi_n = g(\cdot, x_n) / g(x_n, x_n)$.

In case (b) the hypotheses together with property (d) yield immediately
\[ f = \sum f(x_n) \phi_n \quad \forall f \in \mathcal{H}, \]
and hence the result by case (a).

Remarks. Lemma 4 (i) can be deduced from [6; Theorem 2]. It is seen from Lemma 4 (ii) that every complete orthogonal interpolatory set in $\mathcal{H}$ must arise from $g$ and some sequence $\{x_n\}$; by property (c), the existence of such a set evidently depends on the existence of zeros of $g$. If such a set does exist it is not, of course, necessarily unique; we find such a case in the next section.

3. The Bessel–Hankel case

We consider the case where $S$ of (3) is the Bessel–Hankel transform on $L^2(0, \infty)$ defined by
\[ H_v u = \lim_{A \to \infty} \int_0^A (xt)^\dagger J_v(xt) u(t) \, dt, \]
where $J_v$ is the Bessel function of real order $v > -\frac{1}{2}$. If we take $K$ of §2 to be $H_v$ restricted to $L^2(0, 1)$, then $K$ is an isometry and the range $\mathcal{H}_v$ consists of functions $f$ such that
\[ f = \int_0^1 (xt)^\dagger J_v(xt) u(t) \, dt, \quad u \in L^2(0, 1). \]

Since $H_v$ is self-inverse, the functions $f$ of $\mathcal{H}_v$ are just those functions in $L^2(0, \infty)$ whose Bessel–Hankel transform is null outside $(0, 1)$. $\mathcal{H}_v$ has r.k. which can be computed as in Lemma 3 (ii); a well known integral [12; p. 134] yields
\[ G(s, x) = \frac{(sx)^\dagger}{x^2 - s^2} \{xJ_{v+1}(x) J_v(s) - sJ_{v+1}(s) J_v(x)\}. \]

$G(s, x)$ is bounded over any finite interval contained in $[0, \infty)$; hence property (d) applies in respect to convergence in $\mathcal{H}_v$. Now there exist sequences $\{x_n\}$ for which $\{t^\dagger J_v(x_n t)\}$ forms a COS in $L^2(0, 1)$. Two such sequences [12; p. 580] are
(a) the positive zeros of $J_v(x)$, and
(b) the positive zeros of $xJ_v'(x) + hJ_v(x)$, $h + v > 0$.

Taking $K(x, t)$ of Lemma 1 to be the Bessel–Hankel kernel, it is clear that the set $\{s_n\}$ forms a complete orthogonal interpolatory set in $\mathcal{H}_v$, and by Lemma 4 (ii) (b) $s_n$ may be computed by substitution in $G(s, x)$. After some reduction one obtains
\[ \frac{2(x_n x)^\dagger J_v(x)}{J_v'(x_n)(x^2 - x_n^2)} \]
for sequences of type (a), and
\[ \frac{2h(x_n x)^\dagger (xJ_v'(x) + hJ_v(x))}{J_v'(x_n)(h^2 + x_n^2 - v^2)(x^2 - x_n^2)} \]
for sequences of type (b) (cf. [2; p. 125]).
To summarise we have

**Theorem 1.** The class \( \mathcal{J}_v \) of functions in \( L^2(0, \infty) \) whose Bessel–Hankel transforms vanish a.e. outside \((0,1)\) is a Hilbert space with r.k. \( G(s, x) \). The functions \( \{G(x_n, x)/G(x_n, x_n)\}, n = 0, 1, \ldots \), form a complete orthogonal interpolatory set with respect to sequences of type (a) or (b) above, and every function of \( \mathcal{J}_v \) is represented by its Fourier series in such a set, with uniform convergence on any finite interval contained in \([0, \infty)\).

**Remarks.** As is usual with expansion theorems of this type, particular cases lead to expansions for special functions. As an example, let \( \{x_n\} \) be a sequence of the type (a) above and choose

\[
u(t) = \frac{2^{\nu-\mu+1}}{\Gamma(\mu-\nu)} t^{\nu+\frac{1}{2}}(1-t)^{\mu-\nu-1},
\]

which is in \( L^2(0,1) \) if \( \mu > \nu + \frac{1}{2} \). Then (4) becomes [3; vol. 2, p. 48, no. 7]

\[f(x) = x^{\nu-\mu+\frac{1}{2}} J_\mu(x), \tag{5}\]

and by Theorem 1,

\[f(x) = \sum f(x_n) \frac{2(x_n x)^{\frac{1}{2}} J_v(x)}{J_v'(x_n)(x^2 - x_n^2)}. \tag{6}\]

Combining (5) and (6) we obtain the formula

\[x^{\nu-\mu} J_\mu(x) = 2J_v(x) \sum_{n=1}^{\infty} \frac{x_n^{\nu-\mu+1} J_\mu(x_n)}{J_v'(x_n)(x^2 - x_n^2)}, \quad \mu > \nu + \frac{1}{2}. \tag{7}\]

As a special case suppose that \( \mu \) and \( v \) differ by an integer, say \( \mu = \nu + m \). Now by repeated application of recurrence relations for Bessel functions we may write

\[J_{\nu+m}(x_n) = -R_{m-1, v+1}(x_n) J_v'(x_n),
\]

where \( R_{m,v} \) is the Lommel polynomial. Then (7) reduces to

\[J_{\nu+m}(x) = -2x^m J_v(x) \sum_{n=1}^{\infty} R_{m-1, v+1}(x_n) x_n^{1-m}(x^2 - x_n^2)^{-1}. \tag{8}\]

When \( m = 1 \), \( R_{0, v+1} \equiv 1 \) in which case (8) reduces to the well known formula [12; p. 498]

\[J_{v+1}(x) = -2xJ_v(x) \sum_{n=1}^{\infty} (x^2 - x_n^2)^{-1}.
\]

Evidently similar formulae may be obtained with the same choice of \( \nu(t) \) but where \( \{x_n\} \) is taken to be the type (b) above.

A further remark is of interest here. We note that certain classical sets of orthonormal functions associated with the Bessel–Neumann series, namely

\[2(\nu+2n+1)^{\frac{1}{4}} x^{-\frac{1}{4}} J_{\nu+2n+1}(x), \quad n = 0, 1, \ldots, x \in (0, \infty),
\]

and

\[2^{-1}(2n+1)^{\frac{1}{4}}(-i)^n x^{-\frac{1}{4}} J_{n+\frac{1}{2}}(x), \quad n = 0, 1, \ldots, x \in (-\infty, \infty),
\]
are complete in the spaces $\mathcal{J}_v$ and $\mathcal{E}$ respectively. This is verified by using completeness properties of the Jacobi and the Legendre polynomials respectively, and using special transforms ([3; vol. 2, p. 47, no. 5] and [3; vol. 1, p. 122, no. 1]).

Next, let $\mathcal{E}_e$ denote the subspace consisting of all even functions of $\mathcal{E}$. Put $M_v f(x) = x^{-v-\frac{1}{2}} f(x)$; when $f \in \mathcal{E}_e$, (4) shows that $M_v f$ is a function defined on $\mathbb{R}$.

**Theorem 2.** $M_v$ is a closed, bounded transformation of $\mathcal{J}_v$ into $\mathcal{E}_e$. $M_v$ is not "onto".

**Proof.** From (4) we may continue $f \in \mathcal{J}_v$ to the cut $z = x + iy$ plane; thus

$$f(z) = z^\frac{1}{2} \int_0^1 t^{\frac{1}{2}} J_v(zt) u(t) dt.$$ 

Now

$$z^{-v-\frac{1}{2}} f(z) = z^{-v} \int_0^1 t^{\frac{1}{2}} J_v(zt) u(t) dt$$

is an even integral function of $z$. From the well known inequality

$$|J_v(\zeta)| \leq \frac{|\frac{1}{2} \zeta|^v}{\Gamma(v+1)} \exp |\text{Im} \zeta| \quad \text{Re } v \geq -\frac{1}{2},$$

we have

$$|z^{-v-\frac{1}{2}} f(z)| \leq C_v \int_0^1 e^{v|t|} t^{\frac{1}{2} + \frac{1}{2}} |u(t)| dt \leq C_v e^{v|t|} \int_0^1 t^{\frac{1}{2} + \frac{1}{2}} |u(t)| dt,$$

so that $z^{-v-\frac{1}{2}} f(z)$ is of exponential type at most 1; its restriction to the real axis is $M_v f(x)$. On the other hand, $f \in L^2(0, \infty)$; therefore the form of $M_v f(x)$ shows that it lies in $L^2(\mathbb{R})$. It follows from the Paley-Wiener theorem [10; p. 13] that $M_v f \in \mathcal{E}_e$, i.e., that $M_v$ is "onto".

$M_v$ is a closed transformation, for let $\{f_n\}$ be a sequence in $\mathcal{J}_v$ such that $f_n \to f$ and $M_v f_n \to \phi$; by property (d) these limits exist pointwise, hence

$$M_v f = M_v \lim f_n = \lim M_v f_n = \phi.$$

By the closed graph theorem, $M_v$ is bounded.

The following example shows that $M_v$ is not "onto". We note first that the function $J_{v+\frac{3}{4}}(x)$ is not in $L^2(0, \infty)$ for any $v$ and hence not in $\mathcal{J}_v$. On the other hand, the left member of the Fourier cosine transform pair

$$\begin{cases} (1-x^2)^v, & 0 < x < 1; \quad 2^r \Gamma(v+1) x^{-v-\frac{1}{2}} J_{v+\frac{3}{4}}(x) \\ 0, & x > 1 \end{cases}$$

is in $L^2(0, 1)$ if $v > -\frac{1}{2}$, so that $x^{-v-\frac{1}{2}} J_{v+\frac{3}{4}}(x)$ is in $\mathcal{E}_e$, $v > -\frac{1}{2}$; but it has no pre-image by $M_v$ in $\mathcal{J}_v$.

Finally, we remark that it follows from the closure and boundedness of $M_v$ that a function $\phi \in \mathcal{E}_e$ is the image of some function in $\mathcal{J}_v$ if and only if it possesses
a norm-convergent expansion of the form
\[ \phi = \sum a_n M_n \psi_n, \]
where \( \sum |a_n|^2 < \infty \) and \( \{\psi_n\} \) is any CONS in \( \mathcal{F} \).

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References


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