Refined critical balance in strong Alfvénic turbulence

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ABSTRACT

We present numerical evidence that in strong Alfvénic turbulence, the critical balance principle – equality of the non-linear decorrelation and linear propagation times – is scale invariant, in the sense that the probability distribution of the ratio of these times is independent of scale. This result only holds if the local alignment of the Elsasser fields is taken into account in calculating the non-linear time. At any given scale, the degree of alignment is found to increase with fluctuation amplitude, supporting the idea that the cause of alignment is mutual dynamical shearing of Elsasser fields. The scale-invariance of critical balance (while all other quantities of interest are strongly intermittent, i.e. have scale-dependent distributions) suggests that it is the most robust of the scaling principles used to describe Alfvénic turbulence. The quality afforded by situ fluctuation measurements in the solar wind allows for direct verification of this fundamental principle.

Key words: MHD – turbulence – solar wind.

1 INTRODUCTION

Strong plasma turbulence is present in many astrophysical systems, and is directly measured by spacecraft in the solar wind (Bruno & Carbone 2005). The precision and sophistication achieved by these measurements in the recent years have enabled direct observational testing of theories of magnetized plasma turbulence that go beyond crude dimensional scalings – we mean, in particular, measurements of spatial anisotropy (Horbury, Forman & Oughton 2008; Podesta 2009; Wicks et al. 2010; Chen et al. 2011), intermittency (Horbury & Balogh 1997; Marsch & Tu 1997; Carbone et al. 2004; Salem et al. 2009; Zhdankin, Boldyrev & Mason 2012; Osman et al. 2014) and alignment (Podesta et al. 2009; Chen et al. 2012; Wicks et al. 2013a,b) of magnetic and velocity fluctuations. In this Letter, we report a new result, obtained numerically, that elicits a striking but physically plausible relationship between these three aspects of the structure of plasma turbulence.

In a strong mean magnetic field \( B_0 \), Alfvénic fluctuations decouple from compressive ones and satisfy the reduced magnetohydrodynamic (RMHD) equations, which correctly describe Alfvénic turbulence in both strongly and weakly collisional plasmas (see e.g. Schekochihin et al. 2009, and references therein). The equations are best written in Elsasser (1950) variables \( z^\pm = u_\perp \pm b_\perp \), where \( u_\perp \) and \( b_\perp \) are the velocity and magnetic-field (in velocity units) perturbations, perpendicular to \( B_0 \):

\[
\partial_t z^+ = v_\perp \partial_z z^+ + z^+ \cdot \nabla z^+ = -\nabla_\perp p, \tag{1}
\]

where the pressure \( p \) is determined via \( \nabla_\perp \cdot z^\perp = 0 \), \( v_\perp = |B_0| \) is the Alfvén speed and \( B_0 \) is in the \( z \) direction.

The modern understanding of the small-scale structure of Alfvénic turbulence described by equations (1) (and, indeed, the validity of these equations) rests on the fluctuations being spatially anisotropic with respect to the magnetic field, and ever more so at smaller scales – this is supported both by solar-wind measurements and by numerical simulations (see, e.g. Chen et al. 2011 and references therein). The relationship between the parallel and perpendicular coherence scales of the fluctuations is set via the critical balance conjecture (Goldreich & Sridhar 1995), whereby the non-linear interaction and the Alfvén-propagation times,

\[
r^\perp = \frac{\lambda}{\delta z_\perp \sin \theta}, \quad r_\perp = \frac{l^\perp}{v_\perp}, \tag{2}
\]

are expected to be comparable at each scale in some, shortly to be discussed, statistical sense. The Alfvén time is related solely to the scale \( l^\perp \) of the fluctuations along the magnetic field, while the non-linear time depends on the fluctuation amplitudes \( \delta z_\perp \), their scale \( \lambda \) perpendicular to the field and on the angle \( \theta \) between \( \delta z_\perp \) and \( \delta z_\perp \) – when this angle is small, the non-linearity in equations (1) is weakened, which is why we have included \( \sin \theta \) in the definition of \( r_\perp \). This effect that can become increasingly important at smaller scales as envisioned by the ‘dynamic alignment’ conjecture

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\[ \rho \propto z^{\hat{\lambda}} \text{const, analogous to equation (3)} \]

\[ z^{\hat{\lambda}} - \sim r_{\tau} \text{point separations} \]

\[ \tau_{0}^{2} \sim 1.5 k \text{and } v_{\lambda}^2 \sin \theta \text{is scale-independent. Then, by the Kolmogorov argument, the} \]

\[ \text{corresponding to a perpendicular separation } r_{\perp} \text{is defined as the shortest distance along the perturbed field line at which the Elsasser-field increment is the same as } \delta z^{\perp}_{\perp} \]

\[ \text{(Cho & Vishniac 2000; Maron & Goldreich 2001; Mattheus et al. 2012):} \]

\[ \left| z^{\perp}_{\perp} \left( r_{0} + r_{\perp} + l_{\perp}^{-1} b_{\text{loc}} \right) - z^{\perp}_{\perp} \left( r_{0} + r_{\perp} - l_{\perp}^{-1} b_{\text{loc}} \right) \right| \]

\[ = \left| z_{\perp}^{\perp} \left( r_{0} + r_{\perp} \right) - z_{\perp}^{\perp} \left( r_{0} \right) \right|, \]

\[ \text{The parallel coherence length } l_{\perp}^{0} \text{corresponding to a perpendicular separation } r_{\perp} \text{is defined as the shortest distance along the perturbed field line at which the Elsasser-field increment is the same as } \delta z^{\perp}_{\perp} \]

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2 DEFINITIONS

We first define the quantities of interest. The fluctuation amplitudes are measured by increments

\[ \delta z^{\perp}_{\perp} \triangleq |\delta z^{\perp}_{\perp}| \triangleq |z^{\perp}_{\perp} (r_{0} + r_{\perp}) - z^{\perp}_{\perp} (r_{0})|, \quad \lambda \triangleq |r_{\perp}|, \]

\[ \text{where } r_{0} \text{is an arbitrary point (irrelevant under averaging because turbulence is homogeneous) and } r_{\perp} \text{the separation in the plane perpendicular to } B_{0} \text{moments of } \delta z^{\perp}_{\perp} \text{only depend on } \lambda \text{because of global isotropy in the perpendicular plane. The alignment angle is given by} \]

\[ \sin \theta \triangleq \frac{|\delta z^{\perp}_{\perp} \times \delta z^{\perp}_{\perp}|}{\delta z^{\perp}_{\perp} \delta z^{\perp}_{\perp}}. \]

\[ \text{The parallel coherence length } l_{\perp}^{0} \text{corresponding to a perpendicular separation } r_{\perp} \text{is defined as the shortest distance along the perturbed field line at which the Elsasser-field increment is the same as } \delta z^{\perp}_{\perp} \]

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\[ = \left| z_{\perp}^{\perp} \left( r_{0} + r_{\perp} \right) - z_{\perp}^{\perp} \left( r_{0} \right) \right|, \]

\[ \text{where } b_{\text{loc}} = B_{\text{loc}} / |B_{\text{loc}}| \text{is the unit vector along the ‘local mean field’} B_{\text{loc}} \triangleq B_{0} + \left[ b_{\perp} \left( r_{0} \right) + b_{\parallel} \left( r_{0} + r_{\perp} \right) \right] / 2. \text{Note that } l_{\perp}^{0} \text{is a random quantity, not a parameter (unlike } \lambda). \]

\[ \text{At each scale } \lambda, \text{the joint probability distribution function (PDF) } \]

\[ P(\delta z^{\perp}_{\perp}, \delta z^{\perp}_{\perp}, \theta, l_{\perp}^{0}, l_{\parallel}^{0} / |\lambda| \) contains all the information one customarily requires to characterize the structure of Alfvénic turbulence. As we only consider ‘balanced’ turbulence, with equal mean injected power in the + and − fluctuations, } P \text{ is symmetric with respect to the + and − variables. We will use the + mode wherever we need to make a choice. Imbalance leads to further interesting complications, left for future investigations.} \]

3 NUMERICAL EXPERIMENT

We solved equations (1) using the code described in Chen et al. (2011) in a triply periodic box of resolution 1024. In the code units, v_{A} = 1 and the box length = 2\pi in each direction. The RMHD equations are invariant with respect to simultaneous rescaling \( z \rightarrow az, v_{A} \rightarrow av_{A} \) for arbitrary a. Therefore, although in code units the box is cubic and \( \delta z^{\perp}_{\perp} / v_{A} \sim 1 \), in fact the box is much longer in the parallel than in the perpendicular direction and the fluctuation amplitudes are much smaller than v_{A}, while the linear and non-linear terms remain comparable. The energy was injected via white-noise forcing at k_{A} = 1, 2 and k_{A} = 1 and dissipated by perpendicular hyperviscosity \( (v_{L} \nabla_{L}^{2} \text{with } v_{L} = 2 \times 10^{-17}) \) and Laplacian viscosity in z \( (v_{L} \delta^{2} / \delta z^{2} \text{with } v_{L} = 1.5 \times 10^{-4}) \); this is needed for numerical stability and has been checked to dissipate a negligible fraction of energy. The mean injected power was \( \epsilon = 1 \) (balanced, strong turbulence). The forcing was purely in velocity; the magnetic field was not directly forced (we have checked that when the two Elsasser fields are forced independently, all results reported below continue to hold).

The field increments (5), angles (6) and parallel scales (7) were calculated for 32 logarithmically spaced scales, of which 17 were in the inertial range 0.094 \( \lambda \leq 0.92 \). For each \( \lambda, 10^{6} \) point separations were generated by choosing a random initial point \( r_{0} \) on the grid and a random direction for \( r_{\perp} \) uniformly distributed in angle over a circle of radius \( \lambda \) in the perpendicular plane. For each \( \lambda, \) the joint

\[ \text{while still a random variable, has a distribution that is independent of scale. We call this statement, which in the ‘~’ language could be written as } \chi^{\pm} \sim 1, \text{ the refined critical balance (RCB). We interpret it as evidence that critical balance results from a dynamical process that happens to inertial-range fluctuations in a completely scale-invariant way. The presence of the alignment angle } \theta \text{ in equation (4) will turn out to be an essential feature of the RCB. We will also examine how the (non-scale-invariant) distributions of } \tau_{A}^{\pm} \text{ and } \tau_{nl}^{\pm} \text{ combine to give rise to a scale-invariant } \chi^{\pm}. \]

\[ \chi^{\pm} \triangleq \frac{\tau_{A}^{\pm}}{\tau_{nl}^{\pm}} = \frac{l_{\perp}^{0} / v_{A} \sin \theta}{v_{A} \lambda}, \]

\[ \text{while still a random variable, has a distribution that is independent of scale. We call this statement, which in the ‘~’ language could be written as } \chi^{\pm} \sim 1, \text{ the refined critical balance (RCB). We interpret it as evidence that critical balance results from a dynamical process that happens to inertial-range fluctuations in a completely scale-invariant way. The presence of the alignment angle } \theta \text{ in equation (4) will turn out to be an essential feature of the RCB. We will also examine how the (non-scale-invariant) distributions of } \tau_{A}^{\pm} \text{ and } \tau_{nl}^{\pm} \text{ combine to give rise to a scale-invariant } \chi^{\pm}. \]

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PDF $P$ was averaged over 10 such samples of $10^6$, from snapshots separated by approximately one large-scale eddy turnover time.

4 RESULTS

4.1 Intermittency and lack of scale invariance

A standard question of all turbulence studies is how the increments $\delta z^\perp$ depend on $\lambda$. As we anticipated above, the answer depends on which moment of the distribution $P(\delta z^\perp; \lambda)$ we choose to calculate. As shown in Fig. 1 (inset) the rms increment $S_1^{1/2}(\lambda) = \langle (\delta z^\perp)^2 \rangle^{1/2}$ (red dashed line), the fourth-order increment, $S_4^{1/4}(\lambda) = \langle (\delta z^\perp)^4 \rangle^{1/4}$ (black dash–dotted line), and the ‘typical’ increment $\overline{\delta z^\perp}$ (blue solid line); the slopes $\lambda^{1/4}$ (Boldyrev 2006) and $\lambda^{1/3}$ (Goldreich & Sridhar 1995) are given for reference; all increments are normalized to the overall rms fluctuation level.

4.2 Alfvén time and non-linear time

The distribution of $\tau_A = l_A^r/v_A$ is simply the distribution of the parallel coherence length. Its geometric mean is shown in the inset of Fig. 2(a) and appears consistent with the scaling $\tau_A \propto \exp(\ln \tau_A^l) \propto \lambda^{1/2}$, which is the relationship between the parallel and perpendicular scales that would follow from Boldyrev’s phenomenology ($\delta z^\perp \propto \lambda^{1/4} \propto (l_A^r)^{1/2}$; Boldyrev 2006). We see that it holds without being weighted by the fluctuation amplitude; i.e. it is a measure of the prevailing spatial anisotropy in the system. The PDFs of the rescaled quantity $\tau_A^l/\lambda^{1/2}$ for a range of $\lambda$ are shown in Fig. 2(a); at smaller $\tau_A^l/\lambda^{1/2}$ (i.e. relatively shorter $l_A^r$), there appears to be a scale-invariant collapse, but at larger values, the PDF becomes non-scale-invariant – with a systematically shallower tail at larger $\lambda$.

The geometric mean of the non-linear time is the inset of Fig. 2(b) and, like $\tau_A^l$, scales as $\tau_{3\perp}^l \propto \exp(\ln \tau_{3\perp}^l) \propto \lambda^{1/2}$. Note that the presence of the alignment angle $\theta$ in the definition (2) of $\tau_{3\perp}$ is essential because it reduces the strength of the non-linear interaction in a scale-dependent way. The PDFs of the rescaled inverse non-linear time, $\lambda^{1/2}/\tau_{3\perp}^l$, are shown in Fig. 2(b). There is approximate (but clearly not perfect) scale invariance at small values of the rescaled quantity (i.e. relatively longer $\tau_{3\perp}$), and a very non-scale-invariant tail at larger values, systematically shallower at smaller $\lambda$.

4.3 Refined critical balance

The behaviour of the distribution of the non-linear time fits neatly with that of the distribution of the Alfvén time. The cores of both distributions (roughly, $\tau_A^l/\lambda^{1/2} \lesssim 3$ and $\lambda^{1/2}/\tau_{3\perp}^l \lesssim 3$ in Fig. 2) are close to being scale invariant. On the other hand, their tails vary with $\lambda$ in opposite senses, with the tail of $\tau_A^l/\lambda^{1/2}$ (for $\lambda^{1/2}/\tau_{3\perp}^l$) becoming steeper (shallower) as $\lambda$ decreases. Because of this, the distribution of their product $\chi^\perp$, defined in equation (4), does not change at all: $P(\chi^\perp|\lambda)$, shown in Fig. 3, is independent of $\lambda$, across the inertial range and all its moments are constant – e.g. $\langle \chi^\perp \rangle$ is shown in the inset of Fig. 3 alongside it, we show the mean non-linearity parameter without the sin $\theta$ factor, $\langle \chi^\perp/\sin \theta \rangle$; it is not scale-independent, so the alignment is an essential ingredient of the RCB.

That the non-linearity parameter $\chi^\perp$ has a scale-invariant distribution is the main result of this Letter. This is due to the fundamental physical connection between the parallel and perpendicular structures of turbulent fluctuations – they cannot remain coherent beyond a parallel distance that information propagates at the Alfvén speed during one perpendicular non-linear decorrelation time, $\tau_A^l \sim \tau_{3\perp}$.

4.4 Alignment

The role of alignment in giving rise to the RCB deserves further discussion. At every scale $\lambda$, the fluctuation amplitude $\delta z^\perp$ and the alignment angle $\theta$ turn out to be anticorrelated (cf. Beresnyak & Lazarian 2006). This is best demonstrated by the conditional PDF $P(\sin \theta|\delta z^\perp/\overline{\delta z^\perp}; \lambda)$, shown in Fig. 4. We see that fluctuations whose amplitudes are large relative to the ‘typical’ value $\overline{\delta z^\perp}$ (i.e. those giving rise to the shallow intermittent tails manifest in Fig. 1)

1 A more traditional way of extracting parallel scalings (corresponding to what is in fact done in the solar wind; Horbury et al. 2008; Podesta 2009; Wicks et al. 2010; Chen et al. 2011) is to define parallel increments $\delta z^\parallel = \langle \delta z^\parallel(t_0 + \tau^\perp_i b_{\parallel 0} - \tau^\parallel(t_0) \rangle$, where $b_{\parallel 0}$ is the local field direction at $r_0$ and $\tau^\perp_i$ is a parameter, not a random variable. The rms of these increments is $\langle (\delta z^\parallel)^2 \rangle^{1/2} \propto l_A^{\perp 2} \chi^{\perp 2}$ (Chen et al. 2011), which is reassuring as, replacing in equation (3) $\delta z^\perp = \delta z^\parallel + \delta z^\perp$, $l_r \rightarrow l_{\parallel i}$ and averaging, we get $\langle (\delta z_i^\perp)^2 \rangle \sim l_{\parallel i}^2 |e|/v_A$, where the mean injected power $|e|$ is certainly independent of scale.
tend to be well aligned, whereas the weaker fluctuations \( \delta z_{\perp} / \delta z^*_1 \lesssim 1 \) are unaligned. The alignment of the stronger fluctuations appears to get statistically ‘tighter’ at smaller scales.

Thus, for the stronger fluctuations, the non-linear interaction is reduced by alignment more than for the weaker ones. We find the approximately scale-invariant core of the distribution of \( \lambda^{1/2} / \tau_{nl}^+ \) in Fig. 2(b) to contain simultaneously smaller \( \theta \) but relatively larger \( \delta z_{
abla} \), so it is the more aligned fluctuations that give rise to the Boldyrev (2006) scaling \( \delta z_{\perp} \sim \lambda^{1/4} \), as expected. Note, however, that the anticorrelation between alignment and amplitude is somewhat at odds with Boldyrev’s intuitive interpretation of the alignment angle as determined by the maximal angular wander within any given fluctuation \( \theta \sim \delta h_{\perp} / B_{0} \), but rather suggests that alignment might be caused by dynamical shearing of a weaker Elsasser field by a stronger one (Chandran et al. 2014; the anticorrelation holds for both the weaker and the stronger of the two Elsasser fields, but is slightly more pronounced if Fig. 4 is replotted for \( P(\sin \theta | \delta z^*_{\perp} / \delta z^*_{\parallel} \), \lambda) \) with \( \delta z^*_{\parallel} \) the locally stronger field). Qualitatively, this is why measures of alignment weighted by the energy (or higher powers of fluctuation amplitudes) exhibit stronger scale dependence (Beresnyak & Lazarian 2009; Mallet et al., in preparation).

All of these statements must be accompanied by the acknowledgement that a debate continues as to whether the tendency to alignment in Alfvénic turbulence survives at asymptotically small scales, with numerical simulations at resolutions up to 4096^3 falling short of an indisputable outcome (Perez et al. 2012; Beresnyak 2014b). What does, however, appear to be solidly the case is that Alfvénic fluctuations over at least the first two decades below the outer scale do exhibit alignment, even if transiently (cf. Podesta et al. 2009; Chen et al. 2012; Wicks et al. 2013a, b), that they do this in a systematic, scale- and amplitude-dependent fashion and, as argued above, that this effect must be taken into account in interpreting what it means, statistically, for these fluctuations to be in
a critically balanced state. The possible change of regime at even smaller scales (Beresnyak 2014b) is left outside the scope of this work.

5 CONCLUSIONS

The results presented above imply that the structure of Alfvénic turbulence is set by two fundamental effects: the critical balance, which occurs in a scale-invariant fashion (probably due to the upper limit on the parallel coherence length of turbulent fluctuations imposed by causality over a non-linear decorrelation time), and systematic alignment of the higher-amplitude fluctuations (probably due to dynamical mutual shearing of Elsasser fields). The first of these results suggests that critical balance – quantitatively amounting, as we have argued, to the RCB conjecture – is the most robust and reliable of the physical principles underpinning theories of Alfvénic turbulence.

While scale-dependent alignment of inertial-range fluctuations in the solar wind is still in question (Podesta et al. 2009; Chen et al. 2012; Wicks et al. 2013a,b), measurements of the anisotropy/alignment/intermittency of these fluctuations directed at the verification of the RCB might help establish whether numerical and real plasma turbulence share the key structural properties and whether, therefore, debates and insights arising from the former have a useful contribution to make to the understanding of the latter.

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