Calculating turbulent transport tensors by averaging single-plume dynamics and application to dynamos

Hongzhe Zhou and Eric G. Blackman

1 Department of Physics and Astronomy, University of Rochester, Rochester, NY 14627, USA
2 Laboratory for Laser Energetics, University of Rochester, Rochester, NY 14623, USA

ABSTRACT

Transport coefficients in turbulence are comprised of correlation functions between turbulent fluctuations and efficient methods to calculate them are desirable. For example, in mean-field dynamo theories used to model the growth of large-scale magnetic fields of stars and galaxies, the turbulent electromotive force is commonly approximated by a series of tensor products of turbulent transport coefficients with successively higher order spatial derivatives of the mean magnetic field. One ingredient of standard models is the kinematic coefficient of the zeroth-order term, namely the averaged kinetic pseudo-tensor $\alpha$, that converts toroidal to poloidal fields. Here we demonstrate an efficient way to calculate this quantity for rotating stratified turbulence, whereby the pre-averaged quantity is calculated for the motion of a single plume, and the average is then taken over an ensemble of plumes of different orientations. We calculate the plume dynamics in the most convenient frame, before transforming back to the lab frame and averaging. Our concise configuration space calculation gives essentially identical results to previous lengthier approaches. The present application exemplifies what is a broadly applicable method.

Key words: dynamo – MHD – turbulence – methods: analytical – galaxies: magnetic fields – stars: magnetic field.

1 INTRODUCTION

Correlation functions of turbulent fluctuations in fluid dynamics and magnetohydrodynamics (MHD) are needed to characterize mean transport coefficients of turbulent systems. Examples include turbulent diffusion, Reynolds and Maxwell stresses, and kinetic, current, and cross helicities. However, computing these quantities in anisotropic, stratified rotators of astrophysics can be cumbersome. Here we demonstrate a method that simplifies this task, applying it to the helical pseudo-tensor of large-scale dynamo theory as an example.

The context of mean-field dynamo arises because magnetic fields are ubiquitous in stars and galaxies whose ages are usually much larger than the magnetic diffusion time-scales. Consequently, a reliable mechanism is necessary for the in situ generation and sustenance of magnetic fields. In the framework of MHD, dynamo theory describes the conversion of flow kinetic energy into magnetic energy, and mean-field dynamo theory specifically concerns the magnetic fields that have a correlation scale larger than the turbulent scale. In mean field theory, the induction equation is suitably averaged and the correlations between the fluctuating velocity and magnetic fields lead to a turbulent electromotive force (EMF)

$$\mathcal{E} = \mathbf{u} \times \mathbf{b},$$

whose curl appears as a source term in the time evolution of the mean magnetic field. In equation (1), $\mathbf{u}$ and $\mathbf{b}$ are, respectively, the turbulent velocity and magnetic fields, i.e. $\mathbf{u} = \mathbf{U}_{\text{tot}} - \overline{\mathbf{U}}$ and $\mathbf{b} = \mathbf{B}_{\text{tot}} - \overline{\mathbf{B}}$, with $\mathbf{U}_{\text{tot}}$ being the total velocity field, $\mathbf{B}_{\text{tot}}$ the total magnetic field, and $\overline{\mathbf{U}}$ and $\overline{\mathbf{B}}$ the corresponding mean fields. The overline indicates some appropriate mean that satisfies Reynolds rules; hereafter, we will drop the overlines for simplicity.

The turbulent EMF is commonly expanded in powers of the gradients of $\mathbf{B}$ whose spatial scale is assumed to be much larger than the scale of the turbulent fields, i.e.

$$\mathcal{E}_i = \alpha_{ij} B_j - \beta_{ijk} \partial_j B_k + \cdots,$$

where $\alpha$, $\beta$, ..., are the turbulent transport coefficients determined by turbulent velocity and magnetic fields. However, even without including the back-reaction from magnetic fields on the flow, finding robust analytical expressions for the coefficients is not easy. Common approaches involve integrating the equations of motion of $\mathbf{u}$ and $\mathbf{b}$, combined with a closure. For example, Pouquet, Frisch & Leorat (1976) used the eddy-damped quasi-normal Marko-
vian (EDQNM) approximation and obtained $-\delta_{ij} \tau u \cdot \nabla \times u / 3$ for the kinetic part of an isotropic $\alpha$ pseudo-tensor, with $\tau$ being the correlation time of the turbulence. Other analytical methods include the first-order smoothing approximation (FOSA; Krause & Raedler 1980; Moffatt & Proctor 1982), minimal-$\tau$ approximation (MTA; Blackman & Field 2002), second-order correlation approximation (SCOA; Rädler & Stepanov 2006), method of random waves (Walder, Deinzer & Stix 1980), method of individual blobs (Stix 1983; Ferriere 1992a), and path integral approaches (Sokoloff & Yokoi 2018).

These analytical approaches show that the $\alpha$ effect is related to helically correlated turbulent fields. (Note, however, that turbulent flows with no mean helicity may also produce an $\alpha$ effect; see Yousef et al. 2008; Sridhar & Singh 2010; Singh & Sridhar 2011; Sridhar & Singh 2014; Singh & Jingade 2015; Rasskazov, Chertovskih & Zheligovsky 2018.)

In many astrophysical systems, stratified turbulence combined with global rotation is believed to result in a net mean kinetic helicity, as shown in both analytical work (Steenbeck, Krause & Rädler 1966; Moffatt 1978; Ruediger 1978; Rädler & Stepanov 2006) and numerical simulations (Cattaneo & Hughes 2006; Hughes & Cattaneo 2008; Käpylän, Korpi & Brandenburg 2009; Brandenburg, Rädler & Kemel 2012). In particular, Steenbeck et al. (1966) derived their famous result for the $\alpha$ pseudo-tensor in a stratified rotator,

$$\alpha_{ij} = -\frac{16}{15} l^2 \Omega \cdot \nabla \ln(\rho u_{\text{rms}}) \delta_{ij},$$

where $l$ is the correlation length, $u_{\text{rms}} = u^2 / 12$, and $\rho$ is the mean density. Later investigations that allow for anisotropy revealed that the $\delta_{ij}$ in equation (3) should be replaced by the tensor $\tilde{\Omega} \cdot \tilde{D} \delta_{ij} = \mu \tilde{\Omega}_i \tilde{D}_j + \tilde{\Omega}_j \tilde{D}_i$, where $\tilde{\Omega} = \nabla / \Omega$ and $\tilde{D}$ is the unit vector in the direction of $\nabla (\rho u_{\text{rms}})$, and $\mu = 11/24$ in Rädler, Kleeorin & Rogachevskii (2003) and $1/4$ in Brandenburg et al. (2013). There are also debates as to whether the momentum term in equation (3) should be replaced by the kinetic energy $\rho \nabla$, e.g. Brandenburg et al. (2013). For incompressible flows, Rädler & Stepanov (2006) also give a slightly different result.

In this work, we develop a new approach to calculate the kinetic $\alpha$ effect in rotating stratified turbulence. We do not propose a new closure model, but instead focus on calculating the correlation functions, i.e. we start from the result of the MTA closure (Blackman & Field 2002),

$$\alpha_{ij} = \tau \varepsilon_{\text{mta}} u_{\text{rms}} \delta_{ij},$$

where the latter equation defines the pre-averaged pseudo-tensor $\alpha_{ij} = \tau \varepsilon_{\text{mta}} u_{\text{rms}} \delta_{ij}$. We will calculate $\alpha$ using the velocity components of a single plume moving in a certain direction, subjected to the influence of the Coriolis force. The Coriolis force is induced by the expansion or shrinkage of the plume when there is a change in its physical surrounding, typically due to stratified mean fields, e.g. temperature or entropy gradient. We do not specify the detailed modelling of the expansion or shrinkage but rather assume a power-law relation between the dimension of the plume and some relevant mean field, therefore making our scheme generally applicable. The $\alpha$ tensor is obtained by averaging $\alpha$ over all the possible orientations of the motion.

Our approach conceptually differs from previous analytic approaches that start formally from the MHD equations in the frame in which the averaged transport tensors are sought without focusing on the local physics of individual turbulent plumes. We take a ‘bottom-up’, first principles approach, first focusing on the physics of local turbulent plume motions, and then averaging over different realizations of plumes in an ensemble to compute the transport tensor. This helps to provide new insight because the form of the correlation tensor is more transparently connected with the local turbulent plume motions.

Some related approaches have been developed in the literature. Ferriere (1992a,b, 1993) derived a $\omega$-dependent turbulent EMF and its corresponding $\alpha$ tensor for galactic mean-field dynamos, by considering the evolution of a uniform magnetic field due to a single supernova or superbubble explosion and then averaged over a spatial distribution of explosions. The calculation of Ferriere (1992a,b, 1993) does not include non-linear transfer terms in the MHD equations. Here we use the MTA closure and therefore the non-linearities have been taken into account, at least approximately. In addition, our approach is independent of the energy source and the alignment of global rotation and density gradient, and thus can be applied to systems other than galaxies.

For turbulence with a convective origin, which is more relevant to stellar dynamos, Durle & Spruit (1979) assumed that the turbulent velocity is a superposition of eight components, and gave the expressions for the turbulent Reynolds stress and the heat flux. Based on the same method, Durle & Robinson (1982) calculated the $\alpha$ tensor for rotating unstratified turbulence.

Subsequently, Stix (1983) included density stratification in his calculation of $\alpha$, but his helicity density is calculated using the mean velocity, rather than taking the mean of the turbulent helicity density. There are three major differences between our present work and that of Stix (1983): (i) We do not invoke the full non-linear equation of motion, but rather consider only the Coriolis force term that generates vorticity. This greatly simplifies the mathematics; (ii) Following the definition of $\alpha$ in equation (4), we take the average only after $\alpha_{ij}$ is obtained; (iii) We do not make the a priori assumption that the $\alpha$ tensor is isotropic. The anisotropy arises naturally in our approach, and the result agrees very well with previous much lengthier analytic derivations.

In Section 2, we introduce the notion of the (pseudo-)tensorial nature of $\alpha$, and calculate its transformation properties under a coordinate transformation. In Section 3, we describe our physical model of $\alpha_{ij}$ in detail and present our calculation of $\alpha$ using the results of Section 2. Discussions and conclusions are given in Section 4.

### 2 $\alpha$ AS A PSEUDO-TENSOR

Given the expression of the $\alpha$-pseudo-tensor equation (4), it can be readily verified that

$$\varepsilon_{ijk} \alpha_{jk} = \partial_i \nabla u_{ij} - 2 \mu \partial_i u_{ij},$$

Via application of Reynolds’ rules of averaging for homogeneous and incompressible flows, the RHS of equation (5) vanishes and the $\alpha$-pseudo-tensor has to be symmetric, i.e. $\alpha_{ij} = \alpha_{ji}$. Furthermore, even for inhomogeneous or compressible cases, $\alpha$ can always be decomposed into its symmetric and antisymmetric parts,

$$\alpha = \alpha^S + \alpha^A,$$

where $\alpha^S = (\alpha \pm \alpha^T)/2$, and the superscript $T$ indicates transpose. However, in the mean induction equation, the antisymmetric part of $\alpha$ multiplied by $B$ has an advection-like behaviour, and $\alpha^A$ can be rewritten as an effective mean velocity field (Charbonneau 2014). We therefore focus on a symmetric $\alpha$ pseudo-tensor in the rest of the paper.

The components of $\alpha$ in a certain frame constitute a real and symmetric matrix, and thus can be diagonalized by an orthogonal...
matrix constructed from its eigenvectors, e.g.

\[ \alpha'_{ij} = (S^{-1})_{ak}\alpha_{ab}S_{bj}, \]

where \( \alpha'_{ij} \) are the components of a diagonal matrix and \( S \) is real and orthogonal, \( S^{-1} = S^T \). In our formalism, primed quantities are in the frame of diagonalization and unprimed quantities are in the lab frame. The determinant of \( S \) can be either 1 or \(-1\) since \( \det(S^T S) = \det(S)^2 \). The det \( S = -1 \) branch involves reflection operation in configuration space and in order to remove this complexity, we can conveniently normalize \( S \) by its determinant and define \( R = S/\det S \). Notably \( R \) can then be regarded as a rotation matrix or an element of the group \( SO(3) \).

Diagonalizing \( \alpha \) using \( R \) corresponds to a coordinate transformation of configuration space coordinates by rotation of the axes about the origin. This can be readily seen by noting that rotation by a matrix \( W \in SO(3) \) transforms the coordinates and gradients as

\[ x_i \rightarrow x'_i = W_{ij}x_j, \quad \partial_i \rightarrow \partial'_i = W_{ij}\partial_j, \]

and the components of vector and pseudo-vector quantities transform as

\[ u_i \rightarrow u'_i = W_{ij}u_j, \quad B_i \rightarrow B'_i = W_{ij}B_j. \]

Since \( \alpha \) is a rank-2 (pseudo-)tensor, we have

\[ \alpha_{ij} \rightarrow \alpha'_{ij} = W_{ik}W_{jl}\alpha_{kl}. \]

Now if we identify \( W = R^{-1} \), effectively we have rotated the system into a frame where the \( \alpha \) tensor is diagonalized.

We can now show that form of the induction equation is the same in the two frames. We write

\[ \partial_i B'_i = \partial_i(W_{ij}B_j) = W_{ij}\epsilon_{ijk}\partial_k(\alpha_{kl}B_l) = W_{ij}W_{kp}^{-1}W_{pl}^{-1}W_{kr}^{-1}W_{lj}\epsilon_{ijk}\partial_k(\alpha'_{kl}B'_l).
\]

\[ = \delta_{lj}\delta_{kl}\partial_k(\alpha'_{kl}B'_l) = \epsilon_{ijk}\partial_k(\alpha'_{kl}B'_l).
\]

\[ = \epsilon_{ijk}\partial_k(\alpha'_{kl}B'_l), \]

where we have used \((W^{-1})_{ij} = W_{ji}\). The invariance of the form of the induction equation between the lab and diagonalized frames is essential for our method, given that boundary conditions are also accordingly transformed. It allows us to study the local physics of the system (e.g. energy and helicity growth) in the diagonalized frame to construct \( \alpha_{ij} \), and then transform to the lab frame and average to get \( \alpha \).

### 3 \( \alpha \) in a Stratified Rotator

#### 3.1 Method overview

Given the aforementioned invariance of the induction equation, we now derive the \( \alpha \) for turbulent stratified flows. The strategy consists of three steps: (i) Build an appropriate physical model for the flow in a single turbulent cell, and then calculate \( \alpha_{ij} = \chi \epsilon_{ijk}\partial_ku_m^\prime\partial_l^\prime u_n^\prime \) for this local turbulent cell in the preferred, convenient frame \( S \). For example, \( S \) might be a frame in which the velocity components of the flow have simple forms; (ii) Rotate \( \alpha_{ij} \) back to the lab frame \( S \) to get \( \alpha_{ij} \); (iii) Obtain \( \alpha_{ij} \) in the lab frame by averaging \( \alpha_{ij} \) over a distribution of rotational transformation matrices. Equivalently, this means averaging over identical turbulent plumes with different orientations with respect to fixed rotation and density stratification axes.

The theory behind step (iii) above warrants further explanation. First, by averaging over a distribution of \( S \) we are ensemble averaging, since \( a \) is fixed at a given time and location in a real system. However, when the correlation length and time are small compared to the corresponding scales of the mean fields, the ensemble average will deviate only by a small (but quantifiable amount; Zhou, Blackman & Chamandy 2018) amount from spatial and/or temporal averages. Secondly, the realistic distribution of \( S \) is generally non-trivial, and is related to the actively studied question of building an accurate ensemble of turbulence (Kraichnan 1973; Frisch et al. 1975; Shebalin 2013). For simplicity, we choose a uniform distribution of \( S \), and leave other choices for future work.

To understand the averaging procedure mathematically, consider a statistically homogeneous flow where \( \alpha \) in each turbulent cell has a uniform value, and in a given local frame has components \( \alpha_{ij} \). To obtain the components of \( \alpha \) in the lab frame, \( \alpha'_{ij} \) of each cell has to be rotated accordingly, and the result averaged, i.e.

\[ \alpha_{ij} = \sum_{p=1}^{N} f(p)R_{ij}^{(p)}R_{kl}^{(p)}\alpha'_{kl}. \]

where \( p \) labels different turbulent cells, \( N \) is the total number of turbulent cells, \( R^{(p)} \) is the rotation matrix that transforms the \( S \) frame of the \( i \)-th cell to the lab frame, and \( f(p) \) is the fractional multiplicity of \( R^{(p)} \) in the ensemble. In the continuum limit, the summation should be replaced by the integration of the parameters of the rotation group.

#### 3.2 Specific calculation

We now employ the approach to derive the anisotropic \( \alpha \) tensor in a rotating turbulent flow, and compare the result with previous analytical work.

##### 3.2.1 Kinetic helicity of plume in diagonalized frame

We consider a plume in the co-rotating frame to move a distance \( l \) over a turbulent correlation time at a constant velocity of magnitude \( u = l/t \) before dissipating. Motion perpendicular to the contours of mean density or turbulence intensity introduces a lateral expansion or shrinkage of the plume. The Coriolis acceleration resulting from this lateral motion will generate a net vorticity of the plume, parallel or antiparallel to the direction of background global rotation. When the magnetic field is flux-frozen into the conducting medium, the combined rise-and-twist motion of the flow produces a magnetic field component perpendicular to its original orientation. For turbulence represented by an ensemble of randomly oriented plumes, the ensemble average effect gives rise to a mean kinetic \( \alpha \) effect.

To clarify the needed coordinate systems, consider a star with a global rotation \( \Omega \), as in the inset of Fig. 1. The star is given a spherical coordinate system \((\hat{r}, \hat{\theta}, \hat{\phi})\). The gradient of mean density or turbulence intensity introduces a lateral expansion or shrinkage of the plume. The Coriolis acceleration resulting from this lateral motion will generate a net vorticity of the plume, parallel or antiparallel to the direction of background global rotation.

Further, in the Cartesian frame \( S \) at colatitude \( \theta \) whose \( \hat{x}, \hat{y}, \hat{z} \)-axes are, respectively, \( \hat{\theta}, \hat{\phi} \) and \( \hat{r} \), \( \Omega \) has components \( \Omega = (-\sin \theta \cos \phi, -\sin \theta \sin \phi, \cos \theta) \). For galaxies, \( \theta = 0 \) and \( \Omega \cdot \hat{r} = 1 \) everywhere, but for stars, these relations may hold only at the poles.

In the Cartesian frame \( S \), we parametrize the direction of the motion of a plume moving in a straight line as \( \xi = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta) \). Note that \((\theta, \phi)\) and \((\theta, \phi)\) are two different sets of variables: The former denotes the global colatitude and longitude (thus the ‘location’ of \( S \)), whereas the latter...
denote the orientation of the plume in $S$ and are to be averaged over. We use $S'$ to indicate the Cartesian frame whose vertical axis is $\hat{z}'$.

We first study an ascending plume, and thus $\cos \theta > 0$. We assume that the plume expands due to the background mean field gradient (in $\hat{r}$) and internal thermodynamics according to $L(r) \propto Q^q$, where $L$ is a characteristic length-scale of the plume, $Q$ is some mean field (e.g. density, turbulent kinetic energy, etc.), and $q$ is an index determined by a turbulence model. Assuming $Q$ has a large scale height compared to the correlation length, the relative change in $L$ due to the near-axis-of-mass motion of the plume is

$$\frac{\Delta L}{L} = \Delta \ln L = q \Delta \ln Q \approx q l \cos \theta \cdot \nabla \ln Q, \quad \theta \in [0, \frac{\pi}{2}],$$

(13)

where $l \cos \theta$ is the distance the blob travels along $\hat{r}$.

The magnitude of the Coriolis force exerted on the plume during the expansion is $\sim 2\Omega \Delta L$, and thus it produces an angular velocity $-2\Omega \Delta L / L$. The component of the rotation vector perpendicular to $\hat{z}'$ contributes $O(\Omega^2)$ terms to $a'_{ij}$ and therefore will be neglected in the following calculations. To find out the component along $\hat{z}'$, note that the cosine of the angle between $\hat{\Omega}$ and $\hat{z}'$ is

$$-\hat{\Omega} \cdot \hat{z}' = (\sin \theta, 0, -\cos \theta) \cdot (\sin\theta \cos \phi, \sin\theta \sin \phi, \cos \theta)$$

$$= - (\cos \theta \cos \phi - \sin \theta \sin \phi \cos \theta).$$

(14)

Thus in the $S'$ frame, to $O(\Omega)$ order the plume moves with velocity $u_{\hat{z}'}$ and spins with angular velocity $-2\Omega \hat{z}'$, where

$$\Omega' = 2\Omega (\cos \theta \cos \phi - \sin \theta \sin \phi \cos \theta) \Delta L / L.$$  

(15)

Now the components of $a'_{ij}$ can be calculated for the plume using $a'_{ij} = \epsilon_{ijk} u'_k$. Writing out component by component and assuming $\hat{a}'_{ij} = -\hat{\partial}_i u'_j = -\Omega'$ and $\hat{a}'_i = 0$, it follows that in the $S'$ frame the components of $a$ read

$$a'_{xx} = a'_{yy} = -\frac{1}{2} \tau h_p,$$

(16)

with other components all being zero, and

$$h_p = \frac{4q \Omega^2 \hat{r} \cdot \nabla \ln Q}{\tau} \cos \theta (\cos \theta \cos \phi - \sin \theta \sin \phi \cos \theta).$$

(17)

The overall $\cos \theta$ factor assures that both ascending and descending plumes (the latter of which shrink and rotate oppositely to ascending plumes) are accounted for. Equations (16) and (17) are applicable in the slow rotation limit, characterized by low Coriolis number $Co = 2\Omega \tau < 1$. For fast rotators, Taylor-Proudman-like columns form (e.g. Brun & Palacios 2009) and higher order terms in $\Omega$ must be included.

The trace of $a$ is $-\tau h_p$, and we now show that $h_p$ is exactly the kinetic helicity of the plume. The plume vorticity produced by the Coriolis force is

$$\nabla \times u \simeq -2\Omega' \hat{z}'.$$

(18)

The extra factor of two results because $\nabla \times (\Omega' \times r) = 2\Omega' \hat{z}'$, where $\Omega' = \Omega \hat{z}'$. The kinetic helicity of the plume is just $a \hat{z}' \cdot \nabla \times u = -2a \Omega'$.  

(19)

Combining equations (13), (15), (18), and (19) yields equation (17), confirming that the latter is indeed the kinetic helicity.

3.2.2 Transformation to lab frame and computation of $\alpha$

To compute the components of $a$ in the $S$ frame, we make an orthogonal transformation using the matrix $R(\theta, \phi)$ that rotates the vector $(0,0,1)$ (representation for $\hat{z}'$ in the $S'$ frame) to $(\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$ (representation for $\hat{z}$ in the $S$ frame), namely

$$R(\theta, \phi) = \left( \begin{array}{ccc} 1 - 2 \sin^2 \frac{\theta}{2} \cos^2 \phi & -\sin^2 \frac{\theta}{2} \sin 2\phi & \sin \theta \cos \phi \\ -\sin^2 \frac{\theta}{2} \sin 2\phi & 1 - 2 \sin^2 \frac{\theta}{2} \cos^2 \phi & \sin \theta \sin \phi \\ 0 & 0 & \cos \theta \end{array} \right).$$

(20)

Then $a$ in the $S$ frame is obtained by averaging $a$ over all possible orientations, namely

$$a_{ij} = \frac{1}{4\pi} \int_0^{2\pi} \sin \theta d\theta \int_0^{\pi} \sin \theta d\phi \ R_{mn}(\theta, \phi) R_{ij}(\theta, \phi) a'_{mn}. $$

(21)

In this ensemble average, we have assumed that the direction of motion is distributed uniformly on a sphere, i.e.

$$f(\theta, \phi) = \frac{1}{4\pi} \quad \text{for} \quad \theta \in [0, \pi) \quad \text{and} \quad \phi \in [0, 2\pi).$$

(22)

before it is perturbed by the Coriolis force, as expected for slow rotators. For fast rotators the plume would be more cylindrical (Brun & Palacios 2009).

Using equations (16), (17), (20), and (21), we obtain

$$a = \frac{8q}{15} \Omega^2 \hat{r} \cdot \nabla \ln Q \left[ \begin{array}{ccc} \cos \frac{\theta}{2} & 0 & \frac{1}{2} \sin \frac{\theta}{2} \\ 0 & \cos \frac{\theta}{2} & 0 \\ \frac{1}{2} \sin \frac{\theta}{2} & 0 & \frac{1}{2} \cos \frac{\theta}{2} \end{array} \right].$$

(23)

in the $S$ frame. More conveniently, we can also write this as

$$a_{ij} = \frac{8q}{15} \Omega^2 \hat{r} \cdot \nabla \ln Q \left[ \delta_{ij} (\hat{\Omega} \cdot \hat{r}) - \frac{1}{4} (\hat{\Omega} \hat{r}_j + \hat{\Omega}_j \hat{r}_i) \right].$$

(24)

independent of the choice of frame, and it has trace

$$\text{tr} a = a_{ij} = \frac{4q}{3} \left( \hat{r} \cdot \hat{\Omega} \right) (\hat{r} \cdot \nabla \ln Q).$$

(25)
This overall factor of $\text{tr} \alpha$ is determined by the helicity gain due to the motion of the plume and in general would be determined by detailed modelling of turbulence and thermodynamics of the plume motion. In this respect our approach accommodates different detailed models and thermodynamic assumptions for the plume motion. For example, we may consider a simple scaling relation for the plume size $L^2 \propto \rho^{-1} \propto (\rho u) (\mu \tau)^{-1}$ where in the last step we have used $\rho u L = \text{constant}$ in mixing length theory. This gives $Q = \rho u^2$ and $q = -1$, and differs only slightly from previous more rigorous calculations in the literature. For example, Steenbeck et al. (1966) find $q = -2$ and $Q = \rho u L$; Rädler et al. (2003) find $q = 2$ and $Q = u^2$: and Brandenburg et al. (2013) find $q = -2$ and $Q = \rho u^2$.

However, our expression of the tensorial structure of $\alpha$ (the part in the square bracket in equation 24) is purely the result of averaging the motion of the plume over different directions, and only requires the assumption of fully developed isotropic background turbulence. That makes the method refreshingly concise, and reproduces the correct results even without the need to work in Fourier space.

4 CONCLUSION

Even when a closure is assumed, calculating turbulent transport coefficients for practical use in astrophysics can be cumbersome for stratified rotators because of the anisotropies these quantities induce. Here we developed a mathematically simple and physically transparent method and applied it to deriving the kinetic coefficients for practical use in astrophysics. Even when a closure is assumed, calculating turbulent transport coefficients is purely the result of averaging the motion of the plume over different directions, which may not be the case in a more realistic setting. For example, we may consider a simple scaling relation for the plume size $L^2 \propto \rho^{-1} \propto (\rho u) (\mu \tau)^{-1}$ where in the last step we have used $\rho u L = \text{constant}$ in mixing length theory. This gives $Q = \rho u^2$ and $q = -1$, and differs only slightly from previous more rigorous calculations in the literature. For example, Steenbeck et al. (1966) find $q = -2$ and $Q = \rho u L$; Rädler et al. (2003) find $q = 2$ and $Q = u^2$: and Brandenburg et al. (2013) find $q = -2$ and $Q = \rho u^2$.

However, our expression of the tensorial structure of $\alpha$ (the part in the square bracket in equation 24) is purely the result of averaging the motion of the plume over different directions, and only requires the assumption of fully developed isotropic background turbulence. That makes the method refreshingly concise, and reproduces the correct results even without the need to work in Fourier space.

We showed that our result equation (24) matches very well with those from lengthier approaches (Rädler et al. 2003; Brandenburg et al. 2013) that start with the same assumptions, namely: (i) the only two non-trivial vectors in the problem are the rotation vector $\Omega$ and the gradient of a mean-field flow property. Results are obtained to first order in these two vectors. (ii) The background turbulence without the rotation is statistically homogeneous, isotropic, and of high Reynolds number. Previous approaches did not exploit simplifications that arise from first working in the frame of a local plume and then transforming back to the lab frame.

Our approach is quasi-analytic; we do not solve the full Navier–Stokes equation. More accurate results can be obtained by solving for the motion of the plume more rigorously, possibly with numerical simulations. We have also assumed that the direction of the motion of the plume centre of mass has a uniform distribution over all spatial orientations, which may not be the case in a more realistic setting.

Our approach is not restricted to the helicity tensor of mean-field dynamo theories. The method could be extended and applied to the calculation of other correlation functions and transport coefficients.

ACKNOWLEDGEMENTS

We acknowledge support from grant NSF-AST-15156489, and HZ acknowledges support from Horton Fellowship from the Laboratory for Laser Energetics at U. Rochester.

REFERENCES


Cattaneo F., Hughes D. W., 2006, J. Fluid Mech., 553, 401


Sokoloff D., Yokoi N., 2018, J. Plasma Phys., 84, 735840307


This paper has been typeset from a TeX/LATEX file prepared by the author.