SOCIETY OF PETROLEUM ENGINEERS OF AIME
6200 North Central Expressway
Dallas, Texas 75206

PAPER NUMBER SPE 2031

THIS IS A PREPRINT --- SUBJECT TO CORRECTION

# Combination of Numerical and Analytical Techniques To Improve Waterflood Model Efficiency

By

William A. Roper, Member AIME, Jack E. Merchant and Carolyn A. Duvall,
Getty Oil Co., Houston, Tex.







## ABSTRACT

Description of flow near wells is difficult when modeling steady-state flow in an oil reservoir. A technique is presented which avoids this difficulty by describing pressure and velocity in part numerically and in part analytically.

## INTRODUCTION

Numerous mathematical models have been developed to predict water-flood recovery. Often a steady-state or semi-steady-state model will yield adequate predictions. The method described herein is the basis of an improved steady-state reservoir model.

A frequent numerical approach is to represent the reservoir by a grid network and to determine the steady-state pressure distribution by relaxation. The principle weakness of this method is that a rectangular grid is poorly suited to represent the pressure distribution around wells. Flow near wells is radially symmetrical and rapidly changing. Flow in a grid network is assumed to occur only along grid lines. Figure 1 depicts the difficulty of representing flow near a source by a grid network. For this reason most numerical models require a large number of grid points to adequately describe flow.

References and illustrations at end of paper.

The method presented here utilizes pressure distribution and flow velocities which are in part analytical. In essence, the point sources and sinks are subtracted from the original flow problem, leaving a modified flow problem with distributed sources and sinks. The modified flow problem can then be solved numerically with less difficulty than the original flow problem.

## DEVELOPMENT

The mathematical analog to steady-state flow in a reservoir is based on Darcy's Law and the conservation equation.[1] If the three-dimensional physical system is reduced to a two-dimensional model, and if other simplifying assumptions are made, flow in a reservoir can be related to the solution of the following equations.

$$\vec{\nabla} \cdot \vec{q} = \Sigma_i \, Q_i \, \delta(x_i,y_i) \qquad (x,y) \in A \qquad \text{...1.1}$$

$$\vec{q} \cdot \hat{n} = G(x,y) \qquad (x,y) \in L \qquad \text{...1.2}$$

where A is a closed and connected region, L is the boundary of A, and

$$\vec{q} = K \, \vec{\nabla} P \qquad \text{...1.3}$$

In addition,

$$\oint G(x,y) \, dL = \Sigma_i Q_i \qquad \text{...1.4}$$

Basic variables are defined in reservoir terminology at the end of this paper.

Equation 1.1 prescribes volumetric conservation everywhere within Region A except at point sources or sinks. The point sources and sinks are located at $(x_i,y_i)$ for i=1,N. The Dirac delta function is defined as

$$\delta(x_i,y_i) = \begin{cases} \infty & \text{at } (x_i,y_i) \\ 0 & \text{otherwise} \end{cases}$$

Furthermore,

$$\iint_A F(x,y) \; \delta(x_i,y_i) \; dx \; dy =$$

$$\begin{cases} F(x_i,y_i) & \text{for } (x_i,y_i) \in A \\ 0 & \text{otherwise} \end{cases}$$

Equation 1.2 prescribes the normal component of flow on the boundary L.

Equation 1.3 is Darcy's Law if

$$K = -\frac{kh}{\mu}$$

The variable $K \equiv K(x,y)$ is assumed to be a scalar function of position only.

Equation 1.4 states that the net flow across the reservoir boundary and the net influx from sources and sinks must allow volumetric conservation within the reservoir.

Since the advent of digital computers, numerical techniques have been used to solve Equations 1.1-1.4. Usually, the reservoir is overlain with a rectangular grid network as shown in Figure 2. The differential equations are replaced by finite difference equations and these are solved by relaxation or some other scheme.

This solution method has several basic weaknesses. The rectangular grid is poorly suited to describe flow around sources and sinks. This requires the use of a fine grid network which increases computer expense. Furthermore, sources and sinks must be relocated to coincide with grid intersections.

The difficulties enumerated above are illustrated by the solution of a simple flow system. In the example shown in Figure 3, flow is confined in a square area. The two sources have unit strength and the sink has two units strength. Pressure along a line through the sources and sink (dashed line in Figure 3) is shown in Figure 4. The solid line in Figure 4 is the analytical solution to the flow system. Pressures predicted using 49 grid points (triangles) are obviously unsatisfactory. If 729 grid points are used (circles), the predicted pressures are accurate everywhere except at the sources and sink. With 729 grid points, the derivative of pressure is still inaccurate near a source or sink.

If point sources and sinks are replaced by a distributed source (sink), these difficulties do not occur. Heat generated by flow of electricity in a conductor is a distributed source. In particular, the Dirac delta functions are to be eliminated from Equation 1.1.

Therefore, it is desirable to reduce the solution of Equations 1.1-1.4 to the solution of the following equations.

$$\vec{\nabla} \cdot \vec{q}_o = f(x,y) \qquad (x,y) \in A \qquad \text{...2.1}$$

$$\vec{q}_o \cdot \hat{n} = g(x,y) \qquad (x,y) \in L \qquad \ldots 2.2$$

where

$$\vec{q}_o = K \vec{\nabla} P_o \qquad \ldots 2.3$$

and

$$\oint_L g(x,y)\, dL = \iint_A f(x,y)\, dx\, dy \qquad \ldots 2.4$$

To achieve this result, let a set of functions, $P_i$, be defined such that,

$$K \nabla^2 P_i = Q_i\, \delta(x_i, y_i) \qquad \text{for } i=1,N \qquad \ldots 3$$

A set of functions based on the pressure distribution about a point source in an infinite homogeneous system has this property.[2] Let

$$P_i = \frac{Q_i}{2\pi K_i} \ln R_i \qquad \text{for } i=1,N \qquad \ldots 4.1$$

where

$$K_i = K(x_i, y_i) \qquad \ldots 4.2$$

and

$$R_i^{\,2} = (x-x_i)^2 + (y-y_i)^2 \qquad \ldots 4.3$$

Utilizing the pseudo pressure functions, $P_i$, defined by Equation 4.1, the original problem can be reduced to a distributed source problem. Let

$$P = P_o + \Sigma_i\, P_i \qquad \text{for } i=1,N \qquad \ldots 5$$

and

$$\vec{q}_i = K \vec{\nabla} P_i \qquad \text{for } i=0,N \qquad \ldots 6$$

These definitions immediately produce the following relationships:

$$\vec{\nabla} \cdot \vec{q} = \vec{\nabla} \cdot \vec{q}_o + \Sigma_i\, \vec{\nabla} K \cdot \vec{\nabla} P_i + \Sigma_i\, K\, \nabla^2 P_i \qquad \ldots 7.1$$

and

$$\vec{q} \cdot \hat{n} = \vec{q}_o \cdot \hat{n} + \Sigma_i\, \vec{q}_i \cdot \hat{n} \qquad \ldots 7.2$$

As a consequence of the choice of the functions $P_i$, the last sum in Equation 7.1 is zero.

Utilizing Equations 1.1 and 1.2, the following relationships can be obtained:

$$\vec{\nabla} \cdot \vec{q}_o = -\Sigma_i\, \vec{\nabla} K \cdot \vec{\nabla} P_i \qquad \ldots 8.1$$

$$\vec{q}_o \cdot \hat{n} = G(x,y) - \Sigma_i\, \vec{q}_i \cdot \hat{n} \qquad \ldots 8.2$$

Therefore, the solution of the problem initially posed (Equations 1.1-1.4) has been reduced to the solution of the distributed source problem (Equations 2.1-2.4). Since the analytical form of the pseudo pressure functions ($P_i$) are known, f(x,y) and g(x,y) can be evaluated as

$$f(x,y) = -\vec{\nabla} K \cdot \Sigma_i\, \vec{\nabla} P_i \qquad \ldots 9.1$$

$$g(x,y) = G(x,y) - K\, \Sigma_i\, \vec{\nabla} P_i \cdot \hat{n} \qquad \ldots 9.2$$

The development of a numerical technique to solve the distributed source problem is identical to that required to solve the original problem.

The method described above was used to solve the flow system illustrated in Figure 3. The pressures predicted by the improved method are shown in Figure 5. As can be seen, the predicted values are accurate when only 49 grid points are used (triangles). Furthermore, pressure and velocity can be accurately calculated everywhere. This is a result of the smoothness of the grid pressure (the part of the pressure not specified analytically). Grid pressure is shown in Figure 6 along a line through the sources and sink. Interpolation of grid pressure is obviously much more accurate than interpolation of total pressure.

In applying this method, the

following should be noted:

1. To evaluate the source contribution at a grid point, f(x,y) must be integrated over the area associated with that grid point. If A is the area associated with a grid point, and L its parameter, this integral can be obtained conveniently by,

$$\int_A \int f(x,y)\, dx\, dy = \oint_L \hat{n} \cdot \Sigma_i \vec{\nabla} P_i \, dL + \Sigma_j Q_j \quad \ldots 10.1$$

for i=1, N
for all $Q_j$ at $(x_j, y_j) \in A$

Equation 10.1 is a consequence of the conservation theorem (Green's Theorem)[3]

$$\int_A \int \vec{\nabla} \cdot K \vec{\nabla} P_i \, dx\, dy = \oint_L K \vec{\nabla} P_i \cdot \hat{n}\, dL \quad \ldots 10.2$$

2. The evaluation of the integral of f(x,y) should be done carefully. Computer time spent for this purpose is not critical since it is done only once.

3. At first it might appear that f(x,y) is not a significant improvement over its counterpart in Equation 1.1. If the variable K(x,y) is constant near point sources and sinks, the improvement is obvious. In any event, the integral of f(x,y) over the area associated with a grid point approaches zero as the size of the grid mesh approaches zero.

4. Analytical image wells can be used with this method to reduce boundary problems.

## CONCLUSIONS

Untilization of a mathematical model, based on the method presented herein, has proven to be economical, accurate, and convenient. Since the location and number of wells is not related to the grid size, larger grid meshes can be used than would otherwise be possible. Computer rental cost for a typical field study has been less than $1,000.

The successful removal of problems associated with wells has not, however, solved all problems associated with a steady-state reservoir model. The inadequacy of a rectangular grid to represent reservoir boundaries is often a serious problem.

## NOMENCLATURE

| | |
|---|---|
| G(x,y) | Normal component of flow on the reservoir boundary. Frequently G(x,y) = O |
| h | Thickness |
| k | Permeability |
| N | Number of wells |
| $\hat{n}$ | Outer unit normal to a boundary |
| P | Pressure |
| $Q_i$ | Injection or production rate |
| $\vec{q}$ | Flow vector equal to reservoir thickness times Darcy velocity |
| $\mu$ | Viscosity |

## ACKNOWLEDGMENT

The authors wish to extend their appreciation to the management of Getty Oil Company for permission to publish this information.

## REFERENCES

1. Muskat, Morris: Physical Principles of Oil Production, McGraw-Hill Book Co., Inc., New York (1949).

2. Scheidegger, A. E.: The Physics of Flow Through Porous Media, The MacMillian Co., New York (1957).

3. Aris, Rutherford: Vectors, Tensors, And The Basic Equations of Fluid Mechanics, Prentice-Hall, Inc., Englewood Cliffs, N.J. (1962).

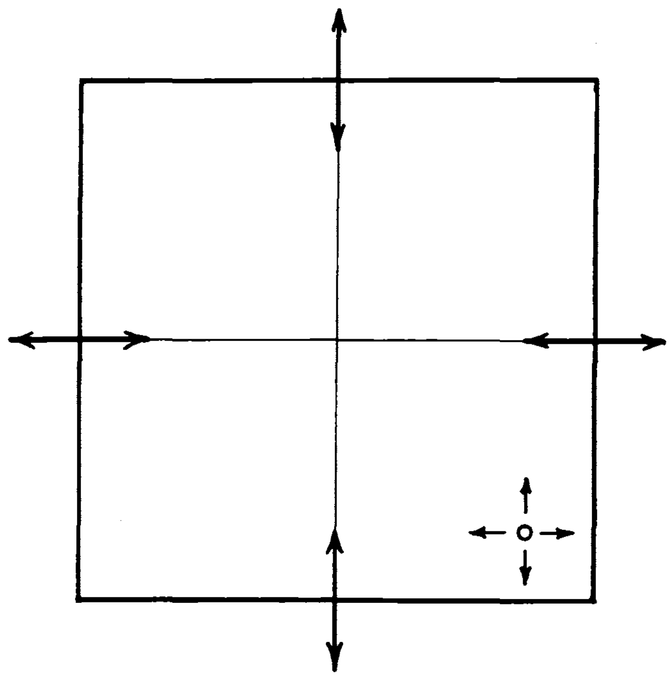

Fig. 1 - Symbolic representation of flow near a source and flow to and from a grid point.

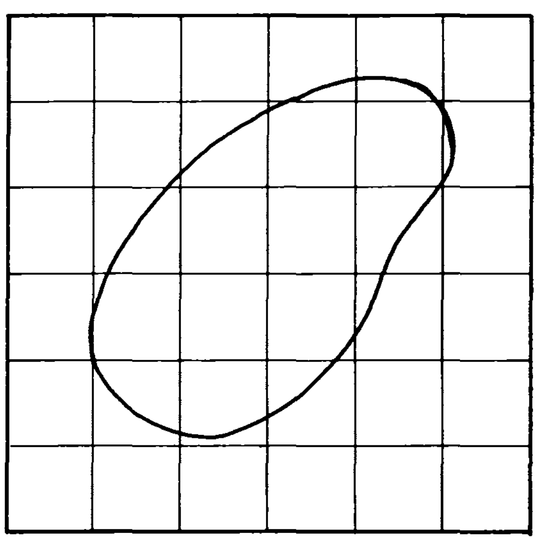

Fig. 2 - Reservoir overlain with a grid network.

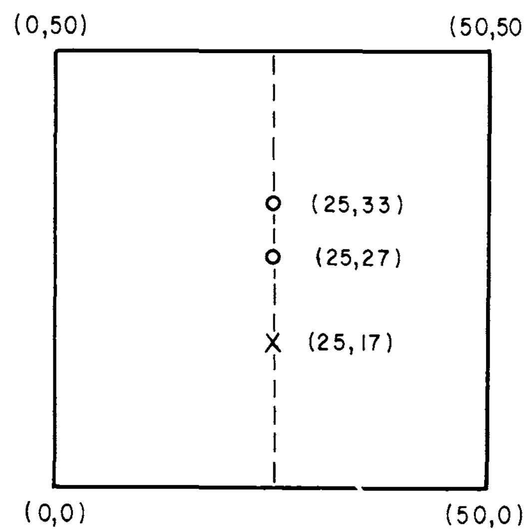


Fig. 3 - Schematic of example flow system. (o - source x - sink)

Pressure and distance units identical in Figs. 4, 5, and 6.

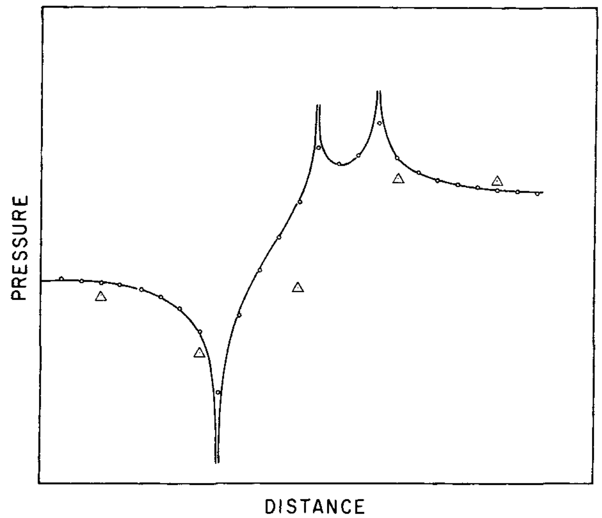


Fig. 4 - Comparison of pressure along the dotted line in Fig. 3 and pressure predicted by the grid method (Δ - 49 grid points  o - 729 grid points).

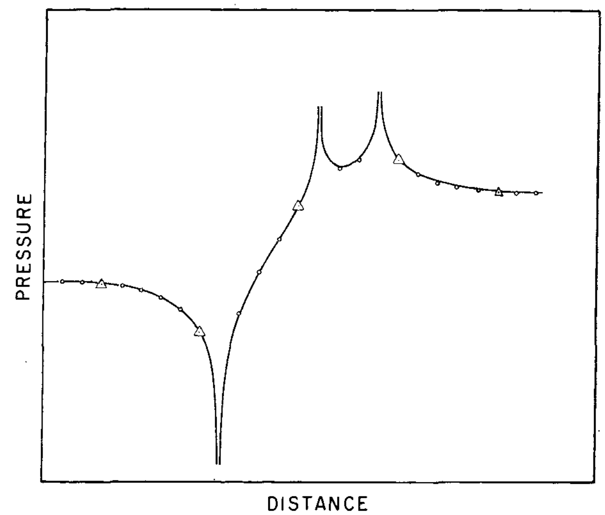


Fig. 5 - Comparison of pressure along the dotted line in Fig. 3 and pressure predicted by the improved method (Δ - 49 grid points  o - 729 grid points).

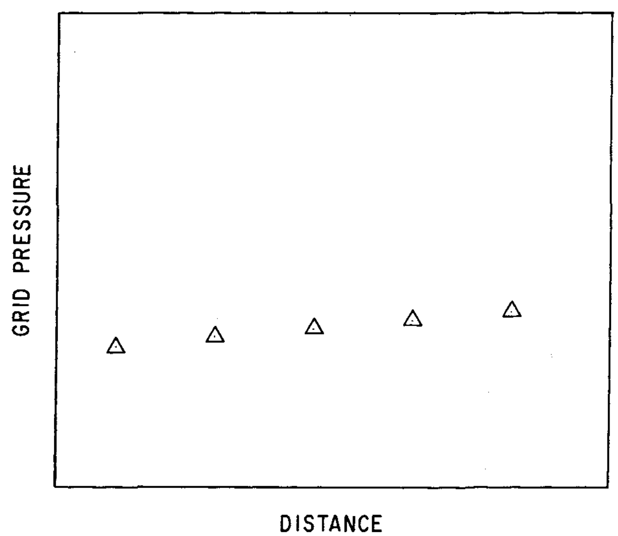


Fig. 6 - Grid pressure along the dotted line in Fig. 3 using the improved method. In this method grid pressure is that part not specified analytically.