Scale invariance, horizons, and inflation

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ABSTRACT

Maxwell equations and the equations of general relativity are scale invariant in empty space. The presence of charge or currents in electromagnetism or the presence of matter in cosmology are preventing scale invariance. The question arises on how much matter within the horizon is necessary to kill scale invariance. The scale-invariant field equation, first written by Dirac in 1973 and then revisited by Canuto et al. in 1977, provides the starting point to address this question. The resulting cosmological models show that, as soon as matter is present, the effects of scale invariance rapidly decline from \( \varrho > \varrho_c \). The absence of scale invariance in this case is consistent with considerations about causal connection. Below \( \varrho_c \), scale invariance appears as an open possibility, which also depends on the occurrence of inflation in the scale-invariant context. In the present approach, we identify the scalar field of the empty space in the scale-invariant vacuum context to the scalar field \( \psi \) in the energy density \( \varrho = \frac{1}{2} \dot{\psi}^2 + V(\psi) \) of the vacuum at inflation. This leads to some constraints on the potential. This identification also solves the so-called ‘cosmological constant problem’. In the framework of scale invariance, an inflation with a large number of e-foldings is also predicted. We conclude that scale invariance for models with densities below \( \varrho_c \) is an open possibility; the final answer may come from high redshift observations, where differences from the ΛCDM models appear.

Key words: dark energy – inflation – cosmology: theory.

1 INTRODUCTION

The laws of physics are generally not unchanged under a change of scale, a fact discovered by Galileo Galilei, as pointed out by Feynman (1963). Feynman also emphasized that the scale references are closely related to the material content of the medium. An empty Universe would be scale invariant, since there would be nothing to define a scale. Indeed, the Maxwell equations in absence of charge and currents are scale invariant. Similarly, the field equation of general relativity (GR), without cosmological constant, is scale invariant for empty space (Bondi 1990), a fact that is rarely mentioned.

Concerning cosmology and the evolution of the Universe, it is however not so clear which amount of matter is necessary to kill scale invariance. Would a single atom in the whole Universe be sufficient to define scales throughout at any time? The problem of scales is related to the existence of physical connection and causality, which relates different regions of the Universe. In this context, the horizons and the inflation play an important role. Here, we examine the domain of cosmological parameters, for which physical connection exists and likely enforces well-defined physical scales. For some cosmological conditions, there may be no causality connection and thus the door could be open to scale invariance. The question is closely related to the problems of horizons and inflation.

In Section 2, we first follow our main objective, i.e. to know what amount matter is killing scale invariance, on the basis of the scale-invariant field equations established by Dirac (1973) and Canuto et al. (1977), as well from the cosmological solutions of these equations. In Section 3, we examine the problem in terms of the particle – and event – horizons. We suggest that another definition, the physical horizon, is more meaningful and places constraints on the mean density of the Universe. In Section 4, we study the scalar field associated to scale invariance in relation with the scalar field of the inflation. We show that the scale-invariant context and equations permits the occurrence of an inflation with a large number of e-foldings. The conclusions and perspectives are given in Section 5.

2 THE ANSWER FOR SCALE-INVARIANT VACUUM COSMOLOGY MODELS

We first explore to what extent the existing studies on scale invariance provide some information on the above mentioned problem: what amount of matter is killing scale invariance? Dirac (1973) and Canuto et al. (1977) have established the basis of scale-invariant cosmology. In addition to the general covariance of general relativity, the field equations are also invariant upon the scale transformation of the form

\[
ds' = \lambda(x^\mu) ds.
\]

There, \( ds' \) is the line element in GR and \( ds \) in the scale-invariant space, which is that of Weyl’s Integrable Geometry (Weyl 1923; Dirac 1973). We will see below (Section 4.2) that the space–time is thus endowed with a scalar field \( \psi \) related to the above \( \lambda(t) \).
2.1 Brief recalls on the scale-invariant cosmology

We limit the recalls to the necessary minimum to follow the developments below. More details have been summarized by Canuto et al. (1977) and Maeder & Gueorguiev (2020). Scale-invariant first and second derivatives, scale-invariant Christoffel symbols, Riemann–Christoffel tensor and total curvature have been obtained. By using the convention that primed quantities are expressions based on the line element in GR, while quantities without prime are the expressions related to the Weyl’s Integrable Geometry and the corresponding relationship for the Ricci tensor (Maeder 2017a), we write

\[
R_{\mu\nu} = R'_{\mu\nu} - \kappa_{\mu\nu} - \kappa_{\nu\mu} - 2\kappa_{\mu}\kappa_{\nu} + 2g_{\mu\lambda}k^\lambda_{\nu} - g_{\mu\nu}k^\alpha_{\alpha} \tag{2}
\]

or explicitly

\[
R = R' + 6\kappa^\alpha_{\alpha} - 6\kappa_{\mu} \tag{3}
\]

The usual Einstein equations can be generalized to the general scale-invariant field equation (Dirac 1973; Canuto et al. 1977)

\[
R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -8\pi G T_{\mu\nu} - \Lambda g_{\mu\nu} \tag{4}
\]

or explicitly

\[
R'_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R' = -8\pi G T_{\mu\nu} - \lambda^2 \Lambda_E g_{\mu\nu}. \tag{5}
\]

This field equation is the generalization of Einstein equation to also account for scale invariance in addition to general covariance. We have in the scale-invariant framework \(\Lambda = \lambda^2 \Lambda_E\), \(G\) is the gravitational constant, taken as a true constant. The above equation contains additional terms depending on the metrical connection \(\kappa_{\mu\nu}\), related to the scale factor \(\lambda\),

\[
\kappa_{\mu\nu} = \frac{\partial \ln \lambda}{\partial x^\nu}. \tag{6}
\]

It is immediately seen that two successive scale transformations \(\lambda = \lambda^{(1)}\lambda^{(2)}\) result in additive expression for the corresponding metrical connection \(\kappa_{\mu\nu} = \kappa_{\mu\nu}^{(1)} + \kappa_{\mu\nu}^{(2)}\). For reasons of homogeneity and isotropy the scale factor \(\lambda\) in cosmology should depend on time only (Maeder 2017a), so that the only component of \(\kappa_{\mu\nu}\) is \(\kappa_{00}\). We have in particular,

\[
\kappa_{\mu\nu} = \kappa_{00}, \quad \frac{d\kappa_{00}}{dt} = \kappa_0 = -\frac{\dot{\lambda}}{\lambda}. \tag{7}
\]

In Weyl’s Integrable Geometry, \(\kappa_{\mu\nu}\) is playing a fundamental role alike the \(g_{\mu\nu}\). If \(\lambda\) is a constant, one is brought back to the usual equations of GR.

With the Friedmann–Lemaître–Walker–Robertson (FLWR) parametrized metric, one is lead to the following differential equations for cosmological models (Canuto et al. 1977):

\[
\frac{8\pi G p}{3} = \frac{a^2 \dot{a}}{a^2} + 2 \frac{\dot{\lambda}}{\lambda} \dot{\lambda} + \frac{\dot{\lambda}^2}{\lambda^2} - \frac{\Lambda_E \lambda^2}{3} \tag{7}
\]

\[
-8\pi G \rho = 2 \frac{a^2}{a^2} + 2 \frac{\dot{\lambda}}{\lambda} \dot{\lambda} + 4 \frac{\dot{\lambda}}{\lambda} \dot{\lambda} + \frac{\dot{\lambda}^2}{\lambda^2} - \frac{\Lambda_E \lambda^2}{3} \tag{8}
\]

These equations contain several additional terms with respect to the standard case, these are those with the scale factor \(\lambda(t)\) and its time derivatives. The equations also contain the Einstein cosmological constant \(\Lambda_E\) that corresponds to the energy density of the vacuum (Carroll92, Press & Turner 1992; see also Appendix B). In these equations, as well as in equation (5), \(\Lambda_E\) is multiplied by \(\lambda^2\), the product of the two represents the cosmological constant \(\Lambda\) in the scale-invariant space (Canuto et al. 1977). A new interpretation of the cosmological constant problem has been proposed within the multiverse approach of quantum cosmology. It reconciles the Planck-scale huge vacuum energy density predicted by quantum physics with the observed small value of \(\Lambda_E\) (Gueorguiev & Maeder 2020).

The field equation (5) and cosmological equations (8) are determined due to their gauge symmetry. The same problem appears in GR where the undeterminacy is resolved by the choice of a line element such as de Sitter, FLWR, etc, which defines the geometry of the system to study. Here, we have to choose some gauging condition to fix the scale factor \(\lambda\). Dirac (1973) and Canuto et al. (1977) were choosing the so-called ‘Large Number Hypothesis’ to fix the gauge. As the cosmological solutions depend on the choice of the gauge, this choice plays a key role, to some extent as the choice of the metric in GR. Here, in the scale-invariant framework, both the metric and the gauging condition play a major role.

We adopt as basic gauging condition the assumption of the scale invariance of the empty space. We notice that this choice is consistent with the remarks by Feynman (1963), that the occurrence of scale references are closely related to the matter content of the system. Thus, the hypothesis of the scale invariance of the empty space (at macroscopic scale) is quite justified. Also, as shown by Carroll et al. (1992) the usual equation of state for the vacuum \(p_{\text{vac}} = -c^2\rho\) is precisely the relationship permitting the vacuum density to remain constant for an adiabatic expansion or contraction. Thus, it is not so surprising that this gauging conditions leads to an analytical relation between the scale factor \(\lambda\) and the cosmological constant that represents the energy density of the vacuum.

Imposing scale invariance of the empty space means to set the Ricci tensor \(R_{\mu\nu} = 0\) and the energy-momentum tensor for matter \((T_{\mu\nu} = 0\) all to zero. By considering the surviving non-zero time and space components (Maeder 2017a; Maeder & Gueorguiev 2020) one arrives at the specific relations between the scale factor and the cosmological constant \(\Lambda_E\),

\[
\lambda^2 = \frac{\Lambda_E}{\lambda^2} \tag{9}
\]

or

\[
\frac{\dot{\lambda}}{\lambda} = 2 \frac{\lambda^2}{\lambda^2} = \frac{\Lambda_E}{\lambda^2}. \tag{10}
\]

The second group is a variant formulation obtained from the first one. The two equations (9) are just what remains from the time and space components of the general scale-invariant field equation (5) expressed for the empty space. It is also worth to emphasize that these equations give a new significance to the cosmological constant. In particular, the cosmological constant appears as defined by the relative variations of the scale factor (cf. the first one in equation 9).

As discussed in Maeder (2017a) the general solution of (10) is of the form \(\lambda(t) = a(t - b)^n + d\) and resolves in \(d = 0\) and \(n = -1\) which by setting \(t = t_0\) results in \(a = (t_0 - b)\lambda_0\). If the units are chosen so that \(\lambda_0 = 1\) then this provides interpretation of \(\lambda\) as the moment in the past where \(\lambda\) blows to infinity and \(a\) is then the time since then. By choosing this special moment to be at \(t = 0\) we can set \(b = 0\) and thus obtaining \(\lambda(t) = \lambda_0(t_0/t)\) as specific choice of time keeping. Thus, these equations impose a variation of the \(\lambda(t)\)-term like \(t^{-1}\), and we choose a normalization constant so that \(\lambda = 1\) at the present time \(t_0\). Thus, we have

\[
\lambda(t) = \frac{t_0}{t}. \tag{11}
\]

In the Appendix A, we further comment on this form of the scale factor, showing it is consistent with the most general possible expression for it.
Interestingly enough, Equations (9) and (10) lead to noticeable simplifications of equations (8):

\[
\frac{8\pi G\dot{a}}{3} = \frac{k}{a^2} + \frac{a^2}{a^2} + 2\frac{\dot{a}\lambda_0}{a\lambda} \quad (12)
\]

\[
-8\pi G \rho = \frac{k}{a^2} + 2\frac{\dot{a}}{a} + \frac{a^2}{a^2} + \frac{4\dot{a}\lambda_0}{a\lambda} \quad (13)
\]

A third equation may be derived from the above two,

\[
\frac{4\pi G}{3} (3p + \rho) = \frac{\ddot{a}}{a} + \frac{\dot{a}\lambda_0}{a\lambda} \quad (14)
\]

Since \(\dot{\lambda}/\lambda\) is negative, the extra term leads to a repulsive force also depending on the expansion rate. This is the force responsible for the acceleration of the expansion of the model Universe illustrated in Fig. 1.

The cosmological constant has now disappeared from the equations. In equation (7), the last two terms have cancelled, and in equation (8) the third and last two terms have done the same. The consequence is that the product \(\dot{\lambda}/\lambda\) now only appears multiplied by the Hubble expansion rate \(\dot{a}/a\). If the factor \(\lambda\) is constant, one is brought back to the Friedman equations. We will see from the solutions in Fig. 1 that, quite consistently, the range of variation of this term is larger for the lower density models and vanishes for models with density tending towards the critical one. The properties of these equations as well as their solutions have been discussed before (Maeder 2017a).

In the case of energy density dominated by radiation and relativistic matter, for flat scale-invariant models with \(k = 0\), analytical solutions for the expansion factor, the matter density, radiation density, and temperature have been obtained by Maeder (2019). The interesting point is that in the early phases, the main functions have the usual dependence on the age \(\tau = t - t_0\) of the Universe as measured from the initial time of the Universe at the Big Bang \(t_0\) when the scale factor is zero \(a(t_0) = 0\). In example, the expansion factor goes like \(a(\tau) \sim \tau^{1/2}\) and then the temperature like \(T \sim \tau^{-1/2}\). The run of other variables and the numerical coefficients of the different analytical relations near the origin are given for density parameters \(\Omega_m\) between 0.04 and 0.50 Maeder (2019).

In the case of flat scale-invariant matter dominated models with \(k = 0\), analytical solutions have been obtained by de Jesus (2018),

\[
a(t) = \left[\frac{t - \Omega_m}{1 - \Omega_m}\right]^{2/3}, \quad (15)
\]

in agreement with numerical solutions (Maeder 2017a). At present, the time \(t_0\) is fixed to \(t_0 = 1\) and the expansion factor to \(a(t_0) = 1\).

The density parameter \(\Omega_m\) and the critical density \(\rho_c\) are defined according to the usual expressions,

\[
\Omega_m = \frac{\rho}{\rho_c} \quad \text{with} \quad \rho_c = \frac{3H_0^2}{8\pi G} \quad (16)
\]

\(\Omega_m\) vary with time. It is generally considered at the present epoch and so is \(H_0 = H(t_0) = \dot{a}/a\). We also write

\[
\Omega_k = -\frac{k}{a^2H_0^2} \quad \text{and} \quad \Omega_{\lambda} = -\frac{2}{H_0} \left(\frac{\dot{\lambda}}{\lambda}\right)_0 = \frac{2}{H_0 t_0} \quad (17)
\]

With these definitions, the cosmological equation (12) leads to an expression that is valid at all times, since the beginning of time \(t = 0\), and in particular at the current time as well \((t = t_0)\),

\[
\Omega_m + \Omega_k + \Omega_{\lambda} = 1. \quad (18)
\]

The terms, respectively, are the contributions to the total energy density of the matter, space curvature, and scale factor \(\lambda\) energy densities as normalized to the critical density \(\rho_c\). In agreement with the usual practice, the \(\Omega\)-parameters of the models (for example in equation 15) generally represent the values at the present time \(t_0\).

Thus, to avoid unnecessary index clutter, we will not be using a sub-index 0 for the present values of these contributions, except for the traditional use of such index in \(H_0\), in favour of carefully paying attention to the time moment to be considered. In a similar manner, we will usually use units such that \(t_0 = 1\), unless we feel that the use of explicit \(t_0\) in the formulas is more appropriate.

### 2.2 Constraints on the mean model density

The scale-invariant models are themselves placing constraints on the validity of the scale invariance hypothesis. Following the Boomerang experiment (de Bernardis et al. 2000), we are considering flat Universe models with \(\Omega_k = 0\). From equation (18), we have then,

\[
\Omega_m + \Omega_{\lambda} = 1. \quad (19)
\]

Let us consider a sequence of models of increasing matter density with \(\Omega_m\) values from 0 to 1.0. For \(\Omega_m = 0\), the cosmological model shows a maximum relative variations of \((\dot{\lambda}/\lambda)\), corresponding to \(\Omega_m = 1\).

For increasing \(\Omega_m\), there is a range of decreasing values of \(\Omega_m\), and thus the relative variations of \(\lambda\) are more and more limited according to the above equation (19). In this domain, there are possible solutions of the scale-invariant equations (12) and (13). For the particular case \(\Omega_m = 1.0\) (at present), equation (19) implies that \(\Omega_m = 0\), which means that \(\dot{\lambda}\) is a constant (at present, but also at all times according to form of \(\lambda\)). Thus, we see that equations (12) and (13) become identical to the Friedman equations, and in this case we would get the same solution for the scale factor \(a(t)\) as function of \(t\), that is, \(a(t) \propto e^{3t/2}\) and \(a(t) \propto t^{1/2}\) for the matter and radiation eras, respectively.

For \(\Omega_m \geq 1\), the situation is completely different; one would have \(\Omega_{\lambda} < 0\), which according to equation (17) leads to

\[
\frac{2}{H_0 t_0} < 0. \quad (20)
\]

Such a condition would imply either a contracting Universe or a negative time, which although formally not impossible, do not correspond to the observed Universe. Thus, we conclude that scale invariance is problematic for a mean density larger than the critical density \(\rho_c\) (unless we are willing to consider local contraction with \(\dot{k}_0 = -\dot{\lambda}/\lambda < 0\) as a process of transfer of energy into the mechanical degrees of freedom related to the gravitational interactions, such as black hole formation).

Let us make a brief comment on the initial time of the Big Bang. It is given by \(a(t_0) = 0\),

\[
t_0 = t_0 \Omega_m^{1/3}, \quad (21)
\]

where we explicitly use \(t_0\) (usually equal to 1) in these expressions to indicate the correctness of the units. In the radiative era, the behaviour of \(a(t)\) close to the origin is \(t^{1/2}\), where \(t\) is the age. In a scale where \(t_0 = 1\), the differences between \(t_m\) given by equation (21) and by a detailed modeling of the radiative era is very small. For example, for \(\Omega_m = 0.20\), equation (21) gives \(t_m = 0.5848036\), while the complete model with the radiative era gives \(t_m = 0.5848043\) (Maeder 2019).

The reason for the very small difference is that the transition time \(t_{eq}\) from the matter to the radiation era occurs very closely to \(t_m\). In the present example, it occurs at \(t_{eq} = 0.5848066\). Thus, the difference of slopes in the time interval \((t_{eq} - t_m)\) only leads to a very tiny
difference in the time at which $a(t) = 0$. We may note that such properties are typical of current cosmological models.

The Hubble expansion rate $H(t) = \dot{a}/a$, based on (15), is

$$H(t) = \frac{2t^2}{t^2 - \Omega_m}, \quad \text{with} \quad H_0 = \frac{2}{1 - \Omega_m}. \quad (22)$$

From equation (22), a value $\Omega_m \geq 1$ would imply a contracting universe since the Hubble rate $H_0$ would be negative. This would be in severe contradiction with all recent determinations, e.g. the SHOES collaboration using the cosmic distance ladder method gives a value $H_0 = 73.2 \pm 1.3$ km s$^{-1}$ Mpc$^{-1}$ (Adam et al. 2021). There would be expanding solutions ($H_0 > 0$) only for $t < \Omega_m^{1/3}$ i.e. before $t_m$, the Big Bang. Even worse, the initial time would appear equal or larger than the present time! Thus, there is clearly no realistic model of the observed Universe, with $k = 0$ and a density equal or larger than the critical density $\varrho_c$. Thus, the overall conclusion is that for flat models, scale invariance is possible for mean densities below the critical density $\varrho_c$, and is forbidden above $\varrho_c$.\footnote{A recession velocity of 70 km s$^{-1}$ Mpc$^{-1}$ corresponds to $2.269 \cdot 10^{-18}$ s$^{-1}$, the inverse of an age of 4.408 $\cdot 10^{17}$ s or 13.97 Gyr. This leads to $\varrho_c = 9.207 \cdot 10^{-30}$ [g cm$^{-3}$].}

This is the answer of the general field equation; however, this does not necessarily mean that this is the case in the real Universe. The definite answer may only come from observations.

Fig. 1 shows the expansion rates in the matter dominated era for the scale-invariant and $\Lambda$CDM model results for $k = 0$ and various values of $\Omega_m$ between 0.01 and 0.80. For $\Omega_m = 0$, the empty scale-invariant model has a growth rate $\dot{a}(t) \sim t^2$. As seen above, a value $\Omega_m = 1$ implies $k = 0$ and the system of scale-invariant equations is brought back to the Friedman system with $\dot{a}(t) \sim t^{2/3}$ in the matter dominated era. Both sets of models containing matter start explosively near the origin with very high values of $H = \dot{a}/a$ and a positive value of $q = -\ddot{a}/(\dot{a})^2$, indicating braking. After the initial braking phase, both sets of models with $0 < \Omega_m < 1$ show an accelerated expansion, which after an inflection point for $q = 0$ goes on all the way. The inflection occurs later for higher densities. In the case of scale invariance with $k = 0$, the inflection point occurs at a time $t_{q=0} = (2 \Omega_m)^{1/3}$,\footnote{The curves are labeled by the values of $\Omega_m$. We notice that the differences between the two sets of models rapidly decline for increasing matter densities.}

$$t_{q=0} = (2 \Omega_m)^{1/3}, \quad (23)$$

and with an expansion factor,

$$a(t_{q=0}) = \left(\frac{\Omega_m}{1 - \Omega_m}\right)^{2/3}. \quad (24)$$

The differences between the SIV and $\Lambda$CDM models in Fig. 1 are rapidly reduced for increasing values of $\Omega_m$ up to 1.0. Above a density equal to about 30 percent of the critical density, the differences become very small. This clearly demonstrates how fast the effects of scale invariance disappear when the matter density progressively increases at very low densities below $\varrho_c$.

### 3 The Horizons and Their Limitation on Scale Invariance

In order to proceed further, we need to specify the notions of horizons for the scale-invariant models. Since scale invariance is likely prevented by causality relations, which permit gravity and light information to be transferred from one place in the Universe to another one. Thus, the occurrence or not of scale invariance may be related to the problem of horizons. We follow the definitions and properties of the horizons as clarified by Rindler (1956, 1969).

#### 3.1 The case of the particle-horizon

At a given age of the Universe, the finite velocity of light limits the distances from where we may receive electromagnetic signals. For a signal emitted in $r_1$ at time $t_1$ and received in $r = 0$ at time $t_0$, one has the following relation, obtained from the equation of photon

\[
rt = \frac{r^2}{c}.
\]

This is the particle-horizon $r = r_1$. This implies that, at $r = r_1$, the light signal is necessarily emitted at $t = t_1$. This is the answer of the general field equation; however, this does not necessarily mean that this is the case in the real Universe. The definite answer may only come from observations.
propagation $ds = 0$,
\[
f \int_0^{t_1} \frac{dr}{\sqrt{1 - kr^2}} = f \int_{t_1}^{\tilde{t}_1} \frac{dt}{a(t)}, \tag{25}
\]
where the FLRW metric has been used. If the integral on the right converges when $t_1$ tends to 0, i.e. at the origin, then the left integral also converges and $r_1$ tends towards a finite value $r_H$, called the particle-horizon (Rindler 1969). Thus, at a given time the limit $r_H$ separates the Universe in two parts, one from which we may receive information and one inaccessible. On the contrary, if the right integral diverges, $r_H$ tends towards infinity and signals may theoretically be received, present, from the whole Universe. As an example, for a law $a(t) \sim t^n$, if $n < 1$, the integral on the right is converging. As an example, the Einstein–de Sitter (EdS) model with $n = 2/3$ is converging (the same for $n = 1/2$). It has a particle horizon at a distance
\[
d_H(t_0) = 3c t_0. \tag{26}
\]
For a static Universe, the distance would just be $ct$, but due to the fast early expansion this distance is larger. The EdS model has a particle-horizon, meaning that at a time $t$ the light of objects more distant than $3ct$ has not yet reached us. The expansion rate $a(t) \sim t^{2/3}$ progresses' slower than the horizon that does it like $t$. It implies that as time goes more distant objects enter our horizon.

The expansion factor $a(t)$ for scale-invariant models is given by equation (15). Let us examine the situation for $0 < \Omega_m < 1.0$. It is sufficient to study the behaviour of equation (15) near the origin. We write the time near the origin as $t = \Omega_m^{1/3} + \delta t$, and get for $\delta t \rightarrow 0$,
\[
(t^3 - \Omega_m)\frac{2}{3/3} = \left(3\Omega_m^{2/3}\delta t + 3\Omega_m^{1/3}\delta t^2 + \delta t^3\right)^{2/3} \sim \left(3\Omega_m^{2/3}\delta t\right)^{2/3}. \tag{27}
\]
Thus, the exponent $n$ in $a(t) \sim t^n$ near the origin, but still in the matter dominated era, is $n = 2/3$, there the scale-invariant model behaves like the EdS model. In the example given in Section 2.2 for $\Omega_m = 0.20$, the transition to the radiation era occurs at a redshift $z = 4028$. In the radiation era, the behaviour of $a(t)$ is in $t^{1/2}$, the conclusion is the same in both cases. The integral is converging, implying that the scale-invariant models with $\Omega_m < \Omega_c$ have a particle-horizon.

### 3.2 The case of the event-horizon

The existence of a particle-horizon implies that there are domains of the Universe that cannot be observed now. However, some domains may progressively become accessible as time goes on. There could also be domains that will never become accessible, even after an infinite time. Let us again consider equation (25). Now, we consider the case where $t_0$ tends towards infinity. If the integral on the right is converging, there is a limit $r_0$ in the left integral, called the event-horizon (Rindler 1969), beyond which the events will never reach us. This occurs for $n > 1$ for an expansion rate $a(t) \sim t^n$. This means that the expansion is accelerating and, thus that some regions of the Universe presently visible (at least in theory) are progressively getting out of accessibility. On the contrary, if the integral on the right is diverging, $r_0$ tends towards infinity and the concerned models have no event-horizon: all the domains of the Universe will become accessible in the future. This occurs if $n < 1$, the horizon 'advances faster' than the expansion.

The scale-invariant models with $0 < \Omega_m < 1.0$, after an initial braking phase, experience an acceleration, alike the $\Lambda$CDM models. As matter, and also radiation as well, become diluted these models are progressively tending towards a behaviour in $t^2$ and $e^H$, respectively, and thus have an event-horizon.

In summary, the scale-invariant models with $0 < \Omega_m < 1.0$, have near the origin a behaviour alike the EdS model (in $t^{2/3}$ and $t^{1/2}$) and thus have a particle-horizon. For large enough times, after an inflection point (which depends on $\Omega_m$), they are accelerating and thus also have an event-horizon. Matter is entering the particle-horizon in the early phases and getting out in the later phases. Thus, these Universe models, on both sides of the arrow of time, have regions that are not causally connected. Apart from the question of inflation (Section 4), this lets open the door for scale invariance.

### 3.3 Physical conditions for scale invariance

We have to understand why the critical density $\Omega_c$ appears as a limit above that the effects of scale invariance are absent from cosmological models. Below $\Omega_c$, the equations are permitting scale invariance. It is an interesting possibility, but not a proof of existence.

Let us consider an observer in an homogeneous and isotropic medium of mean density $\Omega_c$ expanding according to the Hubble–Lemaître law with a present expansion rate $H_0$. At some limiting distance $R_{lim}$ from the observer, the recession velocity becomes equal to the light velocity,
\[
H_0 R_{lim} \simeq c. \tag{28}
\]
In the relativistic context, a recession velocity equal to $c$ corresponds to an infinite redshift, $R_{lim}$ is also a meaningful definition for a horizon, it may be called the 'physical horizon'. No gravity effect, no gravitational or electromagnetic waves from larger distances can reach the observer.

We note that $R_{lim}$ differs from the formal definition of the particle-horizon given in Section 3.1. For example, in the EdS model, one has with $H_0 = (2/3)(1/\Omega_m)$,
\[
\frac{2}{3} R_{lim} = \frac{d_h}{3 t_0}, \quad \text{thus} \quad R_{lim} = \frac{1}{2} d_h. \tag{29}
\]
In general, $R_{lim}$ is smaller than $d_h$. This is due to the fact that the particle-horizon formally goes back to time zero, where the initial expansion rate is extreme, tending towards infinity at the initial singularity (even without inflation). This is the case in the EdS model, as well as in the scale-invariant models with $0 < \Omega_m < 1.0$. Both the particle-horizon and event-horizon depend on model properties and in particular on those of the most extreme phases, the initial one for the particle-horizon, and the final one for the event-horizon. The physical horizon $R_{phys}$ has the advantage of not resting on a particular model of the extreme phases. It is model independent, apart from the fact that the Hubble law assumes an isotropic and homogeneous Universe in expansion. Thus, $R_{lim}$ may be considered as the meaningful, model independent, horizon for causality connection.

Over distances smaller than the physical horizon, causality connection is present, gravity effects are acting, electromagnetic waves are transmitted, etc. This means that for distances smaller than $R_{lim}$, scale invariance is likely forbidden since physical connection is present over the whole domain within this limit. For distances larger than $R_{lim}$, the physical connection is absent and scale invariance might be present. With the expression of the critical density $\Omega_c$ given in equation (16), the above equation (28) becomes
\[
\frac{8\pi}{3} G \Omega_c R_{lim}^2 \simeq c^2. \tag{30}
\]
If \( \rho_c \) is the mean density of the matter within \( R_{\text{lim}} \), i.e. if \( \rho_c = \frac{3M}{4\pi R_{\text{lim}}^3} \), the limiting distance \( R_{\text{lim}} \) is equal to the Schwarzschild radius \( R_S = 2GM/c^2 \). This would correspond to \( \Omega_m = 1 \) and it is also the highest possible matter density for an object in our Universe – a black hole. This can be seen even from classical Newtonian considerations presented by Freeman (1975). Equation (30) leads to the following value of the limit radius in terms of the critical density,

\[
R_{\text{lim}} \simeq c \sqrt{\frac{3}{8\pi G \rho_c}}.
\]

(31)

If the real mean density \( \rho \) of the medium is higher than the critical density \( \rho_c \), the same amount of mass (whatever it is) has a distribution in space, which is contained in a radius \( R \) smaller than \( R_{\text{lim}} \). Thus, the whole volume enclosed within radius \( R \) is causally connected. Thus, physical units are defined throughout. The above simple model suggests that a medium with \( \rho \geq \rho_c \) is unlikely to be scale invariant, since its various parts are physically connected, being enclosed within \( R_{\text{lim}} \). This throws some light on the above results from the scale-invariant equations, which showed the absence of SIV models with \( \Omega_m > 1 \), since such situation will imply an object denser than a black hole. Thus, according to (19) and (17) the conformal factor \( \lambda \) would be increasing, which is the reversal process of the usual behaviour of \( \lambda \) as seen in (11).

At the opposite, when \( \rho < \rho_c \), the considered volume cannot lie entirely within the limit \( R_{\text{lim}} \). This means that, while some parts are connected, causal connection by gravity and light is not present in the whole system. Moreover, as such systems are accelerating, some domains of the accessible Universe will escape in future. This lets open the possibility that space–time is scale invariant for \( \Omega_m < 1 \), a fact that would be in agreement with the existence of solutions to the equations (12) and (13), but as repeatedly mentioned this is not a proof.

The problem is also related to the cosmological history of the Universe. In this respect, near the origin, the expansion rate \( H \) of the models, whether \( \Lambda \)CDM, EdS, or scale invariant are diverging with an expansion faster than that of \( R_{\text{lim}} \). This would favour matter outside the horizon. Such matter outside the horizon is consistent with the black hole universe idea and may not possess many of the problems of the Standard Big Bang model while not necessarily requiring a long period of cosmological inflation (Easson & Brandenberger 2001), and even be also consistent with the multiverse ideas where global structure of the space–time contains an infinite sequence of black and white holes, vacuum regular cores, and asymptotically flat universes (Dymnikova et al. 2001). The region around the singularity at the center of a black hole would naturally provide confined high-energy density and therefore the needed high potential energy for inflation. It has been argued that the coupling between the spin of elementary particles and torsion in the Einstein–Cartan theory of gravity generates gravitational repulsion at extremely high densities in fermionic matter, approximated as a spin fluid, and thus avoids the formation of singularities in black holes. It may even undergo several non-singular bounces until it has enough matter to reach a size at which the cosmological constant starts cosmic acceleration with a finite period of exponential expansion (inflation) of such universe creation (Poplawski 2016). There are many such interesting open questions regarding the singularity. They are, however, beyond the scope of the present work and we now concentrate on the main question concerning the inflation, in particular whether this phase of incredibly fast explosion is compatible and predicted by scale-invariant equations. This critical question is examined next.

4 INFLATION, CONSERVATION LAW, AND SCALE INVARIANCE

The high isotropy of the cosmic microwave background (CMB) radiation has brought a problem for the various current models with a particle-horizon. Regions separated on the sky by an angle larger than about two degrees were outside their own horizons at \( z \sim 1100 \) on the last scattering surface. Thus, the very high isotropy, that the whole CMB sky is presenting, was difficult to explain. The inflation theory (Guth 1981) predicts an initial exponential growth of the initial Universe at Planck length by a factor \( e^{N} \), with typically \( N > 62 \) during the first \( 10^{-32} \) s or so [see reviews by Linde (1995, 1996, 2005) and Weinberg (2008)]. Thus, physical interactions between the different parts of the Universe were possible before the inflation. The inflation also accounts for the observed flatness of the Universe (and for the incredibly higher accuracy with which it had to be satisfied in early stages). Also, the absence of magnetic monopoles is considered as a consequence of the inflation. In addition, the inflationary Universe is the source of the spectrum of primordial fluctuations [Kofman, Linde & Mukhanov (1988); see also review by Coles & Lucchin (2002)]. We first examine the relations between the scalar field associated to the scale invariance and the scalar field of the inflation, and then whether the inflation is compatible and may also occur in the context of the scale-invariant equations.

4.1 The energy density of the vacuum and the cosmological constant

Let \( \ell \) be some constant line element in the space of GR. In the scale-invariant space, the corresponding line element \( \ell \) behaves as \( \ell = \ell /\ell(t) \), where the scale factor \( \ell \) is only a function of the cosmic time \( t \), as said above. The possible variations of the scale factor \( \lambda(t) \) may contribute to the energy density present in the empty space. If \( \lambda(t) \) varies, the energy associated to the length \( \ell \) in the empty space will be given by an expression related to its change

\[
\ell = \ell /\lambda \Rightarrow \ell^2 = \ell^2 /\lambda^2. \tag{32}
\]

The energy density \( \rho \) in the scale-invariant space is obtained by taking the above value by length unit. Thus, if there is no other source of energy in the empty space, its energy density \( \rho \) can be written as

\[
\rho \sim \frac{1}{2} \frac{\ell^2}{\ell^2} \quad \text{and thus} \quad \rho = \frac{1}{2} C \frac{\lambda^2}{\lambda^2}, \tag{33}
\]

where \( C \) is a proportionality constant, which has to be fixed in a consistent way with current definitions.

The Einstein cosmological constant \( \Lambda_E \) is related to the energy density \( \rho ' \) of the empty space in GR (Carroll et al. 1992),

\[
\Lambda_E = 8 \pi G \rho ', \tag{34}
\]

In the scale-invariant system of Weyl’s Geometry, as shown by the field equation (5), and the cosmological equations (7) and (8), the corresponding cosmological constant \( \Lambda \) is

\[
\Lambda = \lambda^2 \Lambda_E. \tag{35}
\]

This is in agreement with the behaviour of the coscalor expressing the relation between the vacuum density \( \rho \) in the Weyl’s space (denoted without prime) and \( \rho ' \) in the Riemann space (denoted with a prime; Maeder 2017a),

\[
\rho = \lambda^2 \rho '. \tag{36}
\]
From equations (9), \( \Lambda_E \) is related to \( \lambda \) and its derivatives by,

\[
\Lambda_E = 3 \frac{\dot{\lambda}^2}{\lambda^2}. \tag{37}
\]

From equations (34), (36), and (37), we get the consistent expression of the vacuum density in the scale-invariant system,

\[
\varrho = \frac{3}{8 \pi G} \frac{\dot{\lambda}^2}{\lambda^2}. \tag{38}
\]

Thus, we see that the two expressions (33) and (38) of the density of the empty space are consistent if

\[
C \frac{\dot{\lambda}}{\lambda} = \frac{3}{4 \pi G}. \tag{39}
\]

in the above expression (33). In Appendix B, we further comment on the expression (38) of the vacuum density in relation with the scale-invariant cosmological equations.

As a side remark, we note that the constancy of \( \Lambda_E \) also implies equations (10)

\[
\frac{d\Lambda_E}{dr} \sim \frac{\dot{\lambda}}{\lambda^2} \frac{\dot{\lambda}^3}{\lambda^3} - \frac{4}{\lambda^2} \frac{\dot{\lambda}}{\lambda} = 0 \Rightarrow \frac{\dot{\lambda}}{\lambda} = 2 \frac{\dot{\lambda}^2}{\lambda^2}. \tag{40}
\]

This is the first of equations (10), and this shows that the present definition of the vacuum density is consistent with the results of the field equations for the empty space.

Both \( \varrho \) and \( \Lambda \) (in the scale-invariant space) behave like \( 1/r^2 \) according to expression (11) based on the field equation of the vacuum. This implies that the energy density of the vacuum, and the cosmological constant \( \Lambda \), in the scale-invariant space become very large near the origin. For example at the Planck time \( t_0 = 5.39 \times 10^{-44} \) s, dominated by quantum effects, the cosmological constant would be a factor \( (\frac{3.55 \times 10^{17}}{5.39 \times 10^{16}})^2 \approx 6.4 \times 10^{121} \) larger than the value at the present cosmic age \( t = 13.7 \text{ Gyr} = 4.323 \times 10^{17} \) s. Thus, as such this may solve the so-called cosmological problem by viewing the Planck-seed universes and the \( \alpha \)-derivable universes as different stages of the same Universe rather than a disconnected universe (Güerogiu & Maeder 2020). In other words, the smallness of the Einstein cosmological constant \( \Lambda \) is naturally related to current age of the Universe, assuming that now \( \lambda = 1 \) by choice of units, because the solution (11) for (9) implies \( \Lambda_E = 3/t_0^2 \approx 1.6 \times 10^{-38} \text{s}^{-2} \).

4.2 The scalar field clock associated to the scale-invariant empty space

Instead of the above particular notations, it is appropriate to express the energy density of the empty space in term of a scalar field \( \psi \),

\[
\varrho = \frac{1}{2} C \psi^2 \quad \text{with} \quad \dot{\psi} = -\frac{\dot{\lambda}}{\lambda}. \tag{41}
\]

with the constant \( C \) given by equation (39). This relation expresses that the empty space is endowed with an energy density related to scale transformations. Relation (11) implies

\[
\dot{\psi} = \frac{1}{\lambda}. \tag{42}
\]

In other conditions, like at inflation, there could be some additional contribution to the energy density of the medium. We note that \( \psi \) is in fact equal to the time component of the metrical connection \( \kappa \) defined by equation (6) and present in the general scale-invariant field equation (5),

\[
\psi = \kappa_0, \quad \text{with} \quad \kappa_0 = -\frac{\dot{\lambda}(t)}{\lambda(t)}. \tag{43}
\]

The scalar field \( \psi \) is thus

\[
\dot{\psi} \sim \ln t, \quad \text{and} \quad t \sim e^{\psi} \tag{44}
\]

Near the origin \( t \to 0 \), \( \psi \to -\infty \) and \( \dot{\psi} \to \infty \). Thus, near the origin the time grows very fast as a function of the field \( \psi \) which evolves at a much slower pace and may thus be the appropriate 'timekeeping field' near the origin as discussed by Weinberg (1989).

We see below that field \( \psi \) can be identified with the 'inflaton', the rolling field during inflation.

4.3 The inflation

Following the first version of inflation by Guth (1981), there were several inflation theories resting on different hypotheses (Linde 1995, 1996, 2005), e.g. phase transition in the vacuum, breaking of grand unified theory, specific conditions producing eternal inflation, and chaotic inflation, etc. Over the years, it has emerged that the key condition to have inflation, is the presence of a scalar field \( \psi \) (called the inflaton) and a potential \( V(\psi) \) that initially contains most of the matter-energy of the Universe (Weinberg 2008). At the end of the inflation, the decay of the potential \( V(\psi) \) is leading to the baryogenesis (Linde 1995), the further evolution of which will form the present Universe. The condition for the existence of an inflation is that at early times \( V(\psi) \) is large and flat. Then, there are various scenarios (Linde 1995).

Typically, the scalar field 'rolls' very slowly at first down this potential, so that the Hubble constant decreases only slowly, and 'the universe experiences a more-or-less exponential inflation before the field changes very much' (Weinberg 2008). However, as pointed out by Brandenberger (2017), the whole picture in its different expressions are not free from remaining problems.

The energy density \( \varrho \) and the pressure \( p \) of the vacuum state during inflation are resulting from the contributions of a term of \( \dot{\psi}^2 \) due to the scalar field \( \varphi \) and of an additional potential \( V(\psi) \), which is the dominant one (Linde 1995; Weinberg 2008),

\[
\varrho = \frac{1}{2} \dot{\psi}^2 + V(\varphi), \quad \text{and} \quad p = \frac{1}{2} \dot{\psi}^2 - V(\varphi). \tag{45}
\]

We propose to relate the scalar field \( \varphi \) of the inflation (the 'inflaton') to the above field \( \psi \) associated with the scale invariance of the ordinary empty space and to examine the consequences of this identification. We establish the correspondence \( \sqrt{C} \dot{\psi} \leftrightarrow \varphi \) and thus we write for the energy density and pressure at inflation,

\[
\varrho = \frac{1}{2} C \left( \dot{\psi}^2 + U(\psi) \right), \quad \text{and} \quad p = \frac{1}{2} C \left( \dot{\psi}^2 - U(\psi) \right). \tag{46}
\]

We also have the correspondence for the potential with \( U(\psi) \leftrightarrow (1/C) \dot{\psi}^2 \).

Usually, the potential \( V(\psi) \) is a potential of high energy during inflation. It is supposed not to vary too much during inflation and various forms have been proposed for it. This also applies to the potential \( U(\psi) \) that only differs by a (large) constant factor. The properties of \( V(\psi) \), and thus of \( U(\psi) \), are supposed to lead to a Hubble–Lemaître constant \( H_{\text{inf}} \) that does not vary too much during some interval of \( \psi \) and thus tends to produce the exponential growth of the Universe, characteristic of the inflation. We note that the relation of \( \psi \) with the time \( t \) would naturally explain why the system is 'rolling' slowly along the flat potential.

As emphasized by Weinberg (2008), the large value of the energy density during inflation does not necessarily rule out the classical treatment of gravitation according to GR. The quantum gravitational effect may be neglected under the assumption that the energy density is already much less than the Planck energy. If the conditions for the applicability of GR are satisfied, as is usually considered, the
conditions for the applicability of equation (5) are also met. Once more, it does not mean that Nature in her wisdom has done it, but it is a possibility to be explored. The point to be verified now is whether the scale-invariant equations and the above identification also permits an inflation.

In the scale-invariant context, the equation of energy conservation contains an additional term with respect to the standard case (Maeder 2017a),

$$\frac{d(\rho a^3)}{da} + 3pa^2 + (\rho + 3p)\frac{a^3}{\lambda} \frac{d\lambda}{da} = 0.$$  \hfill (47)

We want to write it in terms of the inflation quantities $\psi$ and $U$. For this purpose, it is preferable to start from the following form,

$$\dot{\psi} + 3\frac{\dot{a}}{a} (\rho + p) + \frac{\dot{\lambda}}{\lambda} (\rho + 3p) = 0.$$  \hfill (48)

There, we see that the quantities $\rho$ and $p$ only appear linearly, thus the constant $C$ appears as a multiplicative factor in the left member of this equation. This implies that we can ignore it and thus we have,

$$\dot{\psi} + U' + 3H_{\text{infl}} \dot{\psi} - 2(\psi^2 - U) = 0,$$  \hfill (49)

which differs from the usual Klein–Gordon equation (Kofman et al. 1988; Linde 1995) by the presence of the last two terms. The expansion rate during inflation, i.e. when there is a large potential $V(\psi)$ contributing to the energy density of the vacuum, denoted here by $H_{\text{infl}}$. Since $\psi = -\psi'$, the above equation simplifies to

$$U' + 3H_{\text{infl}} \dot{\psi} - 3\psi^2 + 2U = 0.$$  \hfill (50)

Thus, the expansion rate behaves like,

$$H_{\text{infl}} = -\dot{\psi} - \frac{2U}{3\psi} - \frac{U'}{3\psi}.$$  \hfill (51)

The condition to have an inflation with an exponential growth during a very short time implies that the relative change $|H_{\text{infl}}/H_{\text{infl}}|$ during a time $1/H_{\text{infl}}$ should be much less than unity (Weinberg 2008), i.e.

$$|H_{\text{infl}}| \ll H_{\text{infl}}^2.$$  \hfill (52)

The model Universe will thus follow a de Sitter-like exponential expansion.

For our model (51), the time derivative of the expansion rate is

$$\dot{H}_{\text{infl}} = -\frac{2U'\dot{\psi}}{3\psi} + 2U'\frac{\dot{\psi}}{3\psi} - \frac{U''\dot{\psi}}{3\psi} + \frac{U'}{3\psi},$$  \hfill (53)

which simplifies to

$$\dot{H}_{\text{infl}} = -\frac{\psi^2}{3} - \frac{2U}{3} - U' - \frac{U''}{3}.$$  \hfill (54)

We now examine the above condition for the potential $U(\psi)$, for which several forms are existing (Linde 1995, 2005). Let us consider a typical well-known exponential form used in analytical treatment of the inflation (Weinberg 2008),

$$U(\psi) = g e^{\mu \psi},$$  \hfill (55)

where $g$ and $\mu$ are constant, $g$ being positive and $\mu$ generally negative. The product $|\mu \psi|$ needs to be much smaller than 1, as required by the condition that the potential $U(\psi)$ does not vary too much over the inflation period, during which the time $\tau$ varies by less than $10^{-32}$ s. Moreover, we recall that $\psi = \ln(t)$, so that even if $t$ changes by a few powers of 10, $\psi$ changes only by a few unities. For now, we have

$$U'(\psi) = g \mu e^{\mu \psi}, \quad U'' = g \mu^2 e^{\mu \psi}.$$  \hfill (56)

Then relation (44) between $\psi$ and the time $t$ gives $e^{\mu \psi} = e^{\mu \ln t} = t^\mu$, and therefore:

$$U = g t^\mu, \quad U' = g \mu t^{\mu - 1} \quad \text{and} \quad U'' = g \mu^2 t^\mu.$$  \hfill (57)

With the above expressions for the potential and its derivatives, the expansion rate (51) becomes

$$H_{\text{infl}} = \frac{1}{t} - \frac{(2 + \mu)g}{3} t^{\mu - 1}.$$  \hfill (58)

At inflation, the time $t$, expressed in the scale where the present time $t_0 = 1$, is a very small value of the order of $4.4 \cdot 10^{-49}$. For values of $\mu > -1$, the second term on the right in the expression of $H_{\text{infl}}$ would vanish near the origin. Thus, only the first term in equations (58) would be significant. For $\mu = -1$, this second term would be a negative constant of order $-g/3$, it would contribute negatively, but will be dominated by the first term $1/t$ for sufficiently small $t$ near zero. For $\mu = -2$, the second term would vanish, while for $\mu$ more negative than $-2$, it contributes positively and dominates. For the time derivative, we have from equation (54),

$$H_{\text{infl}} = \frac{1}{t^2} - \frac{(\mu + 2)(\mu + 1)g}{3} t^{-\mu - 2}.$$  \hfill (59)

There, the same kind of remarks may be made. For $\mu$ more negative than $-2$, the second term on the right dominates. It is also negative and therefore provides graceful exit from inflation due to its slowing down effect.

Let us consider negative values of $\mu < -2$ that correspond to a typical exponentially decreasing potential in inflation (Weinberg 2008). As seen above, the second terms in both $H_{\text{infl}}$ and its derivative dominate. The critical ratio (52) for the occurrence of inflation becomes with (58) and (59),

$$\frac{|H_{\text{infl}}|}{H_{\text{infl}}^2} = \frac{3(\mu + 1)}{g(\mu + 2)} t^{-\mu - 2}.$$  \hfill (60)

For example, for the case $\mu = -4$, we would get

$$H_{\text{infl}} \approx \frac{2g}{3t^2}, \quad |H_{\text{infl}}| \approx \frac{2g}{t^2}, \quad \text{thus,} \quad \frac{|H_{\text{infl}}|}{H_{\text{infl}}^2} \approx \frac{9t^2}{2g} \ll 1.$$  \hfill (61)

The condition is satisfied since the potential $U(0) = g$ near the origin is a finite quantity while the time becomes vanishingly small. Thus, the relative variation of $H_{\text{infl}}$ over a time $1/H_{\text{infl}}$ is very small. This ensures a de Sitter-like exponential growth of the ‘radius’ $a(t)$ of the Universe. Notice that after a time $t \approx \sqrt[2g/9]{t}$ the inflation is smoothly over.

Let us consider that inflation starts at an initial time $t_1$, larger than the Planck time, and is ending at time $t_2$ that marks the beginning of the so-called phase of reheating, where the energy density of the potential $U(\psi)$ starts being turned to baryogenesis. We have

$$a(t_2) = \exp\left[ \int_{t_1}^{t_2} H_{\text{infl}} dt \right] = \exp\left[ -\int_{t_1}^{t_2} g \frac{(\mu + 2)}{3} t^{\mu + 1} dt \right]$$

$$a(t_2) = \exp\left[ \frac{g}{3} \left( \frac{t^{\mu + 2}}{t_2^{\mu + 2}} - \frac{t^{\mu + 2}}{t_1^{\mu + 2}} \right) \right].$$  \hfill (62)

For example, with $\mu = -4$, we would have

$$a(t_2) = \exp\left[ \frac{g}{3} \left( \frac{1}{t_2^2} - \frac{1}{t_1^2} \right) \right].$$  \hfill (63)

The constant $g$, which represents the maximum of the potential $U$, has a large value in current inflationary models. Interestingly enough, the way the time intervenes in equation (62), in $t^{\mu + 2}$, produces a strong multiplication factor of the effect of the time interval for $\mu$-values $\mu \leq -3$. Thus, depending on the $\mu$ value, it would even
not be necessary that the potential $U$ is large to ensure a large number of e-foldings during inflation. This is an interesting feature of inflation in the context of scale-invariant models with a potential of the form $U(\psi) = e^{a\psi}$. Another specific feature is that the field $\psi$, the inflaton, is related to time by $\psi = \ln t$, which ensures that the model Universe is slowly ‘rolling down’ the potential. In phases following the inflation, when $V(\psi)$ and $U(\psi)$ start changing significantly, the inflation comes to an end by several possibilities (Linde 1995, 2005). At some stage, the energy density of the potential $C U(\psi)$ is turned into the baryogenesis by the decay of supermassive Grand Unified Theory bosons producing the baryon asymmetry (Kolb, Linde & Riotto 1996; Linde 1996).

The above developments show that the scale-invariant equations do not substantially modify the occurrence of the inflation, even if the usual scalar field $\phi$ has been replaced by the scalar field $\psi$, with which the energy density of the ordinary scale-invariant vacuum is expressed. Although the scale-invariant cosmological equations and the energy conservation contain some additional terms, an exponential growth may be ensured with a high number of e-foldings followed by a graceful exit from inflation.

 Coming back to our main concern about the validity domain of scale invariance, we can say that scale invariance does not prevent the existence of inflation. Whether an inflation compatible with the equations of scale invariance permits scale invariance in the resulting Universe remains an open question. There is certainly room for philosophical discussions as to whether scale invariance was present or not in the initial vacuum state before inflation. The only thing we may confirm here is that the mechanism of inflation could also be working in the scale-invariant context.

Thus, the above study does not lead to a change of the previous conclusions we got from the discussion of the physical horizon. Scale invariance is forbidden, or at least has unclear yet interpretation, in cosmological models with $\Omega_m > 1$. However, this may be also the key to reconcile conformal cyclic cosmology (Penrose 2006, 2012) and black hole cosmology (Pathria 1972; Dymnikova et al. 2001; Poplawski 2016; Oshita & Yokoyama 2018) by providing the needed high potential energy density $V(\psi)$ via the confined central singularity of a black hole. The question of scale invariance at low cosmological densities ($\Omega_m < 1$) is an open possibility.

The answer may come from observations, in particular at large redshifts where significant differences with the $\Lambda$CDM models are predicted. Comparisons have been performed in the $m - z$ diagram for the supernova data by Betoule et al. (2014) up to redshift $z = 1$; there the differences between scale-invariant and $\Lambda$CDM models are too small (Maeder 2017a; Maeder & Gueorguiev 2020). Another analysis has been made with the data by Lusso et al. (2019), where Supernova observations have been extended by data from quasars and gamma-ray bursts up to redshift $z = 7$. There some differences appear between the $\Lambda$CDM and scale-invariant models. The scatter of the data prevents a conclusion, but the situation is not far from allowing a decision in near future (Maeder & Gueorguiev 2020).

5 CONCLUSIONS

We have shown on the basis of the scale-invariant equations that scale invariance is forbidden, or at least has unclear yet interpretation, for cosmological models with a mean density equal or above the critical density $\Omega_c$. The absence of scale invariance above $\Omega_c$ is confirmed by the fact that in this case the Universe models have their mass-energy distribution entirely contained within the limiting radius $R_{\text{lim}}$, which may coincide with the Schwarzschild radius, for causal connection.

On the contrary, models with densities lower than $\Omega_c$ have cosmological solutions according to the general scale-invariant field equations by Dirac (1973) and Canuto et al. (1977). The resulting models suggest that the effects of scale invariance undergo a fast decline from density $\rho = 0$ to $\rho_c$. Such low density models have some of their parts not causally connected. This lets open the possibility that scale invariance is present. However, the answer is also related to the cosmological history of the Universe, and in particular to the inflation.

We have examined the occurrence of the inflation in the scale-invariant context by identifying the scalar field $\psi$ of the inflation with the scalar field $\psi$ associated to the energy density of the scale-invariant space. This identification also solves the so-called ‘cosmological constant problem’. Due to the properties of $\psi$, this leads to some change in the equations. For some forms of the potential $U(\psi)$, the scale-invariant inflation is even favoured with a high number of e-foldings. We conclude that the occurrence of scale invariance for models with $\Omega_m \leq 1$ remains an open possibility, the answer of which may come from high redshift observations.

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DATA AVAILABILITY

No new data were generated or analyzed in support of this research.

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APPENDIX A: THE GAUGE FACTOR $\lambda$.

The most general solution for $\lambda$ with $\lambda(t_0) = 1$ has the form

$$\lambda = \frac{1}{1 + \kappa_0 (t - t_0)},$$

(A1)

with $\kappa_0$ given by the expression

$$\kappa_0 = -\frac{1}{\lambda} \frac{d\lambda}{dr} \bigg|_{r = 0}.$$

(A2)

This solution and the one given by equation (11) are equivalent. We had searched for a solution of equation (9) of the general form (Maeder 2017a),

$$\lambda = a(t - b)^n + d.$$

(A3)

The equation (9) imposes $d = 0$ and $n = -1$. Thus, we have

$$\lambda = \frac{a}{t - b}.$$

(A4)

The above solution (A1) can also be written as

$$\lambda = \frac{1}{\kappa_0 [t - (t_0 - \frac{1}{\kappa_0})]}.$$

(A5)

Thus, we have the correspondences $a = \frac{1}{\kappa_0}$ and $b = (t_0 - \frac{1}{\kappa_0})$. As mentioned in Maeder (2017a), we may choose $a$, while any value of $b$ will satisfy the equations. We note that if $\lambda = t_0/t$ and $t_0 = 1$ we get with the expressions (A1) and (A2)

$$\kappa_0 = -\frac{1}{\lambda} \frac{d\lambda}{dr} \bigg|_{r = 0} = t_0(1/t_0^2) = 1/t_0 \equiv 1.$$

(A6)

Thus, we verify that this is consistent with $a = 1$ and $b = 0$. This means that the solution (11) and the one given by equations (A1) and (A5) are identical.

APPENDIX B: THE DEFINITION OF THE VACUUM DENSITY

In the usual cosmological equations from general relativity,

$$8\pi G \rho' = \frac{k}{a^2} + \frac{\dot{a}^2}{a^2} - \frac{\Lambda_E}{a^2} = \frac{\dot{a}^2}{a^2} - \Lambda_E,$$

(B1)

$$-8\pi G p' = \frac{a^2}{a^2} + \frac{2}{a^2} + \frac{\Lambda_E}{a^2}.$$

(B2)

the cosmological constant $\Lambda_E$ expressing the energy density and pressure of the vacuum, explicitly appear in the equations as an additive term with the appropriate sign. It is important to emphasize that $\rho$ and $p$ are the density and pressure without the vacuum contribution. This leads to the identification,

$$\Lambda_E = 8\pi G \rho_{\text{vac}}', \quad \text{and} \quad p_{\text{vac}}' = -\rho_{\text{vac}}'.$$

(B3)

where $\rho_{\text{vac}}'$ and $p_{\text{vac}}'$ are the pressure and density of the vacuum, while $p$ and $\rho$ refer here to the matter and relativistic contributions. Thus, the equation of state is verified. In the scale-invariant forms, the density $\rho_{\text{vac}}$, pressure $p_{\text{vac}}$, and $\Lambda_E$ are the same as in GR, but multiplied by $\lambda^2$ (Canuto et al. 1977).

Now, if we consider the scale-invariant equations (12) and (13) (obtained with the assumption that the vacuum space is scale invariant), the vacuum effects expressed by $\lambda$ and its derivative now only appear as multiplicative factor of the expansion rate $H = a / a$. Thus, the last terms of these equations cannot be identified with the vacuum density or pressure, since they are containing effects from both the vacuum and the dynamical expansion.

As an illustration, let us take $k = 0$ and write equation (12) as

$$H^2 = \frac{8\pi G}{3} \rho - 2H \dot{\lambda}.$$  (B4)

If we would interpret the last term in this equation as the vacuum density, we would get

$$\rho_{\text{vac}} = \frac{3}{8\pi G} H \dot{\lambda}.$$  (B5)

which, with $H = -2\dot{\lambda} / \lambda$, is four times the (true) one given by (38). Proceeding similarly, we get $p_{\text{vac}}$ from equation (13),

$$p_{\text{vac}} = \frac{4}{8\pi G} \frac{H \dot{\lambda}}{\lambda} = \frac{H \dot{\lambda}}{2\pi G} = -\frac{2}{3} \rho_{\text{vac}}.$$  (B6)

Thus, this process is not consistent with the equation of state of the vacuum, a situation that results from the fact that the last terms in equations (12) and (13) are not just the density and pressure of the vacuum. Thus, we should keep the direct estimate given by equation (38),

$$\rho_{\text{vac}} = \frac{3}{8\pi G} \frac{\dot{\lambda}^2}{\lambda^2},$$  (B7)

which verifies the equation of the vacuum state, and does not combine it with the energy of dynamical effects.

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