Contributions to the non-linear integrated Sachs–Wolfe effect: Birkinshaw–Gull effect and gravitational self-energy density

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ABSTRACT
In this paper, we recompute contributions to the spectrum of the non-linear integrated Sachs–Wolfe (iSW)/Rees–Sciama effect in a dark energy cosmology. Focusing on the moderate non-linear regime, all dynamical fields involved are derived from the density contrast in Eulerian perturbation theory. Shape and amplitude of the resulting angular power spectrum are similar to that derived in previous work. With our purely analytical approach we identify two distinct contributions to the signal of the non-linear iSW effect: the change of the gravitational self-energy density of the large-scale structure with (conformal) time and gravitational lenses moving with the large-scale matter stream. In the latter we recover the Birkinshaw–Gull effect. As the non-linear iSW effect itself is inherently hard to detect, observational discrimination between its individual contributions is almost excluded. Our analysis, however, yields valuable insights into the theory of the non-linear iSW effect as a post-Newtonian relativistic effect on propagating photons.

Key words: methods: analytical – cosmic background radiation – large-scale structure of Universe.

1 INTRODUCTION
The tiny fluctuations in the temperature of the cosmic microwave background (CMB), its primary anisotropies, reflect the state of the Universe approximately 400 000 yr after the big bang. Photons emanating from the last scattering surface, however, interact on their way to today’s observer with the intervening large-scale structure (LSS), thereby altering this picture of the early Universe. One of these secondary anisotropies is the integrated Sachs–Wolfe (iSW) effect (Sachs & Wolfe 1967). The interaction of CMB photons with time-evolving gravitational potentials in the LSS causes additional fluctuations in the CMB temperature. While being blueshifted when entering a potential well the photons also experience a redshift on their way out of the potential. A net change in temperature results when blueshift and redshift do not compensate.

Being a secondary effect, the signal of the iSW effect is quite small in comparison to the CMB temperature itself. Its importance, however, arises from its dedicated sensitivity to cosmic fluids with non-vanishing equation of state (Crittenden & Turok 1996) promoting it as a valuable tool in the investigation of dark energy and non-standard cosmologies (Lue, Scoccimarro & Starkman 2004; Zhang 2006), as well as the Universe’s spatial curvature (Kamionkowski & Spergel 1994).

Exploiting its strong correlation with the galaxy distribution the iSW effect has been detected with high significance by various groups (Boughn & Crittenden 2004; Vielva, Martínez-González & Tucci 2006; McEwen et al. 2007; Giannantonio et al. 2008). The constraints they derive for the matter density \(\rho_\text{m} \) and the dark energy equation of state \(w\) give support to the Λ cold dark matter (LCDM) model, which describes a spatially flat universe mainly constituted by cold dark matter (CDM) and cosmological constant Λ.

The non-linear iSW or Rees–Sciama (RS) effect (Rees & Sciama 1968; Seljak 1996; Cooray 2002; Schäfer & Bartelmann 2006) refers to any temperature anisotropy caused by non-linearly evolving gravitational potentials. In contrast to the linear iSW effect it is a small-scale phenomenon dominating the total iSW signal on multipoles \(\ell \geq 100\) but is nevertheless hard to detect because of the primary CMB (Cooray 2002; Schäfer, Kalovichours & Heisenberg 2011).

Resulting from non-linear structure formation the statistics of the RS effect is necessarily non-Gaussian. The non-Gaussianities induced by the RS effect, however, are small. Spergel & Goldberg (1999) showed that the CMB bispectrum due to the RS effect is undetectable. Also mixed polyspectra of the non-linear iSW effect temperature perturbation and the tracer galaxy density field, as recently investigated by Jürgens & Schäfer (2013) in continuation of the work of Schäfer (2008), do not reach sufficiently high signal-to-noise ratios. Therefore, stacking methods (Granett, Neyrinck & Szapudi 2008) as an alternative to a statistical detection of the RS effect are of great interest.
2 COSMOLOGY

2.1 Dark energy cosmologies

In a spatially flat Friedmann–Lemaître–Robertson–Walker universe, the time evolution of the scale factor $a$ is governed by the Hubble function $H(t) = \frac{\dot{a}}{a}$ through
\[
H^2(a) = \Omega_{a}a^{3} + (1 - \Omega_{m})\exp\left(\int_{a}^{1} d\log a \ [1 + w(a)]\right) .
\]

The expansion of the homogeneous background is thus completely described by the matter content $\Omega_{m}$ (in units of the critical density) and the equation of state of the dark energy fluid $w(a)$. Following Chevallier & Polarski (2001) the latter can be phenomenologically parametrized as
\[
w(a) = w_{0} + (1 - a)w_{a} ,
\]

covering a huge variety of different dark energy models. Setting $w_{0} = -1$ and $w_{a} = 0$ one recovers the equation of state of a cosmological constant.

Since photons follow null geodesics, conformal time $d\tau = dt/a$ and comoving distance $\chi$ can be used interchangeably allowing for a simple relation to the scale factor:
\[
\chi = c\int_{a}^{1} \frac{da}{a^{2}H(a)} .
\]

Naturally (comoving) distances are measured in units of the Hubble distance $\chi_{H} = c/H_{0}$ which sets the scale up to which Newtonian gravity is applicable.

2.2 CDM power spectrum and linear structure growth

The fluctuations in the CDM component are described by the density contrast:
\[
\delta(x) \equiv \frac{\rho_{m}(x) - \bar{\rho}_{m}}{\bar{\rho}_{m}} \quad \text{with} \quad \bar{\rho}_{m} = \Omega_{m}\rho_{\text{crit}} .
\]

In linear theory $\delta$ is a statistically homogeneous and isotropic Gaussian random field and thus completely characterized by its power spectrum:
\[
\langle \delta(k)\delta^{*}(k') \rangle = (2\pi)^{3}\delta_{D}(k - k')P_{\delta}(k) .
\]

For the power spectrum we choose a power law modulated by an appropriate transfer function, i.e.
\[
P_{\delta}(k) \propto k^{n_{s}}T^{2}(k) ,
\]

for which we use the fit proposed by Bardeen et al. (1986):
\[
T(q) = \frac{\log(1 + 2.34q)}{2.34q} \left[ 1 + 3.89q + (16.1q)^{2} + (5.46q)^{3} + (6.71q)^{4} \right]^{-(1/4)} .
\]

In this formula the wavenumber $q = k/\Gamma$ is rescaled by the shape parameter
\[
\Gamma = \Omega_{m}h \exp\left(1 + \frac{\sqrt{2}h}{\Omega_{m}}\right) ,
\]

in order to account for a non-vanishing baryon density $\Omega_{b}$ (Sugiyama 1995). Finally, the amplitude of the power spectrum is set by the variance of the density contrast $\sigma_{\delta}^{2}$ smoothed on the scale $R = 8 \text{ Mpc} h^{-1}$:
\[
\sigma_{\delta}^{2} = \int_{0}^{\infty} \frac{dk}{2\pi^{2}} k^{2}P_{\delta}(k)W_{k}^{2}(kR) ,
\]

where $W_{k}(y) = j_{1}(y)/y$ is the Fourier transform of a spherical top hat function $j_{c}(y)$ denoting the $c$th spherical Bessel function of the first kind (Abramowitz & Stegun 1972).

During linear evolution the time dependence of the density contrast is completely encapsulated in the growth function $D_{s}(\eta)$ so that $\delta(k, \eta) = D_{s}(\eta)\delta_{D}(k)$ (normalized to unity today). It is obtained by solving the growth equation (Turner & White 1997; Wang & Steinhardt 1998; Linder & Jenkins 2003)
\[
\frac{d^{2}}{da^{2}}D_{s}(a) + \frac{1}{a} \left( 3 + \frac{d\log H}{d\log a} \right) \frac{d}{da}D_{s}(a) = \frac{3}{2\sigma_{\delta}^{2}}\Omega_{m}(a)D_{s}(a) .
\]

3 NON-LINEAR INTEGRATED SACHS–WOLFE EFFECT

The relative change in temperature CMB photons experience when they traverse time-evolving gravitational potentials on their way from the last scattering surface to today’s observer along the direction $\hat{n}$ is given by the line of sight integral (Sachs & Wolfe 1967):
\[
\tau(\hat{n}) \equiv \frac{\Delta T_{\text{iSW}}(\hat{n})}{T_{\text{CMB}}} = \frac{2}{c^{2}}\int_{0}^{z_{\text{hs}}} d\chi \frac{\partial}{\partial n} (\Phi(\hat{n}, \eta)) .
\]

The gravitational potential is directly related to the density fluctuations via the (comoving) Poisson equation
\[
\Delta \Phi(x, \eta) = \frac{3}{2} H_{0}^{2}\Omega_{m}(\delta(x, \eta)) .
\]
Its time derivative can be most easily accessed in Fourier space where the Poisson equation is readily inverted:

$$\frac{\partial}{\partial \eta} \Phi(k, \eta) = -\frac{3}{2} H_0^2 \Omega_m \frac{1}{k^2} \frac{\partial}{\partial \eta} \delta(k, \eta).$$  \hspace{1cm} (13)

In the derivation of the linear iSW effect one usually inserts \( \delta(k, a) = D_s(a) \delta(k) \) revealing that during matter domination where \( D_s(a) = a \) the iSW effect vanishes. For our purposes, however, it proves advantageous to carry out the time derivative formally:

$$\frac{\partial}{\partial \eta} \Phi(k, \eta) = \frac{3}{2} H_0^2 \Omega_m \frac{1}{k^2} \left[ H(\eta) \delta(k, \eta) - \frac{1}{a} \frac{\partial}{\partial \eta} \delta(k, \eta) \right].$$  \hspace{1cm} (14)

Obviously, the time evolution of the potential is sourced by two different effects: on the one hand the scaling of structures by the Hubble expansion, represented by the first term, and on the other hand by the time varying structure growth. As mentioned before, in epochs dominated by a fluid with vanishing equation of state \((w = 0)\) these two effects just balance.

Since we are interested in non-linear contributions to the iSW effect we express the time derivative of the density fluctuations by means of the continuity equation:

$$\frac{\partial}{\partial \eta} \delta(x, \eta) + \text{div} j(x, \eta) = 0.$$  \hspace{1cm} (15)

Because of matter conservation, the change of the dark matter density in time needs to be compensated by the divergence of the corresponding flux:

$$j(x, \eta) = [1 + \delta(x, \eta)] u(x, \eta).$$  \hspace{1cm} (16)

In the linear regime \( \delta \ll 1 \), thus the momentum density \( \delta(x, \eta) u(x, \eta) \) is negligible and one can immediately read off the familiar Fourier representation of the peculiar velocity field

$$u(k, \eta) = \int d^3x \, u(x, \eta) e^{i k \cdot x} = -i \frac{\partial D_s(\eta)}{\partial \eta} \frac{k}{k^2} \delta_0(k).$$  \hspace{1cm} (17)

In Fourier space, the dark matter current density becomes a convolution:

$$j(k, \eta) = u(k, \eta) + \int \frac{d^3k'}{(2\pi)^3} u(k - k', \eta) \delta(k', \eta),$$  \hspace{1cm} (18)

so that the time evolution of the gravitational potential finally can be written as a function of density contrast and peculiar velocities:

$$\frac{\partial}{\partial \eta} \Phi(k, \eta) = \frac{3}{2} H_0^2 \Omega_m \frac{1}{k^2} \left[ H(\eta) \delta(k, \eta) - ik \cdot \left( u(k, \eta) \right) \right. \hspace{0.5cm} + \left. \frac{d^3k'}{(2\pi)^3} u(k - k', \eta) \delta(k', \eta) \right].$$  \hspace{1cm} (19)

Because of the symmetry of the convolution the expression for the current density is not unique. For example, one could have explicitly symmetrized equation (18). However, for our purpose, aiming at the different contributions to the RS effect, the form given above is most useful as will become clear below.

So far, our derivation is essentially the same as the one presented by Cooray (2002). Cooray (2002), however, focused on the deeply non-linear regime and proceeded by using approximations resulting from the halo model approach. In contrast to this, we are interested in an analytical description of the contributions to the non-linear iSW effect which arise from dark matter currents. Therefore, we restrict our analysis to the moderate non-linear regime. Obviously, in this regime the simple relation between peculiar velocities and density contrast, given in equation (17), is no longer strictly valid.

The CDM particles cease to exclusively follow the gradients of the potential. However, in order to get an analytical estimate of the non-linear contributions, it is still a useful approximation. Thus, we take non-linear corrections to the continuity equation, manifest in the momentum density, into account, but we maintain the linear properties of the underlying fields, namely density contrast and peculiar velocities. Consequently, by virtue of equation (17), there is only one dynamical field, namely the density contrast, involved in our analysis. Equation (18) then may be written as

$$j(k, \eta) = -i \frac{\partial D_s(\eta)}{\partial \eta} \frac{k}{k^2} \delta(k) + D_s(\eta) \times \int \frac{d^3k'}{(2\pi)^3} \frac{k - k'}{|k - k'|^2} \delta_0(k - k') \delta_0(k').$$  \hspace{1cm} (20)

This expression for the current density has already been successfully used in the computation of the power spectrum of the Ostriker–Vishniac effect (Ostriker & Vishniac 1986; Jaffe & Kamionkowski 1998; Castro 2003). Since in our framework all non-linear contributions to the iSW effect are solely sourced by the convolution part of the flux it is convenient to define its rescaled divergence:

$$\Theta(k, \eta) = -\frac{D_s(\eta)}{a} \frac{\partial D_s(\eta)}{\partial \eta} \int \frac{d^3k'}{(2\pi)^3} \frac{k - k'}{|k - k'|^2} \delta_0(k - k') \delta_0(k').$$  \hspace{1cm} (21)

whereas the linear contributions are captured by

$$\tilde{\tau}(k, \eta) = \frac{\partial}{\partial \eta} \frac{D_s(\eta)}{a} \frac{k}{k^2} \delta_0(k).$$  \hspace{1cm} (22)

With this the change of the gravitational potential takes the compact form

$$\frac{\partial}{\partial \eta} \Phi(k, \eta) = -\frac{3}{2} H_0^2 \Omega_m \left[ \tilde{\tau}(k, \eta) - \Theta(k, \eta) \right].$$  \hspace{1cm} (23)

In \( \Omega_m / \chi_3^2 \) we recognize the gravitational coupling constant mediating between the evolving matter fields and the resulting change in the gravitational potential. Equations (21) and (22) underline the different time evolution of the linear and non-linear iSW effect. For illustration we plot the relevant combinations of the growth function and its derivative for three different equation of state parameters of the dark energy component in Fig. 1.

In order to characterize the statistics of the source field \( \tau(k, \eta) \equiv \frac{\partial}{\partial \eta} \Phi(k, \eta) \) we compute its power spectrum next:

$$\langle \tau(k) \tau^*(k') \rangle = (2\pi)^3 \delta(k - k') P_{\tau\tau}(k).$$  \hspace{1cm} (24)

The cross term \( P_{\tau\delta}(k) \) involves the bispectrum of the density contrast. Since we treat the density contrast as a Gaussian random field its bispectrum vanishes identically. Thus, the power spectrum completely separates in its linear and first-order non-linear contributions:

$$P_{\tau\tau}(k) = P_{\tau\tau}(k) + P_{\tau\delta}(k).$$  \hspace{1cm} (25)

The linear contribution is given by \( P_{\tau\tau}(k) = P_{\delta\delta}(k) / k^4 \), while for the evaluation of the second term we invoke Wick’s theorem yielding

$$P_{\tau\delta}(k) = \int \frac{d^3k'}{(2\pi)^3} \left[ \frac{k \cdot (k - k')}{k^2 |k - k'|^2} + \frac{k \cdot k'}{k^2 k'^2} \right] \frac{k \cdot k'}{|k - k'|^2} \times P_{\delta\delta}(k - k') P_{\delta\delta}(k').$$  \hspace{1cm} (26)

At this point we would like to emphasize that our signal of the non-linear iSW effect completely originates from the trispectrum of the
density field representing velocity–density correlations. In contrast to this, contributions of this kind vanish entirely in the approach of Cooray (2002). Under the halo model, the trispectrum formed by velocity–density correlations does not contribute. In the deeply non-linear regime it is independent of the wavevector configuration. Consequently, any anisotropic integration of this quantity cancels. Furthermore, Cooray (2002) argues that at small scales the density fluctuations are independent of the large-scale velocity field. Thus, in his approach there are no contributions from terms involving the cross-power spectra between density and velocity field. Another difference with respect to the work of Cooray (2002) results from correlations of the density field and its (conformal) time derivative. For describing these correlations Cooray (2002) borrows once more results from the halo model assuming that for the fields involved the Cauchy–Schwarz inequality actually becomes an equality. In our approach, however, this far-reaching assumption is not necessary. At our level of approximation, correlations of this form do not contribute to the non-linear iSW effect power spectrum because they involve the bispectrum of the (linear) density contrast which identically vanishes.

The computation of the convolution integral in equation (26) is most readily carried out in a coordinate system where the wavevector $k$ is aligned with the $x$-axis and by introducing spherical coordinates such that $k' = ak(\mu, \sin \vartheta \cos \phi, \sin \vartheta \sin \phi)$ with $\alpha \equiv k/k$ and $\mu \equiv \cos \vartheta$. Expressing the CDM power spectrum in its dimensionless form, we find

$$\Delta_{\Omega_{0}\delta}(k) = \frac{1}{2} \int_0^\infty \frac{d \alpha}{\alpha^3} \int_1^1 d \mu \frac{(\alpha \mu - 1) [\alpha (2\mu^2 - 1) - \mu]}{k^4 (\alpha^2 - 2\alpha \mu + 1)^{7/2}} \times \Delta^2 \left( k \sqrt{\alpha^2 - 2\alpha \mu + 1} \right) \Delta^2(\alpha k),$$

where we have defined $\Delta_{\Omega_{0}\delta}(k) \equiv k^3 P_{\Omega_{0}\delta}(k)/2\pi^2$. The corresponding angular power spectrum is obtained by Limber projection (Limber 1953; Bartelmann & Schneider 2001):

$$\frac{\ell (2\ell + 1)}{4\pi} C_{\ell}^{XX} = \frac{\pi}{\ell} \int_0^{\chi_L} \chi \ d\chi \ W_{XX}(\chi) \Delta^2(\chi k = \ell/\chi).$$

In the case of the linear iSW effect, the weight function is given by

$$W_{\ell}(\chi) = 3 \Omega_m H_0^2 \frac{\partial}{\partial \eta} \frac{D_+ (a)}{a},$$

while for the first-order non-linear one we have

$$W_{\ell}(\chi) = 3 \Omega_m H_0^2 \frac{D_+ (a)}{a} \frac{\partial}{\partial \eta} D_+ (a).$$

In Fig. 2 we plot both weighting functions for varying equation of state parameter of the dark energy fluid. As expected, the weight function of the non-linear iSW effect peaks at comoving distances where the one of the linear iSW effect has already started to decline rapidly. At high redshifts $\Omega_m(a)$ approaches unity so that here growth function and scale factor coincide. Consequently, the linear iSW effect vanishes identically (cf. equation 22).

The resulting power spectra are shown in Fig. 3. Reflecting the additional fluctuation amplitude of the matter distribution on small spatial scales, the spectrum of the non-linear iSW effect is much flatter than the linear one. In the case of the $\Lambda$CDM model the non-linear signal surpasses its linear counterpart at angular scales smaller than about $1^\circ$, i.e. $\ell \gtrsim 100$. For time evolving dark energy models this crossing has shifted to smaller scales. However, being of second order in the density contrast, the amplitude of the non-linear spectrum is much less sensitive to the underlying dark energy fluid.
4 CONTRIBUTIONS TO THE NON-LINEAR INTEGRATED SACHS–WOLFE EFFECT

A deeper analysis of equation (21) reveals some physical insights into the origin of the non-linear iSW effect. Mainly, one can identify two distinct sources. First, we rewrite equation (21) as the sum of two new fields:

\[ \Theta(k, \eta) = \Theta_{SG}(k, \eta) + \Theta_{BG}(k, \eta), \]

with

\[ \Theta_{SG}(k, \eta) = -\frac{D_s(\eta)}{a} \frac{\partial D_s(\eta)}{\partial \eta} \int \frac{d^3k'}{(2\pi)^3} \frac{\delta_0(k - k')}{|k - k'|^2} \delta_0(k') \]

\[ \times \int \frac{d^3k'}{(2\pi)^3} \Phi_0(k - k') \delta_0(k'). \]  

and

\[ \Theta_{BG}(k, \eta) = \frac{D_s(\eta)}{a} \frac{\partial D_s(\eta)}{\partial \eta} \int \frac{d^3k'}{(2\pi)^3} \frac{k \cdot k'}{|k - k'|^2} \delta_0(k - k') \delta_0(k') \]  

Concentrating on the first term, we proceed by making use of the Poisson equation and obtain

\[ \Theta_{SG}(k, \eta) = \frac{H_0^2}{3} \frac{\Omega_m - 1}{\Omega_\Lambda} \int \frac{d^3k'}{(2\pi)^3} \Phi_0(k - k') \delta_0(k') \]

\[ = \frac{H_0^2}{3} \frac{\Omega_m - 1}{\Omega_\Lambda} \left[ \frac{\partial}{\partial \eta} + H(\eta) \right] \Phi(x, \eta) \delta(x, \eta), \]  

likewise in position space

\[ \Theta_{BG}(x, \eta) = \frac{H_0^2}{3} \frac{\Omega_m - 1}{\Omega_\Lambda} \left[ \frac{\partial}{\partial \eta} + H(\eta) \right] \Phi(x, \eta) \delta(x, \eta). \]  

Keeping in mind that we have already absorbed the inverse of the pre-factor in the definition of the weighting function (cf. equation 30), the product \( \Phi(x) \delta(x) \) can be interpreted as the gravitational self-energy density associated with the density contrast and its own gravitational potential. Consequently, the field \( \Theta_{SG} \) may be interpreted as the (conformal) change of the gravitational self-energy density of the LSS together with the variation arising from the homogeneous expansion of the background.

The interpretation of the second term is more involved. First, we rewrite equation (34) in a more suggestive form

\[ \Theta_{BG}(k, \eta) = \frac{H_0^2}{3} \frac{\Omega_m - 1}{\Omega_\Lambda} \int \frac{d^3k'}{(2\pi)^3} \frac{D_s(\eta)}{a} \frac{\partial D_s(\eta)}{\partial \eta} \frac{k}{|k - k'|^2} \delta_0(k'). \]

Using the Poisson equation, we recognize that the first term has the form of a gradient of the potential. The second term, however, resembles the velocity field (cf. equation 17). Thus, we are tempted to write

\[ \Theta_{BG}(x, \eta) \simeq -2 \nabla \Phi(x, \eta) \cdot \mathbf{v}(x, \eta), \]

ignoring that equation (36) does not fully obey the structure of a convolution. Note that we have also restored the pre-factor contained in the weight function. Although being rather symbolic equation (38) has strong illustrative power. In the first term we recognize the lensing deflection angle \( \alpha = -2 \nabla \cdot \Phi \) (Bartelmann & Schneider 2001). Since the CMB temperature fluctuations are measured along the line of sight, only the three-dimensional gradient perpendicular to the light ray is considered. Accordingly, the scalar product just projects out the velocity components in the plane of the sky. Hence, we recover the dipole-like temperature anisotropy pattern which is usually associated with moving gravitational lenses (Birkinshaw & Gull 1983; Gurvits & Mitrofanov 1986). The close relation between this effect and the non-linear iSW effect has already been pointed out by Cooray (2002). However, neglecting the divergence part of the velocity field Cooray (2002) does not account for the contributions of the gravitational self-energy density we discussed before.

For the computation of the power spectra of the two different contributions to the non-linear iSW effect we proceed in complete analogy to Section 3. In terms of the spectra of the two new fields \( \Theta_{SG} \) and \( \Theta_{BG} \) the power spectrum reads

\[ \Delta^2_{\Theta}(k) = \Delta^2_{\Theta_{SG}}(k) + \Delta^2_{\Theta_{BG}}(k) + 2 \Delta^2_{\Theta_{SGBG}}(k). \]
The three different spectra involved can be jointly expressed as

$$\Delta^2_{\chi\chi}(k) = \frac{1}{2} \int_{0}^{\infty} da \int_{-1}^{1} d\mu \ f_{\chi\chi}(k, \alpha, \mu) \Delta^2(a k)$$

with

$$f_{SGSG}(k, \alpha, \mu) = \frac{2 \alpha^2 - 2 \alpha \mu + 1}{\alpha^2 k^2 (\alpha^2 - 2 \alpha \mu + 1)^3}$$

and the cross-spectrum

$$f_{BGBO}(k, \alpha, \mu) = -\frac{\alpha (2 \alpha^2 + 2 \alpha \mu^2 - \alpha^2 \mu - 3 \alpha \mu + 1)}{\alpha^2 k^2 (\alpha^2 - 2 \alpha \mu + 1)^3}$$

In Fig. 5 we plot the three-dimensional power spectra of the gravitational self-energy density and the Birkinshaw–Gull term along with the cross-spectrum of both effects. In addition, we show the resulting power spectrum of the non-linear iSW effect and, for comparison, also the corresponding spectrum of the linear iSW effect in the same figure. From scales $k \gtrsim 0.1 \ (\text{Mpc} \ h^{-1})^{-1}$ on the spectra of the $\Theta_{SG}$ and $\Theta_{BG}$ field, as well as their cross-correlation, exceed the linear signal by several orders of magnitude. However, all three spectra are almost identical, as is demonstrated in the lower panel of Fig. 5. Only on largest scales the relative difference amounts to 10–20 per cent. Therefore, their combination according to equation (39) results in the much lower signal of the non-linear iSW effect, which starts to surpass the spectrum of the linear effect on scales $k \sim 0.2 \ (\text{Mpc} \ h^{-1})^{-1}$. Furthermore, we recognize that the three spectra peak at $k \sim 1 \ (\text{Mpc} \ h^{-1})^{-1}$, whereas the maximum of their combination is shifted to larger scales ($k \sim 0.1 \ (\text{Mpc} \ h^{-1})^{-1}$). Limber projection (cf. equation 28) of the three different spectra results in the corresponding angular power spectra shown at the top of Fig. 6. Since the projection distributes the power of one mode over a wide multipole range the differences in the angular spectra are even smaller than in the case of the three-dimensional power spectra (see lower panel of Fig. 6). From multipoles $\ell \gtrsim 1000$ all three spectra are essentially indistinguishable.

5 PHYSICAL INTERPRETATION

To further motivate our interpretation of the contributions to the non-linear iSW effect we now present some analogies to the theory of gravitomagnetic potentials and classical field theory.

We start with the temperature anisotropy induced by a small lens moving with small (transverse) three-velocity $v$. At lowest order we have (Birkinshaw & Gull 1983; Gurvits & Mitrofanov 1986)

$$\frac{\Delta T_{BG}}{T_{CMB}} = -2 \int d\chi \ v \cdot \nabla \Phi,$$

where the integral is evaluated along the photon path. Albeit treating the involved fields in linear theory, the Birkinshaw–Gull effect is obviously non-linear in the sense that the perturbed quantities enter as a product. In linear theory the velocity field can be obtained as the gradient of the potential (cf. equation 17). Consequently, the Birkinshaw–Gull effect is of order $O(|\nabla \Phi|^2)$. In perturbation theory we can naturally construct one more term of this order, namely $\Phi \Delta \Phi$. By virtue of the Poisson equation this is exactly the gravitational self-energy density we have found in equation (36).

For simplicity we drop all pre-factors in the following. Then, both effects can be obtained from the auxiliary vector potential $A \equiv \nabla \Phi$ by taking the divergence

$$\frac{\Delta T_{\text{iSW}}}{T_{CMB}} = -2 \int d\chi \ \text{div} \ A = -2 \int d\chi \ (\Phi \Delta \Phi + (\nabla \Phi)^2).$$

Rewriting the potential $A$ as

$$A(x) = \int_{V} d^{3}x' \frac{\delta(x') \nabla \Phi(x')}{|x - x'|},$$

one immediately recognizes its similarity to the gravitomagnetic potential:

$$A_{\text{GM}}(x) = \int_{V} d^{3}x' \frac{f(x')}{|x - x'|} \simeq \int_{V} d^{3}x' \frac{\delta(x') \nabla \Phi(x')}{|x - x'|}.$$
Birknshaw–Gull effect, as well as the gravitational self-energy term (at the corresponding order in $\Phi$) since Schäfer & Bartelmann (2006) showed that $\Delta T_{\text{nl}}/T_{\text{CMB}} = 2 \int d^3 x \text{div} A_{\text{GM}}$. There is an important difference with respect to the theory of electromagnetism.

In general, the vector potential describing the magnetic field is not fully determined by the scalar potential of the electric field. The scalar potential only enters implicitly via the charge density. At our level of approximation, however, not only the mass density ($\delta \sim \Delta \Phi$) enters the gravitomagnetic potential but also the velocity field is derived from the scalar potential, too ($\nu \sim \nabla \Phi$). Thus, given the scalar gravitational potential, the gravitomagnetic vector potential is already completely fixed.

It is interesting to note that the contribution of the self-energy density is independent of the assumption $\nabla \Phi \simeq \text{const.}$ for $x \in \mathcal{V}$. This can be seen as

$$
\nabla \cdot \int_{\mathcal{V}} d^3 x \frac{\delta(x') \nabla \cdot \Phi(x')}{|x - x'|} = \nabla \cdot \int_{\mathcal{V}} d^3 x' \left( \Phi(x') \nabla \frac{1}{|x - x'|} + \delta(x') \nabla \frac{\Phi(x')}{|x - x'|} \right)
$$

$$
= -4\pi\delta(x) \Phi(x) + \nabla \cdot \int_{\mathcal{V}} d^3 x' \delta(x') \nabla \frac{\Phi(x')}{|x - x'|},
$$

where we have first integrated by parts and subsequently interchanged the order of differentiation and integration.

The strong similarity to the potential $A$ also indicates why the statistics of the Birknshaw–Gull and the self-energy density term are almost identical (cf. Figs 5 and 6). Integrated over a large volume one expects the divergence of $A$ to vanish due to Stokes’ theorem. Accordingly, both terms, considered over a large volume, must cancel. Thus, one expects that they reveal the same statistical properties, whereas the small deviations point to the actual differences in the two vector potentials $A$ and $A_{\text{GM}}$.

Finally, it is worth mentioning an analogy with the theory of massless scalar fields. Interpreting the potential $\Phi$ as scalar field interacting with an additional scalar field $\delta$ one readily computes its energy momentum tensor (cf. Liddle & Lyth 2000):

$$
T_{\mu\nu} = \partial_{\mu} \Phi \partial_{\nu} \Phi - \frac{1}{2} \delta_{\mu} \Phi \delta^{\nu} \Phi + \delta \Phi \eta_{\mu\nu},
$$

where $\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$ and we apply Einstein’s summation convention. Then the so-called mechanical pressure $P$ of the field $\Phi$ is given by one-third of the trace of the spatial part of the energy momentum tensor, i.e.

$$
P = \frac{1}{3} T^{ij} = \frac{1}{2} \Phi^2 - \frac{1}{6} (\nabla \Phi)^2 - \delta \Phi.
$$

We now see that, at our level of approximation, the non-linear iSW effect measures to some extent the mechanical pressure of the gravitational potential without the contribution arising from its variation in time. That the term including time derivatives does not contribute is somewhat expected since in our derivation of the non-linear iSW effect we used the continuity equation to express the time variation of the gravitational potential by its variation in space. The analogy with the mechanical pressure once more shows that the non-linear iSW effect is naturally constituted by both the Birknshaw–Gull effect, as well as the gravitational self-energy density. The halo model approach of Cooray (2002), however, cannot resolve this close relationship between these two contributions.

At the end of this section we shall comment on the apparent weakness of our approach. Major concerns may arise from the fact that we analyse a non-linear effect starting from linear theory. We exploit the commonly used ansatz in perturbation theory that a product of first-order fields yields a second-order perturbative, i.e. non-linear quantity. This approach, however, reveals at the same time one of the main problems of perturbation theory: it is not clear how to generalize this ansatz to higher order. On the other hand, this disadvantage is compensated for by the extreme simplification provided by this ansatz. It allows for the reduction to one single dynamical field, the density contrast or likewise the gravitational potential. One can then easily identify the terms which contribute at the same order in the perturbative field. Furthermore, the statistical properties may be derived most conveniently by tracing back higher order correlators to the matter power spectrum (cf. Section 4). Thus, our approach facilitates a fully analytical treatment, which in turn offers some direct physical interpretations of the origin of the non-linear iSW effect. These interpretations are widely inspired by analogies and always prefer the illustrative power of physical reasoning.

### 6 SUMMARY

In this paper we revisited the theory of the non-linear iSW effect. We used the continuity equation to express the time evolution of the gravitational potential in terms of the divergence of the dark matter flux density. The momentum density, being the product of density and velocity field, can then be interpreted as source field of the non-linear iSW effect.

(i) Our ansatz treats both dynamical fields, density contrast and velocity, in linear theory. Consequently, the velocity field is completely determined by the density contrast, leaving the latter as the only dynamical quantity involved in our analysis. This simplification allows for a fully analytical treatment in contrast to the previous work of Cooray (2002) which substantially relies on results from the halo model.

(ii) Exploiting Wick’s theorem we derived the angular power spectrum of the non-linear iSW effect. We confirmed the shape and amplitude found by other authors using different approaches (Seljak 1996; Cooray 2002). Especially, we verified that the non-linear signal surpasses that of the linear iSW effect at multipoles $\ell \gtrsim 100$.

(iii) Our analytical ansatz allowed us to reveal two contributions to the non-linear iSW effect: the Birknshaw–Gull effect (as already pointed out by Cooray 2002) and the conformal change of the gravitational self-energy density of the cosmic LSS.

(iv) Computing the three-dimensional power spectra and the corresponding angular power spectra of the individual contributions, we found that they are almost identical, except on large scales, and their individual detection is certainly impossible for ongoing CMB experiments.

(v) We showed by simple arguments from perturbation theory that it is natural that the Birknshaw–Gull term is accompanied by the gravitational self-energy density term. Furthermore, we pointed out several analogies with the theory of gravitomagnetic potentials and scalar fields to highlight the physical meaning of the different contributions to the non-linear iSW effect.

Despite its confinement to linear theory our ansatz allows for a deeper understanding of the physical processes underneath the non-linear iSW effect, especially regarding its relation to other phenomena like the Birknshaw–Gull effect.
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