A two-parameter matching scheme for massive galaxies and dark matter haloes

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ABSTRACT

Halo abundance matching has been used to construct a one-parameter mapping between galaxies and dark matter haloes by assuming that halo mass and galaxy luminosity (or stellar mass) are monotonically related. While this approach has been reasonably successful, it is known that galaxies must be described by at least two parameters, as can be seen from the two-parameter Fundamental Plane on which massive early-type galaxies lie. In this paper, we derive a connection between initial dark matter density perturbations in the early Universe and present-day virialized dark matter haloes by assuming simple spherical collapse combined with conservation of mass and energy. We find that \( z = 0 \) halo concentration, or alternatively the inner slope of the halo density profile \( \alpha \), is monotonically and positively correlated with the collapse redshift of the halo. This is qualitatively similar to the findings of some previous works based on numerical simulations, with which we compare our results. We then describe how the halo mass and concentration (or inner slope \( \alpha \)) can be used as two halo parameters in combination with two parameters of early-type galaxies to create an improved abundance matching scheme. In a forthcoming paper, we will show an application of this scheme to galaxies on the Fundamental Plane.

Key words: galaxies: evolution – galaxies: formation – galaxies: haloes – cosmology: theory – dark matter.

1 INTRODUCTION

Halo abundance matching is one of several methods used to link galaxies with dark matter haloes. It uses the simple assumption that galaxy luminosity (or stellar mass) and halo mass are monotonically related, such that more luminous galaxies reside in more massive haloes, to match observed magnitude-limited samples of galaxies to dark matter halo merger trees from dark matter-only simulations (Kravtsov et al. 2004; Vale & Ostriker 2004, 2006; Guo et al. 2010). Despite the simplicity of its underlying assumption, abundance matching is able to reproduce with surprising accuracy various measures related to the observed physical distribution of galaxies, such as the luminosity functions of different cosmic environments, the occupation numbers of haloes (Vale & Ostriker 2004), galaxy autocorrelation functions (Conroy, Wechsler & Kravtsov 2006; Guo et al. 2010; Nuza et al. 2013), and galaxy–galaxy lensing (Hearin et al. 2014; Fosalba et al. 2015).

Being able to match large samples of galaxies and haloes without the use of complex semi-analytic or numerical hydrodynamic modelling has allowed for a simple probe of the connection between galaxies and their dark matter haloes. Abundance matching has been used to obtain various statistical relationships between galaxies and their host haloes, such as the luminosity–halo mass and stellar–halo mass relations (Shankar et al. 2006; Vale & Ostriker 2006; Conroy & Wechsler 2009; Behroozi, Conroy & Wechsler 2010; Guo et al. 2010; Wake et al. 2011; Leauthaud et al. 2012; Moster, Naab & White 2013), the relationship between galaxy optical circular velocities and the circular velocities implied from their host haloes (Dutton et al. 2010), the halo baryonic mass function (Baldry, Glazebrook & Driver 2008), and the relation between central black hole mass and halo mass (Shankar et al. 2006). It has also been used to examine the dark matter haloes hosting certain types of galaxies, such as quasars (Croton 2009). Furthermore, abundance matching also allows the possibility of assigning observed galaxies at different redshifts to simulated haloes whose mass growth and merger history are known, allowing one to track the evolution of galaxies through time. This has been used to study the fate of satellite galaxies in clusters (Conroy, Wechsler & Kravtsov 2007), the frequency of gas-rich versus gas-poor mergers (Stewart et al. 2009a), the accuracy of observational indicators of halo mergers such as close pair counts (Stewart et al. 2009b), the evolution of the stellar–halo mass relation and velocity dispersion–halo mass relation and the implications for galaxy–halo co-evolution (Firmani &

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Avila-Reese 2010; Chae 2011), and the growth of brightest cluster galaxies (Laporte et al. 2013), among many others.

However, a scheme that treats luminosity as the sole important property of a galaxy cannot be entirely correct. Galaxies are described by at least two parameters, as is apparent for moderate mass systems from the bimodal distribution of galaxy colours at fixed luminosity (e.g., Strateva et al. 2001; Blanton & Moustakas 2009). This implies that at least one other parameter of the galaxy’s halo, aside from its mass, must be relevant to the evolution of the galaxy. Recently, several works have begun to explore abundance matching schemes with a second halo parameter at fixed halo mass to account for the bimodality in galaxy colour (Hearin & Watson 2013; Masaki, Lin & Yoshida 2013; Hearin et al. 2014).

Even considering only massive early-type galaxies, one parameter is inadequate for predicting all of their properties. It has been known for some time that, while a rough one-parameter relation exists relating all variables to the velocity dispersion (the well-known ‘Faber–Jackson relation’; Faber & Jackson 1976), a two-dimensional parametrization called the ‘Fundamental Plane’ (FP) offers a superior description (Dressler et al. 1987). An evolving FP has been detected from $z = 0$ out to $z \sim 2$ (van de Sande et al. 2014). Other properties of early-type galaxies, such as observed galaxy colour, as well as modelled stellar population ages and metal abundances, have been found to be highly correlated with the FP parameters (Graves, Faber & Schiavon 2009a,b; Graves & Faber 2010). This implies that the properties of early-type galaxies may be well-described by two parameters, making them a good sample on which to test a two-parameter matching scheme.

There are also other reasons why a two-parameter matching scheme may work best for massive early-type galaxies. The disc-to-bulge ratios of spiral galaxies are likely to be dependent on their environment (e.g., Hopkins et al. 2009). Also, galaxy colour in star-forming galaxies may not be well-correlated with stellar age because a recent small burst of star formation can make a galaxy significantly bluer while only slightly changing the mean stellar population age.

One halo parameter of physical interest is the redshift at which the halo collapsed. It is well-known that in hierarchical collapse models more massive haloes collapse later than less massive ones, but variations in collapse time at fixed halo mass could influence the properties of the galaxies these haloes host (e.g., Blumenthal et al. 1984). In particular, since the colour of early-type galaxies should be well-correlated with the age of their stellar population (as well as their metallicity), it is possible that the galaxy colour is also correlated with some measure of the collapse time of the host halo, which would determine when gas could collapse and form stars. There are also other galaxy parameters that are likely to be correlated with the collapse time of the host halo at fixed mass, such as the metallicity, stellar surface brightness, and stellar mass-to-light ratio. A present-day halo property correlated with the halo collapse time and a present-day galaxy property could then be used as abundance matching parameters in addition to the halo mass and galaxy luminosity.

In this paper, we use a simple spherical collapse model to derive a present-day halo parameter that is a proxy for the halo collapse time. We adopt two different fitting functions for the $z = 0$ halo density profile, and also consider physical parameters that are independent of the function used to fit the halo profile. We will present the results of a matching scheme utilizing these parameters for the FP, as described briefly in the final section of this paper, in a forthcoming paper.

We first derive a proxy for halo collapse time using simple spherical collapse model in Section 2. We show the results of this model in Section 3. We compare our results with previous parametrizations of halo ‘formation time’ (Bullock et al. 2001; Wechsler et al. 2002; Zhao et al. 2009) derived from dark matter simulations in Section 4. Finally, we describe a scheme for halo matching of the FP of elliptical galaxies (a two-parameter distribution), which is not obviously dependent on environment, that we plan to expand on in a future paper in Section 5.

In all parts of this paper we assume a cosmology consistent with the WMAP nine-year results plus external CMB, BAO, and $\Omega_{0}$ measurements (Hinshaw et al. 2013; table 4); thus we take $\Omega_{m,0} = 0.71$, $\Omega_{b,0} = 0.29$, and $H_{0} = 69$ km s$^{-1}$ Mpc$^{-1}$.

2 SPHERICAL COLLAPSE MODEL FOR HALO COLLAPSE TIME

We would like to choose some property of dark matter haloes that is a good proxy for the halo collapse time and can be easily measured in dark matter-only simulations. We approximate the collapse time of a dark matter perturbation early in the Universe as twice the turnaround time in simple spherical models of collapse. We use a somewhat similar method to that of Rubin & Loeb (2013), who give equations for calculating the virialization density $\Delta$, for arbitrary pre-collapse and post-collapse density profiles by assuming mass and energy conservation. Our method is similar, except that we match the initial and final profiles within their turnaround radii at $z = 0$.

For the purposes of creating a two-parameter matching scheme, we choose final halo profiles that are described by two parameters and that are commonly used to fit numerically simulated haloes – namely, a Navarro-Frenk-White (NFW, Navarro, Frenk & White 1997) profile with parameters $M_\text{200}$ and $c$, and a generalized NFW profile with varying mass $M_\text{200}$ and inner slope $\alpha$, with fixed $c = 5$ (see Section 2.2). We match these final profiles to chosen arbitrary initial profiles (e.g., tophat or Gaussian) that have the same mass and energy within the shell that turns around at $z = 0$. While this model maps final profiles to initial profiles with two parameters uniquely (details below), the corresponding final profile is not actually the profile that the initial profile would evolve to, as evidenced by the fact that profiles of different shapes can be matched to the initial profiles this way. Rather, we choose final profiles that are used as approximations for a variety of dark matter halo profile shapes.

2.1 Initial profile

Here we review the equations for the evolution of the initial profile: the analysis is similar to that in Mo, van den Bosch & White (2010).

We begin with some chosen initial overdensity profile $\rho_i(r)$ at arbitrarily chosen initial time $t_i$. We assume the profile will tend to the mean matter density of the Universe at that time, $\rho_0(t_i)$, for large radii. As long as the density is decreasing or constant with increasing radius, there will be no shell crossing for shells that have not yet collapsed and we can treat them separately. It is assumed that going far enough back in time, the initial perturbation is entirely expanding, and none of its shells have yet turned around. Also, in a $\Lambda$-cold-dark-matter ($\Lambda$CDM) universe, a finite amount of mass will collapse in an infinite time, because for overdensities lower than some value, the shells expand forever due to the $\Lambda$ term instead of collapse. In the appendix to this paper, we present an exact derivation...
of this value for an initial profile assumed to be on the Hubble flow, which tends towards the solution having \( r = 0 \) at \( t = 0 \) for \( t_i \to 0 \).

The collapse of each shell enclosing mass \( M(< r) \) is governed by the following equation:

\[
\frac{d^2r}{dt^2} = -\frac{GM}{r^2} + \frac{\Omega_{\Lambda,0} H_0^2 r}{2}.
\]  

(1)

The above equation once integrated becomes

\[
\frac{1}{2} \left( \frac{dr}{dt} \right)^2 - \frac{GM}{r} - \frac{\Omega_{\Lambda,0} H_0^2 r^2}{2} = \mathcal{E},
\]  

(2)

where \( \mathcal{E} \) is the specific energy of the shell. At the turnaround time \( t_{ta} \) of a given shell, this becomes

\[
\mathcal{E} = -\frac{GM}{r_{ta}} - \frac{\Omega_{\Lambda,0} H_0^2 r_{ta}^2}{2},
\]  

(3)

where the turnaround radius \( r_{ta} \) is the maximum radius attained by each shell. Defining

\[
\zeta = \frac{\Omega_{\Lambda,0} H_0^2 r_{ta}^2}{2GM},
\]

(4)

we see that for \( \bar{r} < 0 \) at \( r_{ta} \) we require \( \zeta < 1/2 \). From this and equation (2) we have a formula relating the radius of the shell and the time, as long as the shell has not yet turned around:

\[
H_0 t = \left( \frac{\zeta}{\Omega_{\Lambda,0}} \right)^{1/2} \int_0^{r/r_{ta}} dx \left[ \frac{1}{x} - 1 + \zeta(x^2 - 1)^{-1/2} \right].
\]  

(5)

Using equation (5), the central collapse time \( (2 \times t_b) \) can be calculated for \( r_{ta} \to 0 \) (as long as the density profile does not have a central cusp), as can the time at which any fraction of the mass at \( z = 0 \) collapsed.

For a selected initial density profile \( \rho(r) \) at time \( t_i \), we can find the energy of each shell from equation (3). We first obtain \( r_{ta} \) for each \( r_i \). To do that we re-express \( \zeta \) as

\[
\zeta = \frac{\Omega_{\Lambda,0} H_0^2}{2 \pi G \rho(y)} \left( \frac{r_{ta}}{r_i} \right)^3.
\]

and insert into equation (5) for \( t_i(r_i) \). This can be solved numerically for \( r_{ta} \) as a function of \( t_i \) and \( r_i \).

By setting \( t \) in equation (5) to the age of the Universe at \( z = 0 \) (referred to here as \( h_0 \)) we can obtain \( \zeta \) of the shell that is turning around at \( z = 0 \). Then we can use this equation to find the ratio of the original radius to the turnaround radius \( r_i/r_{ta} \) of this shell by setting \( t \to t_i \). This combined with equation (6) gives us the \( r_i \) and \( r_{ta} \) for the shell turning around at \( z = 0 \); we will designate these as \( r_{i,\text{max}} \) and \( r_{ta,\text{max}} \). We designate the mass and energy within \( r_{i,\text{max}} \) at \( t_i \) (and \( r_{ta,\text{max}} \) at \( h_0 \)) as \( M_{\text{tot}} \) and \( E_{\text{tot}} \), and these are what we will match to the mass and energy of the final profile. We also define the initial density within \( r_{i,\text{max}} \) to be \( \rho_{i,\text{max}} \), such that

\[
r_{i,\text{max}} = \left( \frac{M_{\text{tot}}}{4\pi \rho_{i,\text{max}}} \right)^{1/3}.
\]  

(7)

We can find \( \mathcal{E} \) for each shell using equation (3). The total energy of all the shells within the maximum radius will then be

\[
E_{\text{tot}} = \int_0^{r_{ta,\text{max}}} \mathcal{E}(r_{ta})dM(r_{ta}).
\]  

(8)

To obtain \( \mathcal{E}(r_{ta}) \) for each shell at initial radius \( r_i \) between \( r_i = 0 \) and \( r_{i,\text{max}} \), one can again insert equation (6) into (5) for \( t = t_i \), and solve for \( r_{ta}/r_i \).

Scaling the size of the profile by the radius \( r_{i,\text{max}} \), so that the coordinate used is \( y = r/r_{i,\text{max}} \), equation (3) gives

\[
\mathcal{E} = -\frac{4}{3} \pi G \rho(y) \frac{r_i}{r_{ta}} y^2 r_{i,\text{max}}^2
\]

\[
- \frac{\Omega_{\Lambda,0} H_0^2}{2} \left( \frac{r_{ta}}{r_i} \right)^2 y^2 r_{i,\text{max}}^2,
\]

(9)

where \( r_{ta}/r_i \) is a function of \( y \) and also depends on \( z_0 \) and \( z_i \).

Then, we can use equation (6) to substitute for \( r_i/r_{ta} \) in the equation for \( \mathcal{E} \); we obtain

\[
\mathcal{E} = -\frac{2}{3} \left[ 8 \frac{\pi G}{\Omega_{\Lambda,0} H_0^2} \right]^{2/3} \frac{3^{1/3} \rho_{i,\text{max}}^{2/3} (y^3 - 1)}{\zeta^{1/3} y^{3/2} r_{i,\text{max}}^2}.
\]

(10)

The total energy is then given by

\[
E_{\text{tot}} = \int_0^{1} \mathcal{E}(y) \rho(y) y^2 dy
\]

\[
= 4\pi \rho_{i,\text{max}} \int_0^{1} \left[ \frac{1}{2} \left( \frac{8}{3} \pi G \right) \frac{1}{\zeta^{1/3} \Omega_{\Lambda,0}^{1/3} H_0^{8/3}} \right] \times \int_0^{1} \rho_{i,\text{max}} (y^3 - 1) \left( \frac{1 + \zeta(y)}{\zeta^{1/3} y^{3/2}} \right) y^2 dy,
\]

(11)

where using equation (7) we then obtain

\[
E_{\text{tot}}/M_{\text{tot}}^{5/3} = \frac{3}{\rho_{i,\text{max}}^{1/3}} \frac{1}{2^{1/3} (H_0 G)^{2/3}} \Omega_{\Lambda,0}^{1/3} \times \int_0^{1} \rho_{i,\text{max}} (y^3 - 1) \left( \frac{1 + \zeta(y)}{\zeta^{1/3} y^{3/2}} \right) y^2 dy.
\]

(12)

This equation holds for any initial density profile \( \rho(y) \) at any chosen \( z_i \), as long as no part of the profile has yet collapsed.

For the initial profile, we also want to consider the limit as \( z_i \) becomes large. Here \( \delta(y, < y) \), where \( \delta(y, < y) = \bar{\rho}(z_i)(1 + \delta(y, < y)) \), approaches (Mo et al. 2010)

\[
\delta(y, < y) = \frac{3}{5} \frac{1 + \zeta(y)}{\zeta^{1/3} y^{3/2}} \left( \frac{\Omega_{\Lambda,0}}{\Omega_{\Lambda,0}} \right)^{1/3} (1 + z_i)^{-1},
\]

(13)

where \( \zeta \) corresponds to a given turnaround (or collapse) time. We want to consider the same overdensity profile shape at all times, i.e., \( \delta(y, < y)/\delta_{\text{max}} \), does not vary with time, where \( \delta_{\text{max}} \) is the overdensity at \( r_{i,\text{max}} \). It is clear that for a shell with a given collapse time, \( \delta \) for that shell evolves with time; however, the above equation shows that for large \( z_i \), a \( \delta(y)/\delta_{\text{max}} \) profile taken to be constant with time is in fact also a constant profile in \( \zeta(y) \) — that is, collapse time at any fixed interior mass.

Here \( \rho_{i,\text{max}} \) changes with \( z_i \), but for large \( z_i \), \( \rho_{i,\text{max}} \to \bar{\rho}(z_i) \) as the perturbation \( \delta \) decreases like \( (1 + z_i)^{-1} \) for fixed \( \zeta \) (equation 13), eliminating \( \rho_{i,\text{max}} \), \( \rho(y) \), and \( \delta(y) \). This gives

\[
E_{\text{tot}}/M_{\text{tot}}^{5/3} = \frac{3}{2^{2/3} (H_0 G)^{2/3} \Omega_{\Lambda,0}^{1/3}} \int_0^{1} \delta(y, < y) \left( \frac{1}{\delta_{\text{max}}} \right) y^2 dy,
\]

(14)

where \( \zeta \) is the value of \( \zeta \) for the shell that turns around at \( z_0 \), in our case taken to be \( z = 0 \).

Since \( \zeta_0 \) is dependent only on the reference redshift \( z_0 \) that we select and we assume that the overdensity profile is a constant shape, this expression approaches a constant value as \( z_i \to \infty \). We see that \( E/M^{5/3} \) is dependent only on the shape of the overdensity profile, which also determines the collapse times relative to the chosen reference redshift at high \( z_i \).
It is also clear that for a constant or decreasing overdensity profile, the integral on the right-hand side of equation (14) is bounded between $1/5$ and $1/2$, meaning that the energy within a shell turning around at a given redshift must be bounded between two values. However, as described in the next section, the final ($z = 0$) halo profiles we consider are common empirical fits to simulated haloes, and thus have unbounded possible values for $E/M^{5/3}$. Therefore, not all conceivable final profiles are able to correspond to a possible initial profile.

As can be seen above, the value of $E/M^{5/3}$ that is matched to that of the final profile depends on the shape of the initial profile. The shape of true cosmological initial perturbations from Gaussian random fields is discussed in Dalal, Litchick & Kuhlen (2010). Ignoring subhaloes, they recover the BBKS profile (Bardeen et al. 1986), which is extremely steep in the outer regions and quickly turns over to become flat in the inner regions, similar to a tophat profile. However, haloes also contain subhaloes whose influence on the halo evolution must be taken into account. Dalal et al. (2010) find that this analysis can be considerably simplified by taking into account the effect of dynamical friction – namely, the fact that the densest subpeak at each scale will be most dragged towards the centre. Using this analysis, they find a significantly modified profile that is similarly steep at large radii but has an inner slope of $\sim -0.25$. In this paper we consider an initial tophat profile (similar to the BBKS profile), and an initial Gaussian profile.

### 2.2 Final profile

We would like to find a corresponding present-day halo density profile that has equal $M_{\text{tot}}$ and $E_{\text{tot}}$ within the turnaround radius at $z = 0 (r = r_0)$ as the initial profile. We note that a specific turnaround time defines a unique density within the turnaround radius at that time (equations 4 and 5), implying that for fixed $M_{\text{tot}}$, the final and initial profiles also have the same turnaround radius.

We choose two different forms for the present-day halo, based on the fact that these shapes are commonly used to fit simulated haloes.

(i) The entire profile is described by an NFW profile over the mean matter density:

$$
\rho(r) = \frac{\rho_0}{r/r_0(1 + r/r_0)^2} + \bar{\rho}(t_0),
$$

As noted above, for large values of the concentration $c$, $E/M^{5/3} \propto c/(\log c)^2$ for the NFW profile, which is not bounded. Therefore NFW profiles above some concentration cannot be the product of a simple spherical collapse model.

(ii) While the NFW profile is the most common parametrization of dark matter halo density, others have found that haloes are equally or better described by profiles that have a varying inner slope (e.g., Subramanian, Cen & Ostriker 2000). In particular, Ricotti, Pontzen & Viel (2007) have found that simulated dark matter haloes at virialization are equally well fit by either an NFW profile with varying $c$, or by a generalized NFW profile with fixed $c = 5$, and varying inner slope $\alpha$. Thus we also match initial profiles to a dark matter density profile given by

$$
\rho(r) = \frac{\rho_0}{(r/r_0)^\alpha(1 + r/r_0)^2} + \bar{\rho}(t_0),
$$

where $r_0$ is related to $M_{200}$ via the fixed concentration $c = 5$. We refer to this profile as an $\alpha$ model to avoid confusion with the NFW profile. This profile and the standard NFW have the same slope of $-3$ for large $r$, making their properties similar at large radii. As for the NFW profile, $E/M^{5/3}$ is unbounded for large $\alpha$.

We assume haloes to be virialized within $r_{200}$, the radius within which the mean density is 200 times the critical density, and that the virial theorem can be used to find the energy within this region. Outside the virial radius, the profile is collapsing, out to a radius $r_a$ at which $t_a = t_0$. While a different radius could be chosen within which the profile is virialized, we note that taking the virial radius to be $r_{200}$ as computed by Bryan & Norman (1998) would produce a negligible difference in our results.

The energy of the NFW profile within the radius turning around at $z = 0$ is the sum of the energy within the virial radius and the energy in the shells turning around. The potential energy within the virial radius is the sum of that from the matter and that from $\Lambda$:

$$
U_m = -\int_0^{r_{200}} 4\pi GM(<r)/r \rho(r)r^2 dr,
$$

and by the virial theorem, $E(r_{200}) = 0.5U_m + 2U_\Lambda$. Because $\bar{\rho}$ is much lower than the virial density, the fact that we take the density profile to be an NFW or an $\alpha$ model in overdensity has an insignificant effect on the virial energy we find within the virial radius. Thus the energies we obtain within the virial radius for the NFW profile are approximately those given by equations 3.33 and 4.20 in Rubin & Loeb (2013).

Outside the virial radius, the region between $r_{200}$ and the turnaround radius $r_a$ is collapsing. To determine the energy for this region we use equation (11). However, because the region is collapsing and not expanding, one must add the turnaround time to the time a shell has been collapsing after $t_a$, thus substituting equation (5) with

$$
H_0 t_a = \left(\frac{\xi}{\Omega M_\odot}\right)^{1/2} \left[\int_0^1 dx \frac{1}{x} - 1 + \zeta(x^2 - 1)\right]^{-1/2} + \int_{r_{200}}^{r_a} dx \frac{1}{x} - 1 + \zeta(x^2 - 1)\right]^{-1/2}.
$$

For a selected initial profile shape (e.g., a Gaussian) at a chosen initial time $t_i$, the above steps will create a one-to-one mapping between initial profiles with mass $M_{\text{tot}}$ and energy $E_{\text{tot}}$ and final profiles at $z = 0$ with the same mass and energy within the turnaround radius. Since both the NFW profiles and $\alpha$ model profiles are a two-parameter family, a unique combination of mass and energy values will correspond to a unique final profile of a given form.

### 3 RESULTS

Applying the above procedure, we are able to match final profiles to initial profiles. It is clear from equation (12) that the value $E/M^{5/3}$ at a fixed redshift is dependent only on the shape of the overdensity profile (and the mean cosmic density at the chosen redshift), as it must be from dimensional arguments. For the final profiles we take to be at $z = 0$, NFW profiles and $\alpha$ models with the same mass and energy, this equation implies that there will be a relationship between $c$ and $\alpha$ that is not mass-dependent. We show this relationship in Fig. 1. More concentrated NFW profiles correspond to $\alpha$ model profiles with steeper inner slope. For comparison, we also show mean fits to the 40 most massive haloes from simulations at different redshifts from Ricotti et al. (2007). Our mapping using
the energy and mass corresponds well to the match between $\alpha$ and $c$ from direct fitting. This is as could be expected, since direct fitting ensures that the profiles will have similar shapes in the region $r \sim r_c$, and both profiles have slope $-3$ at large radii, leading to similar energies at fixed mass for both profile shapes.

Similarly, equation (12) implies that for the initial profiles, $E/M^{5/3}$ is also a function of only the shape of the overdensity profile and the mean density at the chosen initial redshift $z_i$. Furthermore, as seen in equation (14), for a fixed overdensity shape, the value of $E/M^{5/3}$ approaches a constant value as $z_i \to \infty$. However, the collapse time for a mass shell containing a certain fraction of the total mass is also a function of only the chosen initial redshift $z_i$ and the shape of the overdensity profile (equations 5 and 6), and also approaches a constant value for fixed overdensity shape as $z_i \to \infty$ (equation 13). Thus the value of $c$ or $\alpha$ of the final profile will be a function of the collapse time of any chosen fraction of the mass for an initial profile of a fixed shape for fixed $z_i$. We show this correspondence in the top panel of Fig. 2, in which we present the NFW concentration $c$ (or $\alpha$ model inner slope) versus the collapse time of the centre of an initial Gaussian or tophat profile, as well as the time for half the mass collapsed at $z = 0$ of the initial profile to collapse. For the initial profiles we take $z_i = 1000$ and compute the exact collapse times as described in Section 2.1; however, this redshift is large enough that the results will be similar to those calculated with the approximation of $z_i \to \infty$. We find that more concentrated haloes (or those with steeper $\alpha$) have earlier collapse times, as might have been intuitively expected. Thus, either $\alpha$ or $c$ can be used as a proxy monotonically related to collapse epoch that is independent of halo mass.

The results depend on the choice of initial profile. They also depend (weakly) on $z_i$, the initial redshift at which the profile is selected to be the shape of choice. This is because in general the shape of the profile evolves over time, so that the same profile at a later or earlier time does not follow the same functional form. Thus the dependence on $z_i$ can also be seen as equivalent to a dependence on the shape of the profile at any given time. However, as seen in Section 2.1, as $z_i \to \infty$, the dependence on $z_i$ disappears, and the value of $E/M^{5/3}$ depends only on the overdensity profile shape.

Additionally, we present two tables showing the same values as the top panel of Fig. 2 for the initial Gaussian profile; Tables 1 and 2 show the central and half-mass collapse time, respectively, for NFW profiles with concentration $c$ and mass $M_{200}$. While the collapse times do not depend on $M_{200}$, in bold we show the mass–concentration relation for simulated dark matter haloes from Diemer & Kravtsov (2015), in an observationally normalized CDM universe within which both typical values of $c$ and $M_{200}$ are functions of collapse epoch. We select the results from Diemer & Kravtsov (2015) that use the same cosmology as we assume throughout the paper (Hinshaw et al. 2013). The range in $c$ at fixed mass represents their reported one-sigma scatter of 0.16 dex. By nature, our calculation is done for an arbitrary initial profile, so we do not assume cosmological initial conditions. Thus we must take the mass–concentration relation from elsewhere. For the assumptions about the initial profile we have made above, the bold values represent the expected scatter in the central and half-mass collapse redshifts for NFW profiles of...
a given virial mass. The deviation of the concentration of a halo from the mean concentration at its virial mass could be used as a parameter in a two-parameter matching scheme between haloes and galaxies that would be correlated with the halo collapse time and potentially with galaxy properties at fixed mass. This is described further in our discussion of future work (Section 5).

The bottom panel of Fig. 2 shows another parameter of the final profile, the normalized squared maximum circular velocity $v^2_{\text{max}}/(GM_{200}/R_{200})$, versus the central collapse redshift of the same initial profiles as in the top panel. Again, as can be expected from dimensional arguments, this parameter is a function of the collapse redshift. We compare the circular velocity of the final NFW and corresponding $\alpha$ model profiles. The values are similar at low circular velocity (low $c$ or $\alpha$) and diverge for high circular velocity, but are close to one another for the relevant range of circular velocity values seen in dark matter haloes. Thus $v^2_{\text{max}}/(GM_{200}/R_{200})$ could also potentially be used as a parameter in a halo matching scheme, and would have the benefit of not being highly dependent on the fitting function chosen for the final dark matter halo profile.

In Fig. 3, we show the $v^2_{\text{max}}/(GM_{200}/R_{200})$ versus a related parameter, the excess normalized central potential $\delta\Phi/(GM_{200}/R_{200})$, for both NFW profiles and $\alpha$ models. While the values of $v^2_{\text{max}}/(GM_{200}/R_{200})$ are similar for both models at fixed mass and energy, the values of the central potential are significantly different for the two profiles shapes, implying that the maximum circular velocity
Contours of constant \( \delta \Phi / (GM_{200}/R_{200}) \) are related to the formation time via \( t_{2001} \) is the time at which 4 per cent of the mass at \( M_{200} \) is the scale factor, and \( a \) is the scale factor at which \( c \) models, over the range \( \alpha \). Figures 3 and 4 show the central potential is significantly different for the two models.

The normalized squared maximum circular velocity, \( v_{\text{max}}^2/(GM_{200}/R_{200}) \), versus the normalized excess central potential, \( \delta \Phi / (GM_{200}/R_{200}) \), at \( z = 0 \) for an NFW profile with varying \( c \) and an \( \alpha \) model with varying inner slope \( \alpha \). Unlike the maximum circular velocity shown in Fig. 2, the central potential is significantly different for the two models.

The normalized squared maximum circular velocity is a superior ‘common’ parameter between the two types of models to use to predict the initial halo collapse redshift. The central potential is a superior ‘common’ parameter between the two types of models to use to predict the initial halo collapse redshift.

For reference, we show contours of constant \( v_{\text{max}} \) in Fig. 4, in the top panel for varying \( M_{200} \) and \( c \) for NFW profiles, and in the bottom panel for varying \( M_{200} \) and \( \alpha \) for \( \alpha \) models, over the range of physical interest.

### 4 Comparison to Previous Work

Previous papers have investigated parametrizations of the time at which a halo formed, usually based on the results of dark matter simulations. Such papers include Bullock et al. (2001) and Wechsler et al. (2002). The model of Wechsler et al. (2002) improved upon that of Bullock et al. (2001), but obtained similar results.

Wechsler et al. (2002) examined the mass accretion histories of the most massive progenitors of individual haloes from dark matter simulations for \( z < 7 \), defining the halo mass to be the mass within \( \Delta_{\text{vir}} \) (Bryan & Norman 1998). Wechsler et al. (2002) found that the mass accretion histories of haloes at each timestep, despite halo mergers in the simulation, can generally be fit well by a simple analytic form:

\[
M(\alpha) = M_0 \exp \left[ -a, S \left( \frac{a_0}{a} - 1 \right) \right].
\]

(20)

Here \( a \) is the scale factor, \( a_0 \) is a reference scale factor at which the mass of the halo is \( M_0 \), \( S \) is an arbitrarily chosen constant, and \( a \) is taken to be the ‘formation scale factor’ of the halo, given the choice of \( S \). This form is self-consistent for different choices of \( a_0 \).

Taking the formation scale factor \( a_0 \) of the halo as defined above with \( S \) chosen to be 2, the authors assign a formation time to each halo and find that the concentration of each halo at some reference scale factor \( a_0 \) is related to the formation time via

\[
c_{\text{vir}} = 4.1 a_0 / a_c
\]

(21)

where the constant 4.1 is the concentration of haloes ‘forming’ at the present day, given the choice of \( S = 2 \).

A more recent paper looking at mass accretion histories in dark matter simulations is Zhao et al. (2009). The authors claim to obtain more accurate fits to halo accretion histories using a more complex model than Wechsler et al. (2002) and Bullock et al. (2001). Unlike Wechsler et al. (2002), they find that mass accretion histories follow a power-law form as opposed to an exponential. Zhao et al. (2009) find that the concentration of haloes at some observation time \( t \) is a function of the time at which their mass reached 4 per cent of the mass at the observation time, given by

\[
c = \left( 4^8 + \left( t / t_{0.04} \right)^{8.4} \right)^{1/8},
\]

(22)

where \( t_{0.04} \) is the time at which 4 per cent of the mass at \( t \) was reached. Note that halo concentrations in this model cannot be less than 4.

Due to the fact that we begin with arbitrary initial conditions and not cosmological ones, it is difficult to compare our results with those of Wechsler et al. (2002) and Zhao et al. (2009). However, both our model and the results obtained by these two papers find a monotonic relation between the collapse time or ‘formation time’ of haloes and their concentrations at fixed mass at any given time, where a higher concentration implies an earlier formation time. This is crucial as abundance matching-type methods require a monotonic relation between the parameters being matched.

### 5 Future Work

Using a combination of two halo parameters, one can perform two-parameter abundance matching to two observable parameters of galaxies. As shown above, parameters that correlate well with the physically relevant parameter of halo collapse time include \( c \) or \( \alpha \).
or alternatively \( \sigma^2 \sim GM_{200}/R_{200} \). These could be combined with the parameter \( M_{200} \) used in standard abundance matching, although these are not the only possible choices for matching.

Because we only consider two parameters, and standard abundance matching is most effective for massive galaxies, we plan to focus on two-parameter abundance matching to the FP of massive early type galaxies. The FP is an observed relationship between three parameters: the effective radius \( R_e \), the velocity dispersion \( \sigma \), and the surface brightness within \( R_e \) (or alternatively, the luminosity within \( R_e \)). Elliptical galaxies occupy a plane in the space of these three parameters:

$$ \log R_e = a \log \sigma + b \log I_e + c, $$

where \( a \) and \( b \) are constants and \( c \) is a redshift-dependent zero-point. Assuming that the mass-to-light ratio of elliptical galaxies is roughly independent of stellar mass, and that the galaxies are fully virialized, one would expect from the virial theorem that the FP would follow \( R_e \propto \sigma^2 I_e^{-1} \). However, the observed FP has a tilt with respect to this relation (e.g., Bezanson et al. 2013). Although the FP has some thickness and is not completely two-dimensional, this thickness is small (Graves et al. 2009b; Graves & Faber 2010). While we plan to focus on early-type galaxies, interestingly, Hearin, Franx & van Dokkum (2015) find that early- and late-type galaxies fall on the same mass FP.

Some recent works have explored the addition of another halo and galaxy parameter to abundance matching at fixed stellar and halo mass (Hearin & Watson 2013; Masaki et al. 2013; Hearin et al. 2014; Watson et al. 2015). In particular, the scheme presented in Hearin & Watson (2013) and Hearin et al. (2014) matches the colours of galaxies to a proxy for the halo age at fixed stellar and halo mass. The latter is parametrized as the earliest of three times: when the main halo progenitor mass exceeded \( 10^{12} M_\odot \), when the halo became a subhalo, or when the halo transitioned from fast to slow dark matter accretion, which is computed directly from the halo concentration as in Wechsler et al. (2002). In practice, the last, concentration-based, age parameter is the one used for all but the most massive galaxies. The authors find that their method is able to match a number of observables for galaxies separated into blue and red colours, including clustering statistics and the galaxy–galaxy lensing signal.

Our matching scheme is somewhat different in that all the FP parameters depend on both the halo mass and the collapse time of the halo, including the stellar mass. In particular, while more massive haloes are assumed to host more massive galaxies as in standard abundance matching, haloes with earlier collapse times at fixed mass are also assumed to host more massive galaxies. The ratio of galaxy stellar mass to host halo mass has previously been suggested to correlate with halo formation time (Limb et al. 2015). This would be expected on physical grounds, since an earlier collapse corresponds to a higher cosmic baryon density, more rapid cooling and thus more efficient star formation.

Once an ansatz has been chosen for the dependence of halo mass on halo collapse time at fixed galaxy stellar mass, the relationship between all FP parameters and the halo parameters of mass and collapse time can be determined by matching the abundances of galaxies and haloes as in standard abundance matching. The results of this scheme will be presented in a forthcoming paper.

Other galactic parameters have been found to be strongly correlated with parameters of the FP. These include galaxy colour, stellar mass-to-light ratio, and mean stellar population age and metallicity (Graves et al. 2009a,b; Graves & Faber 2010; Porter et al. 2014). Thus matching to the FP would take into account the variance in these parameters as well, potentially allowing a near-complete prediction of the properties of a massive galaxy residing in a halo based on the halo’s properties, and creating a better connection between the properties of the halo (including formation epoch) and the properties of the galaxy.

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**REFERENCES**

APPENDIX A

In Section 2.1, we show the equations determining the evolution of mass shells of an overdensity in a ΛCDM universe, where each shell has initial condition \( r = 0 \) at \( t = 0 \). In a ΛCDM universe, mass shells with overdensity \( \delta \) below some value will never turn around. For the case presented in Section 2.1, there is no simple analytic formula for this overdensity for all times (although it can be found numerically from the equations given therein), but there is one as the initial time approaches 0, given by equation (13). However, for the case in which all mass shells are taken to be initially on the Hubble flow \( \{r_i, t_i\} \) for a chosen initial time \( t_i \), there is an analytic solution for the minimum overdensity to turn around for all initial times. Here we present a short derivation of this value and compare to the solution for initial conditions \( r = 0 \) at \( t = 0 \) at early times.

Once again, we begin with the equation of motion for a shell containing mass \( M \):

\[
\frac{d^2 r}{d t^2} = -\frac{G M}{r^3} + \Omega_{\Lambda,0} H_0^2 r.
\]  

(A1)

Integrated once, we obtain

\[
\frac{1}{2} r^2 - \frac{1}{2} r_i^2 = \frac{G M}{r} - \frac{G \Omega_{\Lambda,0} H_0^2}{2} r^2 = \frac{\Omega_{\Lambda,0} H_0^2}{2} r_i^2.
\]  

(A2)

Here our initial condition is

\[
\frac{r_i^2}{H_0^2 r_i^2} = \frac{2GM}{r_i} + \Omega_{\Lambda,0} H_0^2 r_i^2, \quad \langle M \rangle = \frac{4}{3} \pi G \bar{\rho}(t_i) r_i^3.
\]  

(A3)

Combined with equation (A2), we obtain

\[
r^2 = \frac{2GM}{r} + \Omega_{\Lambda,0} H_0^2 r^2 - \frac{2GM - \langle M \rangle}{r_i}.
\]  

(A4)

Now we define

\[
r / r_i = \chi, \quad \frac{d}{dr} = H_i \frac{d}{d\tau}.
\]  

(A5)

giving

\[
\left( \frac{d\chi}{d\tau} \right)^2 = \frac{8\pi G \bar{\rho} M}{3H_0^2 \langle M \rangle} \chi + \frac{\Omega_{\Lambda,0} H_0^2}{H_0^2} \chi^2 - \frac{8\pi G \bar{\rho} M - \langle M \rangle}{3H_0^2 \langle M \rangle}.
\]  

(A6)

Noting that

\[
1 + \delta = \frac{M}{\langle M \rangle}, \quad \frac{8\pi G \bar{\rho}}{3H_0^2} = \Omega_{m,0}, \quad \frac{\Omega_{\Lambda,0} H_0^2}{H_0^2} = \Omega_{\Lambda,i}
\]  

(A7)

and that for a flat universe \( \Omega_{m,i} + \Omega_{\Lambda,i} = 1 \), we obtain

\[
\left( \frac{d\chi}{d\tau} \right)^2 = \frac{\Omega_{m,i}(1 + \delta)}{\chi} + \Omega_{\Lambda,i} \chi^2 - \Omega_{m,i} \delta.
\]  

(A8)

For any shell that turns around, \( (d\chi/d\tau)^2 = 0 \) for some maximal value of \( \chi \). Therefore we want to find the smallest \( \delta \) such that there exists a positive \( \chi \) where

\[
\chi^3 - \frac{\Omega_{m,i}}{\Omega_{\Lambda,i}} \delta \chi + \frac{\Omega_{m,i}}{\Omega_{\Lambda,i}} (1 + \delta) = 0.
\]  

(A9)

We also require that this cubic to be positive for \( \chi \) between 0 and the maximum value of \( \chi \) in order for \( d\chi/d\tau \) to be a real number. This cubic has inflection points at \( \chi_{\pm} = \pm(\Omega_{m,i} \delta/3\Omega_{\Lambda,i})^{1/2} \); for the previously mentioned conditions to hold, it requires that the value of the cubic at \( \chi_{+} \) be negative or 0:

\[
\left( \frac{\Omega_{m,i}}{\Omega_{\Lambda,i}} \right) \frac{\delta}{3} \left( \frac{3}{2} \right) - \frac{1}{2} \left( \frac{\Omega_{m,i}}{\Omega_{\Lambda,i}} \right) \left( \frac{1 + \delta}{\Omega_{\Lambda,i}} \right) \leq 0.
\]  

(A10)

This reduces to simply

\[
\frac{\delta}{1 + \delta} \geq \frac{27}{4 \Omega_{\Lambda,i}}.
\]  

(A11)

As stated, this is the exact solution for an initial profile that is assumed to be on the Hubble flow at time \( t_i \). This can be compared to equation (13), which is the solution for \( \delta \) with initial conditions \( r = 0 \) at \( t = 0 \) for all shells as \( z_i \to \infty \). Here the boundary between collapsing and forever expanding shells is given by \( \zeta = 0.5 \), by the definition of \( \zeta \). This gives for the minimal overdensity at early times

\[
\delta = \frac{9}{10} \Omega_{\Lambda,i}^{1/3} \left( \frac{\Omega_{m,i}}{\Omega_{\Lambda,i}} \right)^{1/3};
\]  

(A12)

whereas at early times for the solution on the Hubble flow we find

\[
\delta \approx \frac{3}{2} \left( \frac{\Omega_{m,i}}{\Omega_{\Lambda,i}} \right)^{1/3}.
\]  

(A13)

It can be seen that the derived \( \delta \) differs by a value of 3/5. This is the difference in overdensity expected for a matter-dominated universe between considering a hypersurface of constant \( H_i \) and one of constant \( z_i \), as derived in Gunn & Gott (1972). This ratio between the results will not hold at small \( z_i \) due to the effect of \( \Lambda \), but for early times the Universe is increasingly matter dominated and so the two solutions are the same modulo this factor of 3/5.