On the Doppler effect for light from orbiting sources in Kerr-type metrics

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ABSTRACT

A formula is derived for the combined motional and gravitational Doppler effect in general stationary axisymmetric metrics for a photon emitted parallel or antiparallel to the assumed circular orbital motion of its source. The same formula is derived by both the eikonal approximation and Killing vector approaches to elucidate connections between observational astronomy and modern relativity. The formula yields expected results in the limits of a moving or stationary source in the exterior Kerr and Schwarzschild metrics and is useful for broad range astrophysical analyses.

Key words: black hole physics – galaxies: distances and redshifts – galaxies: kinematics and dynamics – cosmology: theory.

1 INTRODUCTION

Light is the predominant means of detection and observation on astrophysical scales. However, light undergoes distortion due to various effects before being received on or near earth. The chief phenomenon in this regard is the Doppler shift in the observed frequency of a photon. We use the term ‘Doppler shift’ in a generalized sense throughout this paper, to refer to any effect that causes a photon’s frequency at detection to differ from that which it had when emitted. Thus, we are interested in a combination of factors. On the one hand, there is the kinematic effect resulting from relative motion of emitter and observer. This special relativistic effect, which commonly is called the ‘Doppler shift’, is analysed using the Minkowski metrics on the tangent spaces at the space–time events where a given photon is emitted and received. On the other hand, there are general relativistic effects due to the photon’s propagation through the intervening region of curved space–time. These are usually called ‘gravitational redshifts’. They include not only the effect of the Coulombic potential well implied by the mass distribution, but also the frame dragging effect caused by its spinning motion. Experimentally, of course, only a combination of these effects can be observed. We treat them all together under the name ‘Doppler shift’ accordingly.

The Kerr–Doppler effect resulting from the combination of effects listed above has been given in Fanton et al. (1997) for the Kerr black hole geometry. We have two aims in this paper. The first is to extend the analysis to arbitrary Kerr-type (i.e. stationary and axisymmetric) space–times, e.g. the coarse-grained, diffuse internal space–time of a spiral galaxy. The extension allows new, high-resolution observations (Lacroix & Silk 2013) to distinguish between different model based predictions. The second is to illustrate full derivations, in parallel formalisms used in the astrophysics and relativity communities, to draw attention to assumptions involved at each step in the analyses and to illustrate the utility of each approach. The astrophysics derivation is based on coordinates and an effective optical index of refraction for wave propagation (Narayan & Bartelmann 1996), and the modern relativity construction on conserved quantities along particle geodesics and Killing vectors (Wald 1984). Naturally, we will see that the two lead to the same final expressions.

Various frequency-shift effects have to be taken into account in a wide array of astrophysical contexts. These include, for example, the interpretation of X-ray spectral distributions originating from black hole accretion discs (Asaoka 1989; Laor 1991; Fanton et al. 1997; Bromley, Miller & Pariev 1998; Cunningham 1998; Fabian et al. 2000; Martocchia, Karas & Matt 2000; Li et al. 2005), and the implementation of satellite navigational systems (Linet & Teyssandier 2002; Bahder 2003; Pascual-Sanchez 2007). Depending on the specific context, emphasis has to be placed on different aspects influencing the shift. For example, where satellite navigational systems are concerned, the gravitational fields near earth are sufficiently weak to allow for the perturbative treatment of general relativistic effects. However, specific details of the Earth’s gravitational field, such as deviation from exact spherical shape, are important. Thus, while Linet & Teyssandier (2002), using a linearized axisymmetric metric, expands the Doppler shift in 1/c to fourth order, it includes not only its dependence on the Earth’s mass and angular momentum, but also the quadrupole moment of the mass distribution. In the context of black holes, the underlying metric has fewer complicating features, but one cannot invoke a perturbative expansion in 1/c, nor may one always neglect deviations of light geodesics from straight lines. Therefore, in general one is forced to integrate the geodesic equations numerically,
The present note is intended for a certain special class of geometries, wherein simple analytical results for the Doppler shift can be obtained in order to give guidance for more complex geometries or to distinguish between different model based assumptions. One such example is treated in Radosz, Augousti & Ostasiiewicz (2009), analysing situations in which the Doppler shift factorizes into the kinematic contribution and the general relativistic contribution (in general, these contributions are intricately entangled). Another example where the formula derived in this paper is useful is in the analysis presented by Schönenbach et al. (2014), contrasting the predictions of emission from the accretion discs of central galactic nuclei in General Relativity versus pseudo-complex General Relativity (pGR).

The geometries under consideration here comprise all stationary axisymmetric metrics of the Kerr type, i.e. all metrics independent of time t and azimuthal angle \( \phi \) in polar (Boyer–Lindquist) coordinates \((t, r, \theta, \phi)\), with \( g_{\theta\theta} = g_{\phi\phi} \) the only non-vanishing off-diagonal elements. In such metrics, we consider a test particle moving in the \( \phi \)-direction (which is the case for emitters in circular orbits in the equatorial plane, and also for emitters at the apsides of other orbits in that plane), and emitting in (or against) that same direction. In this setting, the Doppler shift observed by a receiver in asymptotic flat space can be given without recourse to a perturbative expansion. The treatment does not yield information about the light geodesic; thus, while the result for the Doppler shift derived here in itself is exact, its practical application will usually require complementary information about, say, the position of the emitter. For example, if one is interested in reconstructing a radial velocity distribution from the Doppler-shifted light observed, one has to link the Doppler shift to the radius of the emitter’s orbit, which in general may require a numerical treatment of the deflection of the light ray, as mentioned above. On the other hand, due to the general form of the class of metrics considered, the Doppler formula given here is expected to be of use in the context of interior Kerr-type metrics describing space–time structure close to, or inside, extended rotating matter distributions. Whereas the case which we present here may require further generalization depending on the context of application, we none the less regard it as representative of what we would have further generalization depending on the context of application, we none the less regard it as representative of what we would have.

The Doppler shift in the geometries described above will be derived using two different methods, namely, by an eikonal approach in Section 2, and by employing invariants constructed from the Killing vectors (KV) of the metric in Section 3. The special case of the Kerr metric (Chandrasekhar 1983; O’Neill 1995) and further limiting cases are considered at the end of Section 3.

## 2 EIKONAL APPROACH

This approach arises from two aspects of the physics. The first is that we may define the energy \( E \), and thus the frequency \( \omega \), of a photon as measured by any observer at any location by Hartle (2003) as

\[
\omega_0 = -u \cdot k = -u^\mu k_\mu,
\]

where \( u \) is the 4-velocity of the local observer and \( k \) is the momentum 4-vector of the photon at the observer’s location (natural units).

The second aspect is that, in the approximation we are using, the photon can be propagated out to asymptotic infinity, enduring only negligible bending of its ray path. This argument allows us to use the eikonal approximation to read off an effective index of refraction from the wave equation. We assume that the scalar wave equation will capture the relevant physics in the eikonal limit. This wave equation for a wavefunction \( \Psi(x) \) is

\[
\Box \Psi = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^\mu} \left( g^{\mu\nu} \sqrt{g} \frac{\partial}{\partial x^\nu} \right) \Psi = 0,
\]

where \( g^{\mu\nu} \) are the contravariant metric components, and \( g = \det(g_{\mu\nu}) \) is the determinant of the matrix of covariant components \( g_{\mu\nu} \). We consider general Kerr-type metrics in Boyer–Lindquist coordinates, \((r, \theta, \phi, \psi)\), whereby the non-zero \( g_{\mu\nu} \) are \( g_{rr}, g_{\theta\theta}, g_{\phi\phi}, g_{\psi\psi} \), and \( g_{\phi\psi} \), independent of \( \phi \) and \( t \).

We write an eikonal approximation for the wavefunction,

\[
\Psi(r) = \Psi_o \exp(i \phi),
\]

where \( \Psi_o \) is a constant amplitude and \( \phi \) is the integral along the photon path of the differential phase

\[
\phi = k_o dx^\nu.
\]

We consider a photon emitted in a direction tangent to a circular orbit, so that in the neighbourhood of the emission point, \( k_e = k_0 = 0 \). We also write

\[
k_e = e^\nu = k = k/\sqrt{g_{\nu\nu}},
\]

where we have defined the wavenumber \( k = e^\nu \cdot k = k/\sqrt{g_{\nu\nu}} \).

Here, \( e^\nu = e^\nu/\sqrt{g_{\nu\nu}} \) is simply a unit vector parallel to the covariant basis vector \( e^\nu \); it is not part of an orthonormal tetrad. (Note that while \( k_o \) is dimensionless, \( k \) has dimension of inverse length, appropriate for a wave number.) Also, using equations (3)–(5), and the fact that the metric is independent of \( t \), the wavefunction in a neighbourhood of the photon emission point \((r, \theta, \phi, \psi)\) is

\[
\Psi(x) = \Psi_o \exp(-i \omega t) \exp \left( ik/\sqrt{g_{\nu\nu}}(\phi - \phi_o) \right),
\]

where \( \omega = -k_o > 0 \) is the invariant angular frequency of the wave with respect to the time coordinate. It is the frequency measured by an observer at rest at spatial infinity, where \( k = k_o \).

Inserting equation (6) into (2) yields

\[
\omega^2 - g^{\nu\mu} 2ng_{\nu\mu} \sqrt{g_{\nu\nu}} - n^2 g^{\nu\mu} g_{\nu\mu} = 0,
\]

where the effective index of refraction \( n \) is defined by

\[
v_{\text{photon}} = \omega/k = c/n
\]

and \( v_{\text{photon}} \) is the coordinate light speed at the point of emission. The general solution of equation (7) is

\[
n = -g_{\nu\mu} - (g_{\nu\mu})^2 / (g_{\nu\mu} g_{\mu\nu}).
\]

Note that the denominator is negative, since \( g_{\nu\mu} \approx -1 + 2\Phi/c^2 \) in weak-field Kerr-type metrics, where \( \Phi \) is the Newtonian gravitational potential (Hartle 2003). Therefore, we chose the (−) sign preceding the square root in order to obtain a positive \( n \).

For an orbiting observer, the spatial part of the product in equation (1) is

\[
k_e u^\nu = k/\sqrt{g_{\nu\nu}} \Omega u^\nu,
\]

where we have defined the angular velocity

\[
\Omega = \partial \phi / \partial t = u^\nu / u^\mu,
\]

which is a constant for a circular orbit. Note that (i) \( \Omega \) may be negative or positive, for source motion away from or towards the
asymptotic observer, respectively; and (ii) for sources in circular orbits the magnitudes of these two angular velocities will be different in Kerr-type metrics because of their non-zero $g_{tt}$.

From equations (8)–(11), we obtain

$$\omega_o = ou' \left[ 1 - n\Omega / \sqrt{g_{ee}} \right].$$

We obtain an expression for $u'$ from the constraint $u \cdot u = -1$:

$$u' = 1 / \sqrt{\left( g_{tt} - 2 \Omega g_{t\phi} - \Omega^2 g_{\phi\phi} \right)}.$$

Then, substituting equations (11) and (9) into equation (13) yields the general Kerr–Doppler formula for a source moving directly towards or directly away from the asymptotic observer:

$$\omega_o / \omega = \left[ g_{tt} + \Omega \left( g_{t\phi} + \sqrt{g_{t\phi}^2 - g_{tt} g_{\phi\phi}} \right) / g_{tt} - 2 \Omega g_{t\phi} - \Omega^2 g_{\phi\phi} \right].$$

3 KILLING VECTOR APPROACH

This derivation will take advantage of the conserved quantities of time-like and null geodesic motion which result from the one-parameter families of symmetries of the space–time, described by the KV fields (Wald 1984, see Appendix C.3.1). For stable, circular, equatorial orbits that most closely approximate those in spiral galaxies (Klypin, Zhao & Somerville 2002), the relevant KV are the time-like $\xi = (1, 0, 0, 0)$ and the axial $\eta = (0, 0, 0, 1)$, for Kerr-type metrics in Boyer–Lindquist-type coordinates $(t, r, \theta, \phi)$. The norms of these KV fields are defined as is done in Wald (1984).

The 4-velocities for such particles, $u = (u', 0, 0, u')$, can be expressed in terms of the KV as

$$u = A \xi + B \eta$$

for $A$ and $B$ coefficients determined by the symmetries which give the conserved ‘energy’ $E$ and ‘angular momentum’ $L$ (both per unit mass) of the particles. The conserved quantities $E = -u \cdot \xi$ and $L = u \cdot \eta$ are defined as in Hartle (2003, equation 15.17).

By inspection, $E$ and $L$ can be combined into the system of equations:

$$\begin{pmatrix} -E \\ L \end{pmatrix} = \begin{pmatrix} g_{tt} & g_{t\phi} \\ g_{t\phi} & g_{\phi\phi} \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix}.$$

Matrix inversion then yields $A$ and $B$ in terms of $E$ and $L$, such that equation (15) can be written as

$$u = A \xi + B \eta = g_{t\phi} E + g_{tt} L / k - g_{t\phi} E + g_{tt} L / k \eta$$

for $k = (g_{tt}^2 - g_{t\phi} g_{\phi\phi})$.

For time-like geodesic motion, the condition $u \cdot u = -1$ yields

$$1 = g_{tt} L^2 / k + 2 g_{t\phi} E L + g_{t\phi} E^2 / k.$$

Similarly, for the line-of-sight photons emitted from such particles, the initial 4-momentum is only in the two KV directions $\xi$ and $\eta$. Thus, the photon 4-momentum $k$ can be expressed exactly as is $u$ in equation (17), but replacing the conserved particle quantities $E$ and $L$ with those of the photon $e$ and $l$. The photon null condition $0 = k^2 k_\eta$ then yields a relation between $e$ and $l$:

$$l / e = -g_{tt} - \sqrt{E^2} / \sqrt{g_{tt}}.$$

which connects the ratio $l / e$ to the previously obtained refractive index $n$ of equation (9). Again, the positive root is used because our photon moves in the forward direction.

Then, the inner product of the particle and photon 4-vectors can be expressed as

$$\omega_o = -u \cdot k = c g_{t\phi} E + g_{tt} \left( E \Omega / L + g_{tt} L \right).$$

Finally, given that $k \cdot \xi = \text{const.}$ along the photon path (Wald 1984, C.3.1), the stationary observer at infinity whose 4-velocity is $u_\infty = \xi$ measures the received photon frequency as $\omega = u_\infty \cdot k = e$. This yields from equation (20) the ratio:

$$\omega_o / \omega = 1 / \left[ g_{t\phi} E + g_{tt} (E \Omega \sqrt{g_{ee}} + L) + L g_{t\phi} \sqrt{g_{tt}} \right].$$

Substitutions for $E$ and $L$, from Hartle (2003, equation 15.17), into equation (21) then gives

$$\omega_o / \omega = u' \left( 1 - \Omega n \sqrt{g_{ee}} \right),$$

for $\Omega$ as in equation (11). This is the same result obtained by the eikonal approach in equation (12).

Note, the same result can be derived more quickly in the special case when the relationship between $\Omega$ and $r$ is known, such as when Keplerian orbits become physically reasonable far from the accretion disc or when a normalization condition is applied which reduces equation (1) to $\omega_o = -u \cdot k = u' k_\eta - u' k_\phi = u' e - u' l$. Used together with the null condition for the wave vector, the result in equation (12) can be derived in three lines. However, the relationship between $k_\eta$ and $k_\phi$, and the conserved quantities $l$ and $e$, is in general more complicated (Hartle 2003, see equation 15.7).

In the general case, the form derived in equation (21) is arguably more useful, where the relationship between the orbital frequency $\Omega$ and the $r$ coordinate is most readily expressed by the relationship between $E$, $L$, and $r$. These are the correct sets of variables that you most easily calculate with in General Relativity.

In the special case of the Kerr metric proper (Chandrasekhar 1983; O’Neill 1995),

$$g_{tt} = -(1 - 2Mr / \Sigma), \quad g_{t\phi} = (r^2 + a^2)^{-1} - a^2 \Delta \sin^2 \theta / \Sigma$$

$$g_{\phi\phi} = \Sigma, \quad g_{tt} = \Sigma / \Delta, \quad g_{t\phi} = g_{t\phi} = -2Mr \sin^2 \theta / \Sigma,$$

where

$$\Sigma = r^2 + a^2 \cos^2 \theta, \quad \Delta = r^2 - a^2 - 2Mr,$$

and for an emitter in the equatorial plane, $\theta = \pi / 2$, we have verified that our Doppler formula (14) coincides with the expression given in Fanton et al. (1997), cf. also Asaoka (1989), Cunningham (1998), and Li et al. (2005). Specializing further to the Schwarzschild metric limit by setting $a = 0$, with $M = M_3 / c^3$ for a source of mass $M_3$, one obtains from equations (9) and (13), again for an emitter in the equatorial plane,

$$n = (-g_{tt})^{-1 / 2} \left[ 1 - 2M / r \right]^{-1 / 2}$$

$$u' = c \left[ 1 - 2M / r - \Omega^2 r^2 / c^2 \right]^{-1 / 2},$$

where the appropriate factors of $c$ have been inserted. Then after a little algebra, equation (14) yields

$$\omega_o / \omega = \left[ 1 - 2M / r \right] \left( 1 - 2M / r \right)^{1 / 2} + \Omega \sqrt{1 - 2M / r} / \left( 1 - 2M / r \right)^{1 / 2} - \Omega / c \right]^{1 / 2}.$$

Clearly, this result has the correct limits: one obtains the usual gravitational redshift for non-moving optical sources ($\Omega = 0$),
and the usual longitudinal Lorentz–Doppler ratio for $M = 0$ but $\Omega r/c = v/c$, where the relative source–observer velocity $v = \Omega r$ can be positive or negative. Note that, for a source in circular orbit, $M/r \approx v^2/c^2$, so equation (26) yields the usual Lorentz–Doppler formula to first order in $v/c$.

### 4 SUMMARY AND DISCUSSION

In this paper, we derived a formula for the motional and gravitational ‘Kerr–Doppler’ effect for a photon emitted tangentially to the motion of its source. The source was restricted to be moving azimuthally in any Kerr-type metric, i.e. any metric which when expressed in polar coordinates $(t, r, \theta, \phi)$ has only $g_{tt}$, $g_{rr}$, $g_{\theta\theta}$, $g_{\phi\phi}$, and $g_{t\phi}$ non-zero and functions only of $r$ and $\theta$. The formula, equation (14), provides the frequency of the photon measured by an observer at rest at spatial infinity in terms of that measured by a local observer comoving with the source. We showed that the formula yields expected results in the limits of an orbiting or stationary source in the exterior Schwarzschild metric and a moving source in flat space, and also agrees with the result of Fanton et al. (1997) for the exterior Kerr metric.

In obtaining the formula, we utilized two seemingly different approaches, an eikonal approximation solution to a scalar wave equation in the Kerr-type metric, and a KV representation of both the source circular motion and the photon motion. The two approaches produced the same formula, because despite apparent dissimilarities the underlying physics is the same. For example, the local propagation, or wave 3-vector used in the eikonal approach is (proportional to) the local photon 3-momentum used in the KV approach. While the KV approach is limited to the particular highly symmetric application that we treated, that of a photon emitted tangentially to the circular orbit of a source, it allows analysis in a more modern relativistic context, in which the relationship between the orbital velocity, $\Omega$, and the radius is determined by the conserved quantities of the Lagrangian, $E$ and $L$. On the other hand, the eikonal method should be applicable for a local photon propagation 3-vector in any direction relative to the source motion, cf. Fanton et al. (1997) for the special case of the exterior Kerr metric. The wave equation (2) would then yield a different expression for the local effective refractive index than we obtained in equation (9) for tangential emission.

The categorical derivation of the formula from the two perspectives given here allows construction of model based predictions, such as those made in Schönchenbach et al. (2014), which can then be used to distinguish feature of the models’ physics. In example, the predictions made in Schönchenbach et al. (2014) can distinguish between General Relativity and pcGR in late 2015 when the Event Horizon Telescope (Broderick et al. 2011; Fish et al. 2014) begins to collect data at our Galaxy’s nucleus and that of M87. The advent of such high-resolution observations, where we can distinguish fine features in emission lines from accretion discs, makes the formula given here important as we move forward in our analysis of the cosmological gravity theory.

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### REFERENCES


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