Convergence of Performance Calculation of Twin Spool Turbojet and Turbofan

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ABSTRACT

A method of order reduction of multivariate non-linear system to calculate twin spool turbojet and turbofan performance is recommended. With the proper incorporation of other approaches, it can achieve satisfactory results in any flight condition and throttle setting. Compare with the experimentation data, it can models the engine very well.

LIST OF SYMBOLS

A cross section area
F thrust
H altitude
M mach number
N rotor speed
P pressure
SFC specific fuel consumption
T temperature, control variable
Y variable
Z residue
β auxiliary variable on compressor performance map

Subscripts
c compressor
c1 cooling air
f fuel
h high pressure

INTRODUCTION

Some performance calculation programs of twin spool turbojet and turbofan have been published, such as the calculation programs in references (1)(2)(3)(4)(5). Using these programs, most the engine operation conditions can be calculated but some operation conditions remain unsolved. This is due to the model of engine performance, which is represented by a series of multivariable non-linear equations, it must be linearized during the solution. To solve this problem, we must analyse with respect to mathematical and physical essence. In this paper, we discuss this problem, and describe calculation methods which produce satisfactory results of performance calculation of twin spool turbojet and turbofan at any flight condition and throttle setting.

The methods recommended in this paper may be extended to any calculation of non-linear system.

MATHEMATICAL MODEL

The purpose of the engine performance calculation is to establish the mathematical model, then the engine thrust and the specific fuel consumption under given flight condition (altitude H and speed of aircraft M), and given throttle setting (given one of the following parameters: fuel mass flow mf, gas temperature at the entrance of high pressure turbine Tt3, rotor speed N1 and Nh) can be calculated. Those calculations need to know the performance of each component (include control system) which were obtained from
experiment and were non-linear. It is difficult to write the equation between output parameters (F, SFC) and input parameters (H, M_0, throttle and exhaust area A_9). But it is easy to write a calculation program according input parameters, performance and matching relation of each component. Using this calculation program, we must calculate some components before its working condition is determined. Therefore giving test value for some parameters is necessary, and we can solve it by iteration. According to the working conditions and relations of components, we check whether the test value are correct or not. Conditions are shown as following: compressor rotor speed equal to turbine rotor speed, power of compressor and turbine are equal, and the air mass flow and gas mass flow of each section are continuous.

We discuss how to choose the parameters to which test value may be given during calculation. For a twin spool turbojet, if the input parameters are H, M_0, N_1 (represent throttle), A_9, there are at least three parameters: \( \beta_1 \), \( \eta_h \) and \( T_{t3} \) which require test value to calculate engine thrust and SFC. \( \beta_1 \) represents the auxiliary parameter on the compressor performance map. \( \eta_h \)'s represents test values: \( Y_1 = \eta_1 \), \( Y_2 = \eta_h \), \( Y_3 = T_{t3} \).

According to the following three conditions to check whether the above three test values are correct or not.

(1) Given input, test value of compressor and the performance of compressor and burner, we can calculate the gas mass flow \( m_3 \) at the exit of burner. Given pressure ratio of high pressure turbine and according to its performance, we can get the gas mass flow \( m'_{33} \) in front of this turbine. The pressure ratio of the high pressure turbine is obtained from equilibrium conditions of high compressor power needed and high turbine power supplied, \( m_3 \) and \( m'_{33} \) may be equal, we have

\[
\frac{m_3 - m'_{33}}{m_3} = 0
\]

If they are unequal, mark residue

\[
Z_1 = \frac{m_3 - m'_{33}}{m_3}
\]

(2) Given the pressure ratio of low pressure turbine and according to its performance, we can get the gas mass flow \( m'_{33} \) in front of this turbine. This pressure ratio of low pressure turbine is obtained from equilibrium condition of low compressor power needed and low turbine power supplied. The \( m_{33} \) may be equal to the gas mass flow \( m_{33} \) exhausted from high pressure turbine (the gas mass flow exhaust from burner plus the cooling air \( m_{c1} \) entering the high pressure turbine), we have

\[
\frac{m_{33} - m_{33}}{m_{33}} = 0
\]

If they are unequal, mark residue

\[
Z_2 = \frac{m_{33} - m_{33}}{m_{33}}
\]

(3) We can calculate the area of the exhaust nozzle \( A_9 \) according to the total pressure \( P_4 \), total temperature \( T_4 \), gas mass flow \( m_4 \) at the entrance of exhaust nozzle and its performance. The \( A_9 \) may be equal to the input value \( A_9 \), we have

\[
\frac{A_9 - A_9}{A_9} = 0
\]

If they are unequal, mark residue

\[
Z_3 = \frac{A_9 - A_9}{A_9}
\]

The calculation program and iteration included above three test parameters and three check parameters often fails in the following situations:

(1) The working point on the high compressor performance map which is determined by the equivalent mass flow (calculated by using the performance of low compressor exhaust air) and the equivalent rotation speed (calculated by using the test value of high compressor speed) is often located far beyond the normal working range.

(2) In the process to calculate three check parameters by using three given values of test parameter, we must use some component performance. It often makes the differential of test parameter and true parameter greater than ever, because of the non-linearity of component performance and the error accumulation in the process of calculation. It leads to the failure of the above-stated calculation, or result in a greater residue than ever and makes the next test value exceed the reasonable range.

It is necessary to add auxiliary parameter \( \beta_h \) to mark the position on the map of high compressor performance as a test parameter, and then to add a check equation for the air mass flow \( m_{11} \). The value of \( m_{11} \) (calculated by using the low compressor performance) must equal to the value of \( m'_{11} \) (calculated by using the high compressor performance), i.e.

\[
\frac{m_{11} - m'_{11}}{m_{11}} = 0
\]

So that, six test parameters and six check parameters are necessary to calculate twin spool turbojet performance. For a mixed
flow turbofan, it is necessary to add by-pass ratio as another test parameter. And then add a check equation, i.e. at the entrance of mixture the static pressure of air in the bypass and of gas in the core must be equal.

In summary, the performance calculation of twin spool turbojet and turbofan becomes a problem of solving a multivariable non-linear equation which includes some control variables.

\[
\begin{array}{c}
\sum_{i=1}^{n+1} f_i(Y,T) = 0
\end{array}
\]

In this equation T is a control variable.

\[ T = [t_1, t_2, t_3, t_4]^T \]

Four vectors \( t_1, t_2, t_3, t_4 \) are input parameters \( H, M_0, N_1 \) (or one of the \( N_h, T_t3, \varphi_f \)), and \( A_9 \). \( Y \) is a test parameter, it is an \( n \) dimensional vector.

\[ Y = [Y_1, Y_2, \ldots, Y_n]^T \]

For a twin spool turbojet, \( n=6 \), for mixed flow turbofan, \( n=7 \).

When input parameters are given, parameter \( T \) in equation (1) disappears, and then only variable \( Y \) is included in equation (1).

\[ f = [f_1, f_2, \ldots, f_n]^T \]

It is a vector function, the value of \( f_1, f_2, \ldots, f_n \) are equal to the value of residue \( Z_1, Z_2, \ldots, Z_n \). In this paper, the method of \( n+1 \) points sequential secant (or called \( n+1 \) points residue method) was used to solve the equation (2), residue should be smaller than 0.001.

Iteration to solve multi-variable non-linear system in turbojet and turbofan performance calculation has been extensively adopted. It can be employed to calculate most of the operation conditions, but in fact some operation conditions remain unsolved. Reference(3) and (5) had been used to calculate the engine performance at sea level, it often makes calculation fault when low pressure rotor speed less than 80% design value.

**ITERATION CONVERGENT MEASURES**

In this paper some measures are adopted to make iteration convergent.

Order reduction method

The non-linear system which has 6 or 7 orders is reduced to non-linear system which has 3, 2 and 1 order combined, or has 3 and double 2 order combined. Because the order is reduced, the relationship of unknown values become less sensitive, and then the range of test value may be expanded. It can be proved (see appendix) that the lower is the order of \( n+1 \) point residue method, the higher will be the iteration speed.

The steps of calculation are outlined in "Flow chart of reduced order calculation" Figure 1.

After completing the calculation of two parts mentioned above, three equations will be used to check the values of first layer test parameter. The power of high pressure rotor and low pressure rotor must balance and the gas mass flow \( m_{33} \) which is calculated according to the data of high pressure turbine and the gas mass flow \( m'_{33} \) which is calculated according to the data of low pressure compressor must be equal. Secondly, the result of exhaust area \( A_g \) must be equal to the test value \( A_g \).

In the physical model of multi-layer iteration of twin spool turbojet mentioned above, another test parameter \( P_{t3} \) complemented for reducing order. And then a check equation is added. The result \( P_{t3} \) must be equal to the test value \( P_{t3} \). There are seven test parameters in total.

It is necessary for twin spool turbofan to add by-pass ratio \( \beta \) as a test parameter in first layer of test parameters, and then add a check equation, static pressure equality of the bypass and core streams at the entrance of mixture chamber.

The mathematical model of two layers reduced order iteration mentioned above agrees with the physical model of engine operation very well. This model converges easily during calculation.

**Carefulness of selecting the initial values of test parameters**

Though reduction order method had adopted, we must give initial value of test parameter near its result also.

Because of the variation of flight condition \( H=6-200km, M_0=0.2-2.2 \), the temperature \( T_{T1} \) at the entrance of compressor may change from 216k to 450k. When the low pressure rotor speed \( N_1 \) as an input is given, the high pressure rotor speed \( N_3 \), the total pressure \( P_{t3} \) and total temperature \( T_{t3} \) at the entrance of high pressure turbine have a great range of variation. All of those parameter has its equavalent parameter, there are certain
relations between them. It is necessary to establish the relationship between test value and input by means of similarity principle to cause the test value near the result.

In the second layer of iteration, the test value of $P_{24}, P_{33}$ in compressor burner part may be any value in its reasonable range. In this paper, we take the mean value of reasonable range as initial test value. The initial test value of $P_{24}, P_{33}$ in turbine exhaust part may be a value in the range of $P_{33}$ and $P_{24}$ and take a suitable proportion with them. During the calculation, iteration of these two parts converges easily and rapidly.

A method makes test value approaching its result

The $n+1$ point residual method requires that $n+1$ group of approximate results are given as test values.

$$Y_k = \left[ Y_k^{(1)}, Y_k^{(2)}, \ldots, Y_k^{(n)} \right]^T \quad k = 0 \sim n$$

Solve this equation and $n+1$ group of residuals are obtained

$$Z_k = \left[ Z_k^{(0)}, Z_k^{(1)}, \ldots, Z_k^{(n)} \right]^T \quad k = 0 \sim n$$

to establish linear equations.

In this paper, we have taken some measures to make result take place between the test value of each group. We give first group of test value by using the previous method. By changing one of the test value in order we can get other $n$ groups of test values. For example, the test value of $1$st group may be

$$Y_1 = [Y_1^{(0)}, Y_1^{(1)}, \ldots, Y_1^{(n)}]^T$$

in above equation $Y_1^{(j)} = Y_1^{(j)}$ ($j \neq 1$)

The principle to determine $Y_1$ is that, the residue $Z_1$ of this group is compared with the residue $Z_0$ of the first group, at least one of the component must change its sign, or make one of the component which changes its value most and approaches its value to zero.

After adopting this method, we can decrease the times of iteration, and accelerate the speed of calculation.

Limitation of test value during iteration

The mathematical model of engine performance calculation is of non-linear system, during calculation it may be linearized. The value of test parameter always out of reasonable range and the base number of index number will be negative or the number in square root will be negative. It results in calculation stop and failure. Therefore, it is necessary to restrain the value of test parameter with a reasonable range. If the test value is limited several times in repeated calculation, the test value must be different from each other, otherwise the linear equations will be equivalent equations, and the coefficient matrix of system will be singular, resulting in calculation failure.

EXAMPLE

The J-7 turbojet engine performance at several throttle settings and several flight conditions has been calculated. Figure 2 shows the result of calculation at sea level condition which is compared with the performance of engine test value. Every calculation point can be solved satisfactorily. The accuracy is determined by the requirement of residual value during iteration, in this program residuals are less than .001. The difference between the result of calculation and the engine test value is due to the inaccuracy of component model and the error of measurements.

This program is implemented on UNIVAC 1100 and its capacity is about 200K, the time to calculate one operating point is about two seconds.

CONCLUSIONS

According to the physical model of twin spool turbojet and turbofan performance, reducing the order of performance calculation non-linear system from 6-7 orders to 2-3 orders, and using other auxiliary measures, calculation can converge at any conditions (any flight condition and throttle, but the working point of each component must be in reasonable range).

The method mentioned above has been used to calculate the twin spool turbojet performance at any flight conditions and any throttle setting, and the results has been obtained efficiently and satisfactory. During calculations the maximum fuel mass flow of mean fuel pump and afterburner fuel pump had been limited. During afterburner operating condition, according to the static characteristic of afterburner fuel control system, control afterburner fuel mass flow by using static pressure $P_2$ at the exit of compressor and $P_4$ at the exit of turbine. The results coincide satisfactorily with engine performance tested at sea-level condition.
Input \((H, W_0, N_1, A_9)\)

Calculate test value \(N_h, T_{t3}, P_{t3}\)

Check and readjust \(N_h, T_{t3}, P_{t3}\) in reasonable range by means of similarity principle

Calculate test value \(\beta_1, \beta_h\)

Check and readjust \(\beta_1, \beta_h\) in reasonable range

Calculate operation condition of compressor part

Given test value \(\beta_1, \beta_h\)

Calculate operation condition of burner

Calculate test value \(P_{t33}, P_{t4}\)

Check and readjust \(P_{t33}, P_{t4}\) in reasonable range

Calculate operation condition of turbine, afterburner and exhaust part

Given test value \(P_{t33}, P_{t4}\)

Calculate operation condition of burner

Output

Fig. 1 Flow chart of order reduction calculation
REFERENCE


APPENDIX

For

\[ f(x) = 0 \]  \hspace{1cm} (1)

which is a system of \( n \) nonlinear equations in \( n \) unknowns, where \( f: \mathbb{R}^n \rightarrow \mathbb{R}^n \), after its \( n+1 \) approximate solution vectors \( x_k (i=1,2,...,n+1) \) are given, use following iteration formula

\[
\begin{align*}
X_k &= \left[ x_k, x_{k+1}, \ldots, x_{k+n} \right], \\
e_{n+1} &= \left[ 0, \ldots, 0, 1 \right]^T.
\end{align*}
\]

where \( k = n+1, n+2, \ldots \).

It is easy to prove, if the initial points \( x_i (i=1,2,\ldots,n+1) \) are just same, the sequence (4)-(5) and the sequence (2)-(3) are just same also, and equal to \( \{x_k\} \rightarrow x^* \).

Theorem

Let \( x^* \) be the solution of system (1), and let \( f: \mathbb{R}^n \rightarrow \mathbb{R}^n \) be continuously differentiable on the neighborhood \( \mathcal{O} \subset \mathbb{R}^n \) which include \( x^* \), and assume that \( \frac{df(ax)}{dx} \) is nonsingular. Assume, further, that the sequence \( \{x_k\} \) of (2)-(3) is well defined, and for all \( k \geq n+1 \) satisfy

\[
\left\| \begin{bmatrix}
\frac{f(x_k)}{\|f(x_k)\|} \\
\vdots \\
\frac{f(x_{k+n})}{\|f(x_{k+n})\|}
\end{bmatrix} \right\| \leq c \tag{6}
\]

where \( c \) is a constant, \( f_k = x_{k+1} - x_k \neq 0 \) (\( j=1,2,\ldots,n \)). Then there exists a closed ball \( \mathcal{S}_0 = \{x | x-x^* \leq \delta, \delta > 0 \} \subset \mathcal{O} \) and constant \( r > 0 \), so that if the initial points \( x_k \in \mathcal{S}_0 \) (\( i=1,2,\ldots,n+1 \) and \( \|f_k\| < c (\forall k \geq n+1, j=1,2,\ldots,n) \), then \( \|x_k-x^*\| \leq \delta \), \( \|x_k-x^*\| \leq c \), and \( \lim_{k \to \infty} x_k = x^* \). If, in addition, \( r=\infty \)

\[
\|Df(x)-Df(y)\| \leq L \|x-y\|, \quad \forall x, y \in \mathcal{O} \tag{7}
\]

holds, where \( L \) is a constant, then the rate of convergence of \( \{x_k\} \) is \( \lambda \)-order where \( \lambda \) is the positive root of \( t^{n+1} + t^n - 1 = 0 \).

Proof

Proof of the first part of theorem is in Reference 6. Now assume that the condition (7) holds. It follows from the first part of theorem that, if the initial points \( x_k \in \mathcal{S}_0 \) (\( i=1,2,\ldots,n+1 \) and \( \|f_k\| < c (\forall k \geq n+1, j=1,2,\ldots,n) \), then for any \( k \geq n+1 \) we obtain

\[
\|x_{k+1}-x_k\| = \|x_k-x^*-A_k f(x_k)\| \\
= \|A_k \left( (A_k-Df(x_k))f(x_k) + Df(x_k)(x_k-x^*) \right) \| \\
= \|A_k \| \| (A_k-Df(x_k))f(x_k) + Df(x_k)(x_k-x^*) \| \|
\]

Set

\[
G_k = \frac{f(x_k)}{\|f(x_k)\|}.
\]

Then due to the continuously differentiable property of \( f \) on \( \mathcal{O} \), and by (7), we obtain

\[
\|G_k\| \leq \sup_{s \in \mathcal{O}} \|Df(x_1+x^*)-Df(x_1)\| \|x_1-x^*\| \|
\]

and

\[
\|f(x_k)-f(x^*)-Df(x^*)(x_k-x^*)\| \\
\leq \sup_{s \in \mathcal{O}} \|Df(x_1+x^*)-Df(x_1)\| \|x_1-x^*\| \\
= \|Df(x^*)\| \|x_1-x^*\|. \tag{8}
\]

(1) \( Df(x^*) \) is Jacobi matrix of \( f(x) \) at \( x^*. \)
\[ \| x - x^* \|_2^2. \]

And by (6) and (9), we can obtain
\[
\| A_{k-i} f(x_k) \| \leq \| G(R_i^k) \| \| A \| \| x_k - x^* \|_2 \leq C \max_{i \leq j \leq n} G(R_j^k) \| x_k - x^* \|_2
\]
\[
\leq C \beta L \| x_k - x^* \| + C \beta L \sum_{j=1}^{n} \| x_{k-j} - x^* \|, \]

where \( \beta \) is a constant which is related with used norm only.

Thus the inequality (8) becomes
\[
\| x_{k+i} - x^* \| \leq 2 \eta L \| x_k - x^* \|_2 + n \beta L \| x_k - x^* \|_2
\]
\[
+ C \beta L \| x_k - x^* \| + C \beta L \sum_{j=1}^{n} \| x_{k-j} - x^* \|, \]

where \( \eta = 2 \eta L + n \beta \eta L, \ a_j = \beta \eta \) \( j=1, 2, \ldots, n \).

Let
\[ a = \frac{\eta}{\beta} \]

obviously \( a > 0 \), and let
\[ \alpha_j = \frac{a_j}{a}, \quad \xi_k = \| x_k - x^* \|, \quad \eta_k = a \xi_k. \]

Then by (10), we obtain
\[ \eta_{k+i} \leq \eta_k + \sum_{j=0}^{n} \alpha_j \eta_{k-j}, \quad \forall k \geq n+1. \]

Because \( \lim_{k \to \infty} \eta_k = 0 \), therefore for positive number \( \epsilon \), \( \epsilon \) there exists a \( k' > n+1 \) such that
\[ \eta_k < \epsilon, \quad \forall k \geq k', \]
so that obtain
\[ \eta_{k+i} \leq \epsilon, \quad (i=0, 1, \ldots, n). \]

And by (11), obtain
\[ \eta_{k+n+1} \leq \eta_{k+n} + \sum_{j=0}^{n} \alpha_j \eta_{k+n-j} \leq \sum_{j=0}^{n} \alpha_j \epsilon \leq \epsilon^2, \]
\[ \eta_{k+n+2} \leq \epsilon^2 + \frac{\alpha_j \epsilon}{\beta} \eta_{k+n-1} \leq \epsilon^2 (\alpha \epsilon + \frac{\alpha_j}{\beta}). \]

Proved by induction, we obtain
\[ \eta_{k+i} \leq \epsilon^{2i}, \quad (i=0, 1, \ldots) \]

where \( \mu_n = \mu_1 = \mu_2 = \ldots = \mu_n = 1, \mu_{n+1} = \mu_1 + \mu_2 + \ldots + \mu_n \) \( (i=n, n+1, \ldots) \).

Assume that \( \lambda_n \) is a positive root of \( t^{n+1} - t^n - 1 = 0 \). It is easy to prove that \( \lambda_n \) exist exactly one, and \( \lambda_n > \lambda_n, \lim_{n \to \infty} \lambda_n = 1 \).

Let \( \lambda = \lambda_n \), then obviously for \( i = 0, 1, \ldots, n \) we have
\[ \mu_i \geq \lambda^{-i}. \]

If (13) holds until some \( i \geq n \), then
\[ \mu_{i+1} = \mu_i + \mu_{2i} \geq \lambda^{i+1} + \lambda^{2i} \geq \lambda^{i+1}(-\lambda^{i} + \lambda^{2i}) = \lambda^{i+2}. \]

Thus, by induction, (13) holds for any \( i \geq 0 \).

By (12) and (13), we obtain
\[ \xi_{k+i} = \frac{1}{\alpha} \eta_{k+i} \leq \frac{1}{\alpha} \epsilon^{2i}. \]

We have also
\[ \lim_{i \to \infty} \epsilon^{2i} = 0, \]
this is certainly true because of \( 0 < \epsilon < 1 \) and \( \lambda > 1 \). Inequality (11) expresses the order of rate of that, the sequence \( \{ \xi_k \} \) converges to zero, is same with the sequence \( \{ \xi_k \} \) at least. Because
\[ \lim_{i \to \infty} \epsilon^{2i} = 0, \]

therefore the sequence \( \{ \epsilon^{2i} \} \) possess \( \lambda \)-order-rate of convergence so that the sequence \( \{ \xi_k \} \) possess also.

The smaller is the \( n \), the positive root of \( t^{n+1} - t^n - 1 = 0 \) is more approaching \( 2 \), and the sequence \( \{ x_k \} \) converges to \( x^* \) more rapidly.